SUVTiop 25/02/15

Lukas Brantner - Formality of Kähler manifolds

Smooth projective varieties = complex submanifolds of $\mathbb{P}^n\mathbb{C}$

Q: What shapes can they have?

Thm: Let X be a complex submanifold of $\mathbb{P}^n\mathbb{C}$ $\quad T_{X,\mathbb{R}}$ = tangent bundle

$$A^{\bullet}_{X,\mathbb{R}} := \Gamma(\Lambda^{\bullet} T_{X,\mathbb{R}}) \simeq C^{\bullet}(X;\mathbb{R})$$

is always formal as a dga.

(I) Review of complex geometry

X complex manifold

$T_{X,\mathbb{R}}$ = real tangent bundle

$I \circlearrowright T_{X,\mathbb{R}}$ $\quad I^2 = -1$ (locally multiplication by i)

$T_{X,\mathbb{C}}$ = complex tangent bundle = $T_{X,\mathbb{R}} \otimes \mathbb{C} \circlearrowleft I \otimes id$ and this splits in eigenspaces

$$T_{X,\mathbb{C}} = \underbrace{T^{1,0}_{X,\mathbb{C}}}_{I \text{ acts as } i} \oplus \underbrace{T^{0,1}_{X,\mathbb{C}}}_{I \text{ acts as } -i}$$

(holomorphic tgt bundle)

The natural map $T_{X,\mathbb{R}} \hookrightarrow T_{X,\mathbb{C}} \twoheadrightarrow T^{1,0}_{X,\mathbb{C}}$ is an equivalence

Sections of $T^{1,0}_{X,\mathbb{C}}$ are locally $\sum_{i=1}^{n} f_i \, \partial_{z_i}$

(p,q)-decomposition

$$\Lambda^k T^*_{X,\mathbb{C}} = \bigoplus_{p+q=k} \left(\Lambda^p T^{1,0*}_{X,\mathbb{C}}\right) \otimes \left(\Lambda^q T^{0,1*}_{X,\mathbb{C}}\right)$$

$$A^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} A^{p,q}_{X,\mathbb{C}}$$

A typical section of $A^{p,q}$ locally looks like $\sum \underset{\mathcal{C}^\infty}{f_{I,J}} \, dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$

We extend d on $A^{\bullet}_{X,\mathbb{C}}$ by $\mathbb{C}$-linearity and this splits

$$d(A^{p,q}) \subseteq A^{p+1,q} \oplus A^{p,q+1}$$

we can write $d = \partial + \bar\partial \Rightarrow \partial^2 = \bar\partial = \partial\bar\partial + \bar\partial\partial = 0$

We can also define rotated d $\quad d^c := i(\bar\partial - \partial)$

$$d^c(A_{X,\mathbb{R}}) \subseteq A_{X,\mathbb{R}} \quad \text{and} \quad (d^c)^2 = 0$$

② Basics of Hodge theory

Hermitian manifold: Is a complex manifold + an hermitian metric on $T_X^{1,0}$.

$$h(\xi,\eta) = G(\xi,\eta) + i\,A(\xi,\eta)$$

FACT: There's exactly one $\omega \in A^{1,1}$ s.t. $A(\xi,\eta) = \omega(\xi,\bar\eta)$

Def: An hermitian metric is Kähler if $d\omega = 0$

A Kähler manifold is a manifold admitting a Kähler metric

FACT: An hermitian metric is Kähler iff locally there's a coordinate system

$$h(\partial_{z_i}, \bar\partial_{z_j}) = \delta_{ij} + O(z^2)$$

Theorem: Complex projective is Kähler w/ the Fubini Study metric $\bar\partial\partial \log|z|^2$

$\Rightarrow$ All smooth proj. varieties are Kähler.

Laplacians

Start w/ a compact hermitian manifold $(X,h) \Rightarrow \operatorname{Re} h$ is a riemannian metric

on $T_{X,\mathbb{R}} \Rightarrow$ we get a metric H on $T_{X,\mathbb{C}}$

Define an hermitian metric H on $\Lambda^k T^*_{X,\mathbb{C}}$ by usual std nonsense.

$\Rightarrow$ We get an inner product on $A^k_{\mathbb{C}}$

$$\langle\alpha,\beta\rangle = \int_{x\in X} H(\alpha_x,\beta_x)\, d\mathrm{Vol}_g$$

FACT: $d, \partial, \bar\partial, d^c$ have all adjoints $d^*, \partial^*, \bar\partial^*, (d^c)^*$

Q: Given $\alpha \in A^{\cdot}_{X,\mathbb{C}}$ is a closed and, if so, is it a shortest representative of $[\alpha]$?

Consider $\|\alpha + d\beta\|^2 = \|\alpha\|^2 + \|d\beta\|^2 + 2\,\mathrm{Re}\langle d^*\alpha, \beta\rangle$

α is minimal in $[\alpha]$ iff $d^*\alpha = 0$

Def: A form $\alpha \in A^{\cdot}$ is d-harmonic if $d^*\alpha = d\alpha = 0$ or equivalently
$\Delta_d \alpha := (dd^* + d^*d)\alpha = 0$.

