ALGEBRAIC K-THEORY AND SPECIAL VALUES OF L-FUNCTIONS: BEILINSON'S CONJECTURES. (TALK NOTES)

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Contents

The following notes are expanded from a talk I gave in the graduate topology seminar "Juvitop" at MIT. The main reference is [Nek94]. L-functions form a central topic in algebraic number theory, and their special values at integer arguments contain rich arithmetic information. Beilinson's conjectures offer a framework to understand the somewhat mysterious transcendental nature of these special values. The central concept is that of a regulator map from a K-theoretical object to a Hodge-theorical object. Please inform me of corrections and comments at yihang@math.harvard.edu.

Notation: If $f(s)$ is a meromorphic function near $s = s_0 \in \mathbb{C}$, we denote by $f^*(s_0)$ the leading coefficient of the Laurent expansion of f at s_0. We call this the *special value* of f at s_0.

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1. Classical motivation

1.1. **Some classical identities.** The following identities are instances of special values of L-functions:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6} \tag{1.1}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90} \tag{1.2}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4} \tag{1.3}$$

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\cdots=\frac{\log(1+\sqrt{2})}{\sqrt{2}}. \tag{1.4}$$

Here (1.1) and (1.2) are due to Euler, (1.3) is due to Leibniz, and (1.4) is due to Dedekind (1839).

1.2. **Riemann's zeta function.** In modern number theory, L-functions play a central role. Roughly speaking, they are Dirichlet series $\sum a_n n^{-s}$ whose coefficients a_n contain arithmetic information. The first example is Riemann's zeta function:

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p(1-p^{-s})^{-1}.$$

The infinite sum and product converge absolutely for $\Re s>1$. It is classical (due to Riemann) that $\zeta(s)$ has a meromorphic extension to $\mathbb{C}$, with a simple pole only at $s=1$, and it has a functional equation relating its values at s and $1-s, s\in\mathbb{C}$.

We are interested in the special values (i.e. values at integer arguments) of $\zeta(s)$. Euler shows the following formula at positive even integers:

$$\zeta(2m)=\frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!},\ m\in\mathbb{Z}_{>0}, \tag{1.5}$$

where $B_{2m}\in\mathbb{Q}$ is the Bernoulli number given by the following expansion

$$t(e^t-1)^{-1}=\sum_{k\geq 0}B_k\frac{t^k}{k!}.$$

In particular, we get (1.1) and (1.2).

On one hand, the rational numbers B_{2m} contain interesting global information about the arithmetic of cyclotomic fields. (c.f. Kummer's Theorem and Theorem of Herbrand-Ribet. This circle of ideas developed into what is known as Iwasawa theory.) On the other hand, $\zeta(2m)$ has π^{2m} as its "transcendental part", which we would like to understand in general.

Using the functional equation, we easily get

$$\zeta(1-2m)=\frac{B_{2m}}{2m},m\in\mathbb{Z}_{>0}.$$

For example, we have $\zeta(-1)=-1/12$, which can be suggestively written as

$$1+2+3+\cdots=-1/12.$$

Historically, Euler observed this identity using formal operations.

The rest cases are the positive odd integers and the negative even integers. They are of course again related by the functional equation. The nature of these zeta values remain very mysterious. It is known that infinitely many of $\zeta(2n+1), n\in\mathbb{Z}_{>0}$

are irrational (Rivoal 2000), and $\zeta(3)$ is irrational (Apéry 1979). But it is unknown, for instance, whether $\zeta(5)$ is irrational.

At the even negative integers, the functional equation shows that ζ has simple zeros there, and the leading Taylor coefficients $\zeta^*(-2m)$ are easily related to $\zeta(2m+1)$.

The beauty of the Beilinson-type conjectures is that they offer a recipe for understanding these zeta values as elements of $\mathbb{R}^\times/\mathbb{Q}^\times$, not only at positive even integers but also at positive odd integers, in terms of algebraic K-theory.

1.3. **Dedekind zeta functions.** Generalizing Riemann's zeta function, there are Dedekind zeta functions $\zeta_F(s)$ associated to any number field F. By definition,

$$\zeta_F(s) = \sum_{\mathfrak{a}} |\mathcal{O}_F/\mathfrak{a}|^{-s} = \prod_{\mathfrak{p}} (1 - |\mathcal{O}_F/\mathfrak{p}|^{-s})^{-1},$$

where $\mathfrak{a}$ runs through the ideals of $\mathcal{O}_F$ and $\mathfrak{p}$ runs through the non-zero prime ideals of $\mathcal{O}_F$. The infinite series and the infinite product converge absolutely for $\Re s > 1$. Like Riemann's zeta function, the Dedekind $\zeta_F(s)$ has meromorphic extension to $\mathbb{C}$ with a simple pole only at $s = 1$, and there is a functional equation relating s and $1 - s$.

