8. Akhil Mathew, 11/04

This is about the rational version of trace methods.

Let A be a simplicial ring, not necessarily commutative (this is the same as an E_1-algebra over $\mathbb{Z}$ that is connective). We can associate to this $K(A)$, the connective algebraic K-theory spectrum, and we care about it s homotopy groups. This is hard. Recall we also had $THH(A)$, a connective spectrum and also a $\mathbb{Z}$-module. This is $A \wedge_{A \wedge A} A$. I'm thinking of A as something over $\mathbb{Q}$.

In previous talks, we've seen that there's a natural map of spectra $K(A) \to THH(A)$, There's an S^1-action on $THH(A)$, which comes from the fact that you can realize it as the realization of a cyclic spectrum. So we get a lift $K(A) \to THH(A)^{hS^1}$. The theorem for today is that this is locally constant when you tensor with $\mathbb{Q}$.

Definition 8.1. Write $HC^-(A)$ for the negative cyclic homology of A; this is defined to be $HH(A)^{hS^1}$. (Here $HH(A)$ is the $\mathbb{Z}$-version.)

Theorem 8.2 (Goodwillie)**.** *Suppose $A \to A'$ is a morphism of simplicial rings such that $\pi_0 A \twoheadrightarrow \pi_0 A'$ is surjective, and the kernel is nilpotent. Then there is a homotopy pullback square:*

$$\begin{array}{ccc} K(A) \otimes \mathbb{Q} & \longrightarrow & HC^-(A \otimes \mathbb{Q}) \\ \downarrow & & \downarrow \\ K(A') \otimes \mathbb{Q} & \longrightarrow & HC^-(A' \otimes \mathbb{Q}) \end{array}$$

In relative K-theory, this says $\mathrm{fiber}(K(A) \to K(A')) \otimes \mathbb{Q} \simeq \mathrm{fiber}(HC^-(A) \to HC(A')) \otimes \mathbb{Q}$*.*

This is not Goodwillie's original statement; it's weird because HC^- is not a connective spectrum.

8.1. Norm maps.

Recall that for a spectrum X with G-action (for finite G), there is a norm map $X_{hG} \to X^{hG}$. This is an equivalence if G acts freely, i.e. $X = G_+ \wedge Y$. If X is acted on by S^1, you can do the same thing, but the norm map you get is $\Sigma X_{hS^1} \to X^{hS^1}$. If $X = S^1_+ \wedge Y$ for some Y, then this dimension-shifting norm is an equivalence.

(If $X = S^1_+$, then $X_{hS^1} = S^0$ but $X = \Sigma F(S^1_+, S^0)$, so you definitely need a shift in dimension. . .)

Part of Goodwillie's theorem is that the right square in

$$\begin{array}{ccccc} K(A) \otimes \mathbb{Q} & \longrightarrow & HC^-(A \otimes \mathbb{Q}) & \longleftarrow & \Sigma HC^+(A \otimes \mathbb{Q}) = HH(A \otimes \mathbb{Q})_{hS^1} \\ \downarrow & & \downarrow & & \downarrow \\ K(A') \otimes \mathbb{Q} & \longrightarrow & HC^-(A' \otimes \mathbb{Q}) & \longleftarrow & \Sigma HC^+(A' \otimes \mathbb{Q}) \end{array}$$

is also a homotopy pullback.

Example 8.3. Consider $A = \mathbb{Q} \oplus \mathbb{Q}[1]$. I want to compare $K(A)$ to $K(\mathbb{Q})$. Start by computing $HC^-(A)$. This is over $\mathbb{Q}$, so $HC^-(A) = HH(A)^{hS^1}$. A is the free E_∞-ring on X_1 (on S^1), and $HH(A) = \mathrm{Free}_{E_\infty}(S^1_+ \wedge S^1) = \mathrm{Free}_{E_\infty}(x_1, y_2)$, where the homotopy groups are $A(x_1) \otimes \mathbb{Q}[y_2]$. (Tensoring with S^1 commutes with any free construction.)

