JUVITOP 11/02/22

Rational homotopy theory

§1 An algebraic model for rational spaces

Def: X is a rational space if $\pi_* X$ are f.d. $\mathbb{Q}$-vector spaces & X is simply connected

$\Rightarrow \mathcal{S}^{rat} \subseteq \mathcal{S}$ full ∞-subcategory

Let $\mathrm{CAlg}_{\mathbb{Q}} = E_\infty$-algebras in $\mathrm{Mod}_{\mathbb{Q}}$

We have a functor
$$F : (\mathcal{S}^{rat})^{op} \longrightarrow \mathrm{CAlg}_{\mathbb{Q}}$$
$$X \longmapsto C^*(X;\mathbb{Q})$$
singular chains

Thm: F is a fully faithful embedding w/ image is A s.t. A is coconnective & $\pi_{-1} A = 0$, $\pi_* A$ is f.d.

F has a left adjoint $A \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathbb{Q})}(A, \mathbb{Q})$

Thm: For pullback diagram of spaces s.t. some finiteness hyp. holds

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Then we have a pushout diagram in $\mathrm{CAlg}_{\mathbb{Q}}$

$$\begin{tikzcd} C^*(X';\mathbb{Q}) \arrow[r] \arrow[d] & C^*(Y';\mathbb{Q}) \arrow[d] \\ C^*(X;\mathbb{Q}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C^*(Y';\mathbb{Q}) \end{tikzcd}$$

(Known as Eilenberg - Moore spectral sequence)

(note: this in fact works over any field: no use of char 0 yet)

We have a unit map
$$X \longrightarrow \mathrm{Map}(C^*(X;\mathbb{Q}), \mathbb{Q})$$
$$x \longmapsto (C^*(X;\mathbb{Q}) \longrightarrow C^*(\{x\};\mathbb{Q}))$$

The unit map is an equivalence.

IDEA: Induction on the Postnikov tower.

$$\begin{tikzcd} \tau_{\leq n} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ \tau_{\leq n-1} X \arrow[r] & K(\pi_n X, n+1) \end{tikzcd} \rightsquigarrow \begin{tikzcd} C^*(\tau_{\leq n} X; \mathbb{Q}) \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ C^*(\tau_{\leq n-1} X; \mathbb{Q}) \arrow[r] \arrow[ur, phantom, "\lrcorner"] & C^* K(\pi_n X, n+1) \end{tikzcd}$$

So it is enough to prove the thm. for E-McLANE SPACES.

$$C^*(K(V, n+1); \mathbb{Q}) = \mathrm{Sym}^* V[-n-1]$$

and this gives the thm.

<u>§2: From E_∞-algebras to cdga</u>

<u>FACT</u>: You cannot strictify in char p because of Steenrod operations.

In char 0 you have polynomial deRham functor.

<u>Def</u>: A simplicial cdga $/\mathbb{Q}$ $A^{PL}_\bullet$

$$A^{PL}_n = \mathbb{Q}[x_0, \ldots, x_n] \otimes \Lambda[y_0, \ldots, y_n] \Big/ \left(\sum x_i = 0, \sum y_i = 0\right)$$

$$|x_i| = 0, \quad |y_i| = 1 \qquad dx_i = y_i$$

and this is a simplicial cdga $A^{PL}_\bullet$

$\phi : [m] \to [n]$

$$\phi^* : A^{PL}_n \to A^{PL}_m \qquad \phi^*(x_i) = \sum_{\phi j = i} x_j$$
$$\phi^*(y_i) = \sum_{\phi j = i} y_j$$

So we have a functor $sSet^{op} \to$ cdga given by right Kan extension

$$K \longmapsto \mathrm{Hom}_{sSet}(K, A^{PL}_\bullet)$$

<u>Prop</u>: There's an integration map for a Kan complex

$$\int : A^{PL}(K) \longrightarrow C^*(K; \mathbb{Q}) \quad \text{which is a q-iso.}$$

and $\int$ is an isomorphism of algebras on H^*.