Elliptic operator theory:

Thm: If X is compact hermitian there is a decomposition

$$A_{X,\mathbb{C}} = \ker(\Delta_d) \oplus \mathrm{Im}(\Delta_d)$$
$$id = \mathcal{H}_d(-) + \Delta\, G_d(-)$$

Green operator (commutes w/ d, d^*)

Parallel theorem holds for d^c

$$id = \mathcal{H}_{d^c} + \Delta_{d^c} G_{d^c}$$

If X is Kähler then

① $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial} = \Delta_{d^c}$
② $[\bar\partial, \partial^*] = 0 = [\partial, \bar\partial^*]$
$(\Rightarrow dd^{*c} = -d^c d^*)$

Kähler identities

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③ Formality

Lemma (dd^c lemma): Let X be Kähler. If $\alpha \in A^k_{X,\mathbb{C}}$ is

- d-closed
- d^c-closed
- d-exact or d^c-exact

$\Rightarrow$ $\alpha = dd^c\beta$ for some β

Proof: Assume α d-exact. Kähler $\Rightarrow$ $\Delta_d = \Delta_{d^c}$, $\mathcal{H}_d = \mathcal{H}_{d^c}$, $G_d = G_{d^c}$
Write $\alpha = d\gamma$

Observe $\mathcal{H}_d\alpha = 0$ as $\langle \mathcal{H}_d\alpha, \mathcal{H}_d\alpha\rangle = \langle \mathcal{H}_d\alpha, \alpha\rangle$

$$= \langle d^*\mathcal{H}_d\alpha, \gamma\rangle = 0$$

$\Rightarrow \alpha = (dd^* + d^*d)\, G_d\alpha = dd^* G_d\alpha + d^* G_d d\alpha = dd^* G\alpha$

and similarly $\quad \alpha = d^c (d^c)^* G\alpha$

$$\Rightarrow \quad \alpha = dd^* G\left(d^c (d^c)^* G\alpha\right) = dd^c\left(-d^* G\, d^{c*} G\alpha\right)$$

<u>Thm</u>: If X compact Kähler $(A^\bullet_{X,\mathbb{R}}, d)$ is formal

<u>Proof</u>: Let ${}^cA_{X,\mathbb{R}} \subseteq A_{X,\mathbb{R}}$ subcomplex of d^c-closed forms

Let $Q_X = {}^cA_{X,\mathbb{R}} \big/ d^c(A_{X,\mathbb{R}})$ be the quotient

$$Q_X \longleftarrow {}^cA_{X,\mathbb{R}} \xrightarrow{i} A_{X,\mathbb{R}}$$

0-differential   Quasi-isomorphism

i_* is onto because if α is d-closed $\Rightarrow d^c\alpha$ is d^c-exact & d-closed

$\Rightarrow d^c\alpha = dd^c\beta \Rightarrow \alpha' = \alpha + d\beta \in {}^cA_{X,\mathbb{R}}$.

and so on and so forth.

Why $d = 0$ on Q_X

Pick $y \in {}^cA_{X,\mathbb{R}}$ $\quad dy$ is d-exact & d^c-closed

$\Rightarrow dy = dd^c z = d^c dz \Rightarrow [dy] = 0$. □

<u>Application</u>: Consider the space

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{C} \right\} = \text{complex Heisenberg group}$$

Let $\Gamma \in \mathcal{H}$ be the discrete subgroup of matrices w/ $x, y, z \in \mathbb{Z}[i]$

We can consider $X = \Gamma\backslash\mathcal{H}$ complex manifold

- $\dim_{\mathbb{C}} X = 3$
- $X \to \mathbb{C}/\mathbb{Z}[i]^2$

$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \longmapsto (x,y)$ shows that X is a 1-torus bundle over a 2-torus

$b_1 = 4$, $b_2 = 8$, $b_3 = 10$ (no Hodge theory obst. to Kähler)

<u>But</u> Consider the $(1,0)$-forms $\alpha = dx$, $\beta = dz$, $\phi = dy - x\,dz$ on $\mathcal{H}$

<u>FACT</u>: These are left-Γ-invariant $\Rightarrow$ descend to forms on X

<u>FACT</u>: $[\alpha], [\bar{\alpha}], [\beta], [\bar{\beta}]$ form a basis for $H^1(X, \mathbb{C})$

Observe
$$\alpha \wedge \beta = -d\phi \quad \Rightarrow [\alpha][\beta] = 0$$
$$\beta \wedge \beta = 0 \quad \Rightarrow [\beta][\beta] = 0$$

$$\langle [\alpha], [\beta], [\beta] \rangle = [-\phi \wedge \beta] \in H^2(X,\mathbb{C}) / \underbrace{\alpha H^1(X,\mathbb{C}) + H^1(X,\mathbb{C}) \cdot \beta}_{A}$$

observe if $[\gamma] \in A$, then

$t = [\gamma \wedge \alpha] =$ $[\bar{\alpha} \wedge \alpha \wedge \beta]$ and $[\bar{\beta} \wedge \alpha \wedge \beta]$

$\Rightarrow t \wedge \bar{t} = 0$

but $\eta = [+\phi \wedge \beta \wedge \alpha \wedge \bar{\phi} \wedge \bar{\beta} \wedge \bar{\alpha}]$

pulls back to a nonzero constant multiple of the volume form on $\mathcal{H}$

$\Rightarrow \mathrm{Vol}(X) = \int_X \eta \neq 0 \Rightarrow \eta \neq 0$. $\square$

$\Rightarrow X$ is not Kähler.