Example 1.3.1. We have

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(s, \chi_{\mathbb{Q}(i)}) = \zeta(s)(1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots).$$

$$\zeta_{\mathbb{Q}(\sqrt{2})}(s) = \zeta(s)L(s, \chi_{\mathbb{Q}(\sqrt{2})}) = \zeta(s)(1 - 3^{-s} - 5^{-s} + 7^{-s} + 9^{-s} - 11^{-s} - 13^{-s} + \cdots).$$

To illustrate once more that the special values of L-functions contain interesting global arithmetic information, we have the following

Theorem (Class number formula)**.** *The residue of $\zeta_F(s)$ at $s = 1$ is given by*

$$\frac{2^{r_1}(2\pi)^{r_2}}{|d_K|^{1/2}\, w(K)} h(F)R(F).$$

Here as well as in what follows we set r_1 and r_2 to be the number of real and complex places of F, respectively. (i.e. real embeddings and complex embeddings up to complex conjugation). We define

$$h(F) = |\mathrm{Pic}(\mathcal{O}_F)|,$$

$$R(F) = \left| \det(\log |u_i|_j)_{1 \le i,j \le r_1+r_2-1} \right|,$$

where $\{u_i\}_{1 \le i \le r_1+r_2-1}$ is a $\mathbb{Z}$-basis for $\mathcal{O}_F^\times/\operatorname{tors} \cong \mathbb{Z}^{r_1+r_2-1}$, and $|\cdot|_j$ is the absolute value induced by the j-th archimedean place. (We are omitting one archimedean place.)

As examples, identity (1.3) corresponds to the fact that $\mathbb{Z}[\sqrt{-1}]$ is PID, and (1.4) corresponds to $\mathbb{Z}[\sqrt{2}]$ being PID.

We see that once again the special values of ζ_F contain global information such as $h(F)$. This is remarkable because ζ_F is defined by simply multiplying together local information: we have the terms $|\mathcal{O}_F/\mathfrak{p}|$ which can be thought of as a kind of mod p point counting.

For this talk the most interesting term is $R(F)$, called the regulator of F. It determines the "transcendental part" of the zeta value. For example, for $F = \mathbb{Q}(\sqrt{2})$, we have

$$u = 1+\sqrt{2},\ R(F) = |\det(\log|u|)| = \log(1+\sqrt{2}).$$

This is the same $\log(1+\sqrt{2})$, so to speak, as in (1.4).

We can regard $R(F)$ as follows. The maps $\log|\cdot|_j$ form a regulator map

$$r : \mathcal{O}_F^\times \to \mathbb{R}^{r_1+r_2-1}. \tag{1.6}$$

The domain of r is $\mathcal{O}_F^\times \cong K_1(\mathcal{O}_F)$, and it is known to be of rank r_1+r_2-1 (Dirichlet's Unit Theorem). The map r is of full rank, and $R(F)$ is equal to the covolume of r.

1.4. **Higher regulators.** One might ask whether the higher K groups $K_i(\mathcal{O}_F)$ can be related to other values of ζ_F.

Theorem (Borel, 1972)**.** *Let $m \in \mathbb{Z}_{>1}$. Set*

$$d_m := \begin{cases} r_2, & m \text{ even} \\ r_1+r_2, & m \text{ odd} \end{cases}.$$

Then

$$\dim_{\mathbb{Q}} K_{2m-1}(\mathcal{O}_F)\otimes\mathbb{Q} = d_m.$$

We remind the reader that for $i \in \mathbb{Z}_{\geq 3}$, we have $K_i(\mathcal{O}_F) \cong K_i(F)$. Here the significance of the number d_m is that it is also equal to the order of vanishing of $\zeta_F(s)$ at $s = 1-m$. By the functional equation, $\zeta_F^*(1-m)$ is related to $\zeta(m)$.

Lichtenbaum asked the following question: For $m \in \mathbb{Z}_{>1}$, can we define a regulator map $r_m : K_{2m-1}(\mathcal{O}_F) \to \mathbb{R}^{d_m}$, generalizing (1.6), in such a way that the covolume of r_m is related to $\zeta_F^*(1-m)$ (or $\zeta_F(m)$) ?

This project was accomplished by Borel in the 1970's. Since it is known that Borel's approach is equivalent to Beilinson's more general approach, we will not discuss the former here.

2. Motivic L-functions

Number fields are zero dimensional arithmetic objects. We move forward to higher dimensions. Conjecturally, one can associate L-functions to *motives*, the latter being objects of a conjectural nature that should capture cohomological information from algebraic geometry in a universal way. The resulting L-functions are commonly referred to as *motivic L-functions*. These represent the rough idea of "L-functions of a geometric origin", as opposed to other interesting L-functions of an analytic origin, namely the automorphic L-functions. Of course Langlands' program conjectures that the motivic L-functions form a well characterized subset of the automorphic L-functions.

