Formality of DGA's (everything / k fld)

$A^\bullet$ dga. $\mu: A^\bullet \otimes A^\bullet \to A^\bullet$.

$H^\bullet(A)$.

Defn: A dga $A^\bullet$ is <u>formal</u> if there is a chain of q-iso's

$$A^\bullet \xleftarrow{\sim} Z^\bullet \xrightarrow{\sim} H^\bullet(A).$$

For every $A^\bullet$, $A^\bullet = \underbrace{B^\bullet \oplus H^\bullet}_{Z^\bullet} \oplus L^\bullet$ as graded vector spaces.

$$d: L^\bullet \xrightarrow{\sim} B^\bullet[1]$$

$H^\bullet \to A^\bullet$ not a map of dga's.

Def: An A_∞-algebra $A^\bullet$ is a graded v.s. w/

$\mu_n: A^{\otimes n} \to A[2-n]$. s.t.

$$(R_n) \quad \sum_{\substack{r+s+t=n \\ s\geq 1}} (-1)^{r+st} \mu_{r+1+t}(1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = 0.$$

(R_1) $\mu_1\mu_1 = 0$.

(R_2) $0+2+0$, $0+1+1$, $1+1+0$. $\quad \mu_1\mu_2 \mp \mu_2(\mu_1 \otimes 1) - \mu_2(1 \otimes \mu_1) = 0$.

Leibniz rule.

(R_3) $0+3+0$, $0+2+1$, $1+2+0$, $2+1+0$, $0+1+2$, $1+1+1$

$$\mu_1\mu_3 + \overbrace{\mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2)}^{\text{associator}} \pm \mu_3(\mu_1 \otimes 1^{\otimes 2}) \pm \mu_3(1^{\otimes 2} \otimes \mu_1) \pm \mu_3(1 \otimes \mu_1 \otimes 1) = 0.$$

measures failure of associativity on chain level.

Def: A map $A^\bullet \xrightarrow{f} B^\bullet$ is a collection $f_n: A^{\otimes n} \to B[1-n]$ satisfying

$$(S_n) \quad \sum_{\substack{r+s+t=n \\ s\geq 1}} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t})$$
$$= \sum_{\substack{i_1+\dots i_\ell = n \\ i_j \geq 1}} (-1)^{\sum_j (\ell-j)(i_j-1)} \mu_\ell (f_{i_1} \otimes \dots \otimes f_{i_\ell})$$

$$(S_1) \quad f_1 \mu_1 = \mu_1 f_1$$

$$(S_2) \quad f_1 \mu_2 - f_2(\mu_1 \otimes 1) - f_2(1 \otimes \mu_1)$$
$$= \mu_1 f_2 + \mu_2(f_1 \otimes f_1)$$

Prethm: If $A^\bullet$ is a dga, there exists a unique A_∞-structure on $H^\bullet(A)$ with $\mu_1 = 0$ and an quasi-isom $H^\bullet(A) \to A^\bullet$.

Lemma: $f: A^\bullet \to B^\bullet$ where $m_1 = 0$ is an isom iff f_1 is invertible.

Thm: Fix $A^\bullet = B^\bullet \oplus H^\bullet \oplus L^\bullet$. $\exists!$ $(\mu_n), (f_n)$ s.t.

1) f_1 is chosen inclusion $H^\bullet \hookrightarrow A^\bullet$.
2) $\mu_1 = 0$
3) $\operatorname{im} f_n \subseteq L^\bullet \ \forall n \geq 2$
4) (μ_n) is an A_∞-str & (f_n) is a map of A_∞-algs.

Pf: (S_1) f_1 is chain map ✓.

(S_2). $\underset{\substack{\cap \\ H}}{\underline{f_1 \mu_2}} + \overset{\mu_1=0}{\cancel{(\text{stuff w/ } \mu_1)}} = \underset{\substack{\cap \\ Z}}{\mu_2(f_1 \otimes f_1)} + \underset{\substack{\cap \\ B^\bullet}}{\mu_1 f_2}$.

Define: $\mu_2(\alpha, \beta) = [f_1(\alpha) \cdot f_2(\beta)]$.

$df_2(\alpha, \beta) = f_1(\alpha\beta) - f_1(\alpha) \cdot f_1(\beta)$.

$$(S_3) \quad \underbrace{f_1 \mu_3}_{\in H} + f_2(\mu_2 \otimes 1) - f_2(1 \otimes \mu_2)$$

$$= \underbrace{\mu_1 f_3}_{\in B} + \mu_2(f_1 \otimes f_2) - \mu_2(f_2 \otimes f_1)$$

$$(S_n) \quad f_1 \mu_n + (\dots) = \mu_1 f_n + (\dots) \qquad \square$$

<u>Thm</u>: $A^\bullet$ is formal iff $\exists\, (f_n) : H^\bullet A \to A$ lifting the identity s.t. $H^\bullet A$ has the trivial A_∞-structure.

$A^\bullet$ is formal iff $\exists\, (f_n) : H^\bullet A \to A$

$$f_{n-1}(\Sigma \pm 1 \otimes \mu_2 \otimes 1) = \mu_2(\Sigma \pm f_i \otimes f_j) \,. \quad \left(\begin{array}{c}\text{equational criterion}\\ \text{for formality}\end{array}\right)$$

<u>Massey products</u>

A dga. $\alpha_1, \alpha_2, \alpha_3 \in H^\bullet(A)$. $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$.

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \left\{ [\bar{a}_{01} a_{13} + \bar{a}_{02} a_{23}] \;\middle|\; \begin{array}{l} [a_{01}] = \alpha_1 \\ [a_{12}] = \alpha_2 \\ [a_{23}] = \alpha_3 \end{array} \quad \begin{array}{l} d(a_{02}) = \bar{a}_{01} \cdot a_{12} \\ d(a_{13}) = \bar{a}_{12} \cdot a_{23} \end{array} \right.$$

$$(\bar{a} = (-1)^{|a|+1} a)$$

$$\langle \alpha_1, \dots, \alpha_n \rangle = \left\{ \left[\sum_{i=1}^{n-1} \bar{a}_{0i}\, a_{in} \;\middle|\; \begin{array}{l} [a_{i,i+1}] = \alpha_{i+1} \\ da_{ij} = \sum_{l=i+1}^{j-1} \bar{a}_{il}\, a_{lj} \end{array} \right] \right\}$$

(need to quotient by indeterminacy, e.g. for $n=3$, $\bar{\alpha}_1 H^\bullet(A) \oplus H^\bullet(A)\alpha_3$)

<u>Thm:</u> $A^\bullet$ dga., $H^\bullet A \xrightarrow{f} A$.

If $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \phi$, $\pm \mu_n(\alpha_1, \ldots, \alpha_n) \in \langle \alpha_1, \ldots, \alpha_n \rangle$.

<u>Pf:</u> Put $a_{ij} = \pm f_{j-i}(\alpha_{i+1}, \ldots, \alpha_j)$. Use (S_n). $\square$

<u>Example:</u> $\mathbb{Q}[x,y] \otimes \Lambda[\phi, \psi]$. $|x| = |y| = 2$ $dx = dy = 0$
$|\phi| = |\psi| = 3$ $d\phi = x^2$ $d\psi = xy$.

$\langle [x], [x], [y] \rangle \ni [\phi y + x\psi] \neq 0$.

$\mu_3([x],[x],[y]) \neq 0$. $\Rightarrow$ not formal.