## 7. Adam Al-Natsheh, 10/28

This is all based on Goodwillie's paper "Relative algebraic $K$-theory and cyclic homology".

Let $sAb$ denote the category of simplicial abelian groups. $A \in sAb$ gives rise to two different chain complexes:

- $Ch(A)_p = A_p$ with differential $da = \sum_{i=0}^{p} (-1)^n d_i a$ (where $d_i$ are the face maps) for $a \in A_p$;
- the normalized chain complex $N(A) \subset Ch(A)$ where $N(A)_p = \bigcap_1^p \ker(d_i) \subset A_p$. The other way to view this is $N(A) = Ch(A)/D(A)$ where $D(A)_p = \sum_1^p \operatorname{im} s_i$ where $s_i$ are the degeneracies.

**Theorem 7.1** (Dold-Kan correspondence)**.** *$N : sAb \to Ch_{\geq 0}(A)$ is an equivalence of categories.*

**Fact 7.2.** *If $G \in sGp$ is a simplicial group, the underlying simplicial set is a Kan complex.*

**Theorem 7.3.** *Let $A \in sAb$. Then $\pi_* A \cong H_*(N(A)) \cong H_*(Ch(A))$.*

A bisimplicial object in $C$ is a functor $X : \Delta^{op} \times \Delta^{op} \to C$. These have geometric realizations. The standard way to define this is in terms of some coend.

Now imagine $C = Set$.

**Definition 7.4** (Diagonal)**.** The diagonal is a simplicial set $d(X)_k = X_{k,k}$.

This is equivalent to the realization of $X$.

These constructions extend to simplicial rings without issues. In particular, $H_n(Ch(R)) = \pi_n R$.

**Definition 7.5** (Free simplicial ring)**.** Say that a simplicial ring $R$ is free if the following conditions hold:

(1) $R_p$ is free (i.e. it's $F(S)$ for some set, where $F$ is left adjoint to the forgetful functor... it's like words and stuff);
(2) we can choose bases such that $s_i : R_p \to R_{p+1}$ sends bases to bases.

I want to associate some free simplicial ring $\varphi(R)$ to $R$, with a map $\varphi(R) \to R$. Do this by defining

$$\varphi(R)_p = (FU)^{p+1} R_p$$

where the maps come from the adjunction.

If $G \in sGp$, let $BG$ be the classifying space, i.e. the simplicial set constructed by:

(1) take the nerve of each $G_n$ (this gives a bisimplicial set);
(2) take the diagonal.

In particular, $BG_n = (BG_n)_n$.

We want to know under what conditions we have $|G| \cong \Omega|BG|$. This holds if $G$ is a grouplike simplicial monoid, i.e. $\pi_0 G$ is a group.

7.1. **Algebraic $K$-theory of $BG$.** Let $R$ be a simplicial ring. Let $M_n(R)$ be the simplicial ring given by $(M_n(R))_p = M_n(R_p)$. Note that this gives $n^2$ copies of $R$. In particular, $\pi_*(M_n(R)) \cong M_n(\pi_* R)$.

Define $\widehat{GL_n}(R)$ as the pullback in:

$$\begin{tikzcd}
\widehat{GL_n}(R) \arrow[r] \arrow[d] & M_n(R) \arrow[d, "R\mapsto\pi_0 R"] \\
GL_n(\pi_0 R) \arrow[r] & M_n(\pi_0(R))
\end{tikzcd}$$

Let $\widehat{GL}(R) = \bigcup \widehat{GL_n}(R)$. I want to see that we can do the plus construction: note that $\pi_1(B\widehat{GL_k}(R)) = GL_k(\pi_0 R)$, so define

$$K_i(R) = \pi_i(B\widehat{GL}(R)^+).$$

I'll define cyclic homology for:

(1) rings and bimodules
(2) cyclic cohomology groups
(3) cyclic chain complexes
(4) simplicial rings.

Let $R$ be a ring, and $B$ a bimodule over $R$. (We care most about the $B = R$ case.) Then construct $Cyc(R; B)_n = B \otimes R^{\otimes n}$, with differential:

$$d_i(b \otimes r_1 \otimes \ldots \otimes r_n) = \begin{cases} br_1 \otimes \ldots \otimes r_n & i = 0 \\ b \otimes r_1 \otimes \ldots r_i r_{i+1} \otimes \ldots \otimes r_n & 0 < i < n \\ r_n b \otimes r_1 \otimes \ldots \otimes r_{n-1} & i = n \end{cases}$$

and $s_i$ inserts a $\otimes$ after $r_i$. If $R = B$ then $\mathbb{Z}/(n+1)$ acts on $Cyc(R)_n$, where the generator $t_{n+1} \in \mathbb{Z}/(n+1)$ sends $r_0 \otimes \ldots \otimes r_n \mapsto r_n \otimes \ldots \otimes r_{n-1}$ (cyclic permutation of factors).