Let S^1 act on a spectrum X. Then $b : \pi_* X \to \pi_{*+1} X$ satisfies $b^2 = \eta b$, where η be the Hopf element. But we're working rationally, so b is a differential. $b(x_1) = y_2$ so $b(x_1 y_2^k) = y_2^{k+1}$.

Lemma 8.4. *Let S^1 acts on X where X is rational. Suppose b is exact on $\pi_* X$. Then $X^{hS^1} \to X$ induces an injection on π_* with image* $\ker(b)$. *Also, $\Sigma X_{hS^1} \simeq X^{hS^1}$.*

We'll take relative Hochschild homology, so we're ignoring the part in degree zero. The conclusion is that

$$\pi_* \operatorname{fiber}(HH(A) \to HH(\mathbb{Q}))^{hS^1} = \begin{cases} \mathbb{Q} & \text{if } * > 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Question 8.5. How do we compute $K(A) \otimes \mathbb{Q}$?

Answer: if X is a connective spectrum, then

$$\pi_* X \otimes \mathbb{Q} = \pi_*(\Omega^\infty X) \otimes \mathbb{Q} \overset{Hur}{\to} H_*(\Omega^\infty X, \mathbb{Q})$$

is an inclusion and the image are the primitives. So to take the K-groups, take Ω^∞, take the homology, and compute primitives.

We have a model for $\Omega^\infty K(A)$; ignoring the connected components, this is $BGL_\infty(A)^+$. We care about the homology, so we can ignore the $+$. So we're looking for the primitives in $H_*(BGL_\infty(A); \mathbb{Q})$.

A is not a discrete ring, so let me remind you what $BGL_\infty(A)$ is. A has a nontrivial π_0 and a π_1. Then $\pi_1 BGL_\infty(A) = GL_\infty(\mathbb{Q})$ and $\pi_2 BGL_\infty(A) = M_\infty(\mathbb{Q})$, and there are no other homotopy groups. Moreover the action of π_1 on π_2 via the adjoint action. We want the homology of this.

First, we're really interested in the relative K-theory w.r.t. $\mathbb{Q}$. We're ignoring things that come from $H_* GL_\infty(\mathbb{Q})$. We could look at $BGL_\infty(A) \to BGL_\infty(\mathbb{Q})$ that kills π_2. There's a fiber sequence $K(M_\infty(\mathbb{Q})) \to BGL_\infty(A) \to BGL_\infty(\mathbb{Q})$. We know the homology of the fiber, so try to do this using the Serre spectral sequence:

$$H_i(GL_\infty(\mathbb{Q}); \operatorname{Sym}^j M_\infty(\mathbb{Q})) \implies H_{i+2j}(BGL_\infty(A)).$$

Goodwillie has a preceding paper where he proves that the trace map $M_\infty(\mathbb{Q}) to \mathbb{Q}$ induces an isomorphism on $H_*(GL_\infty)$. So if you take $j = 1$, you're getting a contribution corresponding to $H_*(GL_\infty(\mathbb{Q}))$, which we're pretending we know. In Goodwillie's Relative K-theory paper, he finds a way of dealing with Sym^j.

8.2. **Another way of thinking about cyclic homology.** The idea is to replace group homology with Lie algebra homology, and you get something that looks like cyclic homology.

Let A be a discrete $\mathbb{Q}$-algebra. Instead of the general linear group, consider the general linear Lie algebra $\mathfrak{gl}_n(A)$, and $\mathfrak{gl}_\infty(A) = \varinjlim \mathfrak{gl}_n(A)$. The object of study is $H_*(\mathfrak{gl}_\infty(A); \mathbb{Q})$ (Lie algebra homology); this is a Hopf algebra, because there are maps $\mathfrak{gl}_n(A) \oplus \mathfrak{gl}_m(A) \to \mathfrak{gl}_{n+m}(A)$. But we just need the coalgebra structure.