Or you could say that there's a q-iso of simplicial cdgas between $A^{PL}_\bullet$ and the simplicial cdga which represents cochains

(that is A^{PL}_n is a contractible Kan complex for $n>0$)

<u>Def of $\int$</u>: Consider $\sigma: K \to (A^{PL})[n]$ in $A^{PL}(K)$

$$\forall x \in K_n \quad \left(\int f\right)(x) = \int_{\Delta^n} f(x)\, dx_0 \cdots dx_n$$

<u>Upshot</u>: There's an equivalence S^{rat} to 1-connected cdgas

<u>§3</u>: Minimal model

<u>Def</u>: A cdga$/\mathbb{Q}$ is <u>minimal</u> if $\exists$ a filtration satisfying the following conditions

① $A_{i+1} = A_i \otimes \mathrm{Sym}(V_i) \qquad dV_i = A_i$

② $dx \in A^{>0}\, A^{>0}$

<u>Note</u>: A is, in particular, a free dg algebra

A minimal model of A is a minimal A' together w/ a q-iso $A' \xrightarrow{\sim} A$.

<u>Thm</u>: Minimal models exists and are unique up to isom. Indeed q-iso's between min. models are isos.

<u>Pf</u>: Taking the minimal model is cofibrant replacement in a certain model category on cdga's. $\square$

<u>Example</u>: For odd spheres S^{2n+1} $\quad H^*(S^{2n+1}; \mathbb{Q})$ is a minimal model for $A^{PL}(S^{2n+1})$

$$H^*(S^{2n+1};\mathbb{Q}) = \Lambda[x]$$

$$H^*(S^{2n+1};\mathbb{Q}) \xrightarrow{\sim} A^{PL}(S^{2n+1})$$ given by choosing a representative for x.

For the even sphere $H^*(S^{2n};\mathbb{Q})$ is not free anymore

$\mathbb{Q}[x] \otimes \Lambda[y] \quad |y| = 4n+1 \quad dy = x^2$ is a minimal model.

Thm If A is a minimal model of $X \in \mathcal{S}^{rat}$ $A = (\mathrm{Sym}\, V, d)$

$$\pi_* X = \left(A^{>0} / A^{>0} \cdot A^{>0} \right)^\vee = V^\vee$$

because
$$\mathrm{Map}(S^{2n+1}, X) = \mathrm{Map}(A, \quad \Lambda[x]) = V^{2n+1}$$
$$\mathrm{Map}(S^{2n}, X) = \mathrm{Map}(A, \quad \mathbb{Q}[x] \otimes \Lambda[y]) = A^{>0} / A^{>0} \cdot A^{>0}$$

Let's consider the restriction of d to the indecomposables

$$d : A^{>0} / A^{>0} \cdot A^{>0} \longrightarrow A^{>0} / A^{>0} \cdot A^{>0} \otimes A^{>0} / A^{>0} \cdot A^{>0}$$

This yields the Whitehead product $\pi_* X \otimes \pi_* X \to \pi_* X$

$\Rightarrow \pi_*(\Omega X)$ is a Lie algebra

Ex: $dy = x^2 \Rightarrow [x, x] = y$ in S^{2n}

Ex: $X = \mathbb{C}P^n$, $H^*(X) = \mathbb{Q}[c_1] / c_1^{n+1}$ $\quad |c_1| = 2$

$\Rightarrow$ A minimal model for $\mathbb{C}P^n$ is

$$\mathbb{Q}[c_1] \otimes \Lambda[y_{2n+1}] \qquad dy_{2n+1} = c_1^{n+1}$$

$\Rightarrow \pi_* \mathbb{C}P^n \otimes \mathbb{Q} = \mathbb{Q}\langle c_1, y \rangle \qquad n > 1$

Ex: X H-space $\quad H^*(X; \mathbb{Q})$ = commutative Hopf algebra

$\Rightarrow$ we have a structure thm $\Rightarrow \mathrm{Sym}\, V = H^*(X; \mathbb{Q})$ for some V

$\Rightarrow \mathrm{Sym}\, V$ is a minimal model (w/ 0 differential)

$\Rightarrow \pi_* X \otimes \mathbb{Q} = V$

For every space X ΩX is a loop space

$$\pi_*(\Omega X) = V[-1]$$
$$H^*(\Omega X; \mathbb{Q}) = \mathrm{Sym}\, V[-1]$$

$X = SU(n)$ $\qquad H^*(SU(n);\mathbb{Q}) = \Lambda[x_3, \dots, x_{2n-1}]$

$$\Rightarrow \pi_*(SU(n)) \otimes \mathbb{Q} = \mathbb{Q}\langle x_3, \dots, x_{2n-1}\rangle$$