Write $Z(R)$ for $Cyc(R)$. These are called cyclic abelian groups. A cyclic abelian group $X$ defines a double chain complex $B_{**}(X)$, where

$$B_{p,q} = \begin{cases} 0 & q < p \\ \overline{X}_{q-p} & q \geq p \end{cases}$$

where $\overline{X}_n = X_n / \sum_0^{n-1} s_i(X_{n-1})$. The vertical boundary (i.e. the boundary map on $B_{p,*}(X)$) is $b = \sum_0^{q-p} (-1)^i d_i$, and the horizontal boundary is $b' = t_{n+2} s_n \sum_1^{n+1} ((-1)^n t_{n+1})^k$ (this is the place where the cyclic structure gets used!).

Now let $\alpha \leq \beta$. Let $T_*^{\alpha,\beta}$ be the complex just coming from the columns between $\alpha$ and $\beta$, i.e.

$$T_n^{a,\beta} = \prod_{\alpha \leq p \leq \beta} B_{p,n-p}(X).$$

Now define the Hochschild homology:

$$H_n(X) = H_n T_*^{0,0}(X),$$

the cyclic homology

$$HC_n(X) = H_n T_*^{0,\infty}(X),$$

the periodic homology

$$HP_n(X) = H_n T^{-\infty,\infty}(X)$$

and the negative cyclic homology

$$HC_*^-(X) = H_n T_*^{-\infty,0}(X).$$

The trace map is a map from $K$-theory to Hochschild homology that factors through negative cyclic homology. The map from $HC_*^-$ to $H_n$ just takes $T_*^{-\infty,0} \to T_*^{0,0}$.

If $R$ is a simplicial ring, apply $Z$ to each dimension, to get $Z(R)$. There's a functor $Ch : Ab^{\Delta^{op}} \to Ch(Ab)$ such that $Ch(Z(R))$ is a cyclic chain complex – a simplicial chain complex with an action of $\mathbb{Z}/(n+1)$ on the $n^{th}$ dimension. We can do everything we did before with cyclic chain complexes; you get hypercomplexes, and you can take the homology there with the same definitions as before.

7.2. **The trace.** I think you need some flatness assumption but for the constructions here I won't need it, so I'll ignore it. (But really, you should first define it for simplicial rings that are flat over $\mathbb{Z}$, ...)

Let $R$ be a simplicial ring. Then $T_*^{0,0}R$ is the Hochschild double complex from before: $\{T_p^{0,0}R_q\}$. If $G$ is a simplicial abelian group, define $C_*BG = \{(Ch_pB(G_q))\}$. Define a map $t : C_*BG \to T_*^{0,0}(\mathbb{Z}G)$ (here $\mathbb{Z}$ means taking the free abelian group levelwise). This map will be constructed in two steps:

(1) If $H$ is an ordinary group, I have a map $\mathbb{Z}BH \to Cyc(\mathbb{Z}H, \mathbb{Z}H)$ (on the right, $\mathbb{Z}$ means the group ring, and on the left, levelwise abelianization) defined by $BH_p = H^p$, i.e. take $(h_1, \ldots, h_p) \mapsto (h_p^{-1} \ldots h_1^{-1}) \otimes h_1 \otimes \ldots \otimes h_p$.
(2) Apply to chain complexes: $Ch_*(\mathbb{Z}BH) \overset{t(H)}{\to} N_*(Cyc(\mathbb{Z}H, \mathbb{Z}H))$

This was for discrete groups. For simplicial groups, apply this to each level.

Write $G := \widehat{GL}(R)$, where $R$ is a simplicial ring. Before, I had a free resolution $\varphi(G) \to G$. You also get $\varphi(G) \to \langle \varphi(G) \rangle$, which is given by formally adding inverses. (You can do this for any simplicial monoid.)

$$\begin{tikzcd}
C_*BG & C_*B\varphi(G) \arrow[l] \arrow[r] & C_*\langle\varphi(G)\rangle \arrow[d] \\
T_*^{0,0}\mathbb{Z}(G) \arrow[d, "\varepsilon"] & T_*^{0,0}\mathbb{Z}(\varphi(G)) \arrow[l] \arrow[r] & T_*^{0,0}\mathbb{Z}\langle\varphi(G)\rangle \\
T_*(R) & &
\end{tikzcd}$$

I claim that all the backwards maps are quasi-isomorphisms. Except for $\varepsilon$, this is basically gotten from

$$\begin{tikzcd}
\varphi(H) \arrow[r] \arrow[d] & H \\
\langle\varphi(H)\rangle &
\end{tikzcd}$$

Now apply homology: $H_i : B\widehat{GL}(R) \xrightarrow{\tau} H_i(R)$. This is the trace.