Definition 8.6. Define additive K-theory to be $K^+(A) = \operatorname{Prim} C_*(\mathfrak{gl}_\infty(A); \mathbb{Q})$.

Theorem 8.7 (Loday-Quillen-Tsygan)**.** *There is a natural identification* $K^+(A)_{\geq 1} \simeq \Sigma HC(A)$.

K^+ works better than K; it commutes with geometric realization.

Now we restate the Goodwillie theorem in the fase where A is a $\mathbb{Q}$-algebra. Let $I \subset A$ be a nilpotent ideal. Then there is a natural identification

$$\operatorname{fiber}(K(A) \to K(A/I)) \otimes \mathbb{Q} \simeq \operatorname{fiber}(K^+(A) \to K^+(A/I)).$$

In general $K(A)$ and $K^+(A)$ aren't directly related, but they are related in the relative case when you quotient by a nilpotent ideal. Similarly, in general you can't expect $H_* GL_\infty$ to be related to $H_* \mathfrak{gl}_\infty$.

The goal is to find some way of comparing $H_*(GL_\infty(A); \mathbb{Q})$ with $H_*(\mathfrak{gl}_\infty(A); \mathbb{Q})$. There's a theory of Malcev and Quillen (that you can read about in the appendix to Goodwillie's relative K-theory paper) that says that there is an equivalence of categories between nilpotent groups which are uniquely divisible, and nilpotent Lie algebras over $\mathbb{Q}$. This comes from Lie theory. There's a canonical way to associate a nilpotent group to a nilpotent Lie algebra over $\mathbb{R}$, given by using the Baker-Campbell-Hausdorff formula (and things converge because of nilpotence). We're going to send a nilpotent Lie algebra $\mathfrak{g}$ to the grouplike elements in $(U\mathfrak{g})^\wedge$ (completion w.r.t. the augmentation ideal). (In the other direction, take primitives.)

Suppose V is a representation of a uniquely $\mathbb{Q}$-divisible nilpotent group. Say that V is nilpotent if the $\mathbb{Q}[G]$-action on V annihilates a power of the augmentation ideal. In this case, $H_*(G; V) \simeq H_*(\mathfrak{g}; V)$.

New setting: A is a ring, $I \subset A$ is a nilpotent ideal, and we want to know $K^+(A)$. So we're looking at

$$\operatorname{Prim} C_*(GL_\infty(A); \mathbb{Q}) \to \operatorname{Prim} C_()GL_\infty(A/I); \mathbb{Q}) \tag{8.1}$$

which relates to K. On the other hand, we're looking at

$$\operatorname{Prim} C_*(\mathfrak{gl}_\infty(A); \mathbb{Q}) \to \operatorname{Prim}(C_*(\mathfrak{gl}_\infty(A/I); \mathbb{Q})). \tag{8.2}$$

which relates to K^+. You can't compare them directly, because things aren't nilpotent.

Idea: let S be a finite subset of $\mathbb{N}$, and give it a total ordering. Consider $\Gamma_S \subset GL_S(A)$ (matrices which are congruent modulo I to a strictly upper-triangular matrix). I is nilpotent, so this is a nilpotent group. This is some system of subgroups of $GL_\infty(A)$.

Let X_1 be the fiber of (8.1), and X_2 the fiber of (8.2). You can construct a Lie algebra version of Γ_S, where you impose the same condition, but strictly upper triangular means zeros on the diagonal, instead of 1's.

Fact 8.8.

$$X_1 = \varinjlim \operatorname{Prim} C_*(\Gamma_S; \mathbb{Q})$$
$$X_2 = \varinjlim \operatorname{Prim} C_*(\Gamma_S^{\mathrm{Lie}}; \mathbb{Q})$$

The theorem is that there is an equivalence of these.