2.1. **Realizations of motives.** We will consider motives of the form

$$M = h^i(X)(n) \tag{2.1}$$

where X is a smooth projective variety over $\mathbb{Q}$. We say M is a pure motive of weight

$$w = i - 2n.$$

Here the symbol (n) is called the n-th Tate twist, c.f. Example 2.1.3 below. The reader can think of (2.1) as a formal symbol. What is important are the various realizations of M.

- Betti realization: Consider $M_B = H^i(X(\mathbb{C}), \mathbb{Q}(n))$. Here the cohomology is taken with respect to the analytic topology on $X(\mathbb{C})$, and $\mathbb{Q}(n) := (2\pi i)^n\mathbb{Q}$. The space M_B is a $\mathbb{Q}$-Hodge structure pure of weight w, equipped with a $\mathbb{Q}$-linear involution F_∞ on M_B, called the *infinite Frobenius*, given by the action of the complex conjugation simultaneously on $X(\mathbb{C})$ and $\mathbb{Q}(n)$. These structures are compatible in the sense that the action of $F_\infty \otimes c$ (where c is the complex conjugation in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$) on $M_B \otimes_{\mathbb{Q}} \mathbb{C}$ preserves the Hodge decomposition
$$M_B \otimes \mathbb{C} = \bigoplus_{p+q=w} H^{p,q}.$$
- l-adic realization. Let l be a prime number. Consider $M_l = H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(n)$. This is a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on a finite dimensional $\mathbb{Q}_l$-vector space, pure of weight w. The twist (n) means that the Galois representation is twisted by the n-th power of the l-adic cyclotomic character of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.
- De Rham realization: Consider $M_{\mathrm{dR}} = H^i(X, \Omega^\cdot)(n)$. This is a finite dimensional $\mathbb{Q}$-vector space with a decreasing filtration. Here $H^i(X, \Omega^\cdot)$ is the hypercohomology of the complex
$$\Omega^\cdot : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{Q}} \xrightarrow{d} \Omega^2_{X/\mathbb{Q}} \xrightarrow{d} \cdots$$
of sheaves w.r.t. the Zariski topology of X. The Tate twist (n) here simply means shifting the indexing of the filtration, so that the filtration on M_{dR} is given by
$$F^k M_{\mathrm{dR}} := H^i(X, \Omega^\cdot_{\geq k+n}),$$
where $\Omega^\cdot_{\geq j}$ is the complex obtained from $\Omega^\cdot$ by replacing the terms $\Omega^i_{X/\mathbb{Q}}$ by zero for $i < j$.

There are canonical comparison isomorphisms:

- Betti to de Rham:
$$I : M_B \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}.$$
Under this isomorphism, $(F^k M_{\mathrm{dR}}) \otimes \mathbb{C} \subset M_{\mathrm{dR}} \otimes \mathbb{C}$ corresponds to $\bigoplus_{p \geq k} H^{p,q} \subset M_B$. The action $\phi_\infty \otimes c$ on LHS corresponds to the action of $\mathrm{id} \otimes c$ on RHS.
- Betti to l-adic:
$$I_l : M_B \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} M_l.$$
This implicitly depends on the choice of an embedding $\bar{\mathbb{Q}} \to \mathbb{C}$. With such a choice, we can view complex conjugation as an element $c' \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then the action of $\phi_\infty \otimes \mathrm{id}$ on LHS corresponds to the action of c' on RHS, through the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on RHS.

We will think of the data $(M_B, M_l, M_{\mathrm{dR}}, I, I_l)$ as associated to M.

Definition 2.1.1. For any subring A of $\mathbb{R}$, define $\mathcal{MH}^+_A$ to be the category of mixed A-Hodge structures together with an A-linear involution ϕ_∞ such that ϕ_∞ preserves the weight filtration and $\phi_\infty \otimes c$ preserves the hodge filtration. Let
$$\mathcal{M}_B := \mathcal{MH}^+_{\mathbb{Q}}$$

and

$$\mathcal{M}_\infty := \mathcal{MH}_{\mathbb{R}}^+.$$

Tensoring with $\mathbb{R}$ naturally induces a functor $\mathcal{M}_B \to \mathcal{M}_\infty$, denoted by $\cdot \otimes \mathbb{R}$

Definition 2.1.2. Let l be a prime number. Define $\mathcal{M}_l$ to be the category of continuous representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over finite dimensional $\mathbb{Q}_l$-vector spaces.

Thus we have $M_B \in \mathcal{M}_B$ and $M_l \in \mathcal{M}_l$.

Example 2.1.3. We can apply the usual linear algebra operations to get new motives, e.g. tensor products, direct sums, taking duals. These can be understood simply on the level of the realizations. For instance, the so-called *Tate motive*

$$M = \mathbb{Q}(1) = h^0(\operatorname{Spec} \mathbb{Q})(1)$$

is the dual of $h^2(\mathbb{P}^1_{\mathbb{Q}})$. It is a pure motive of weight -2. We have

$$M_l = (\varprojlim_k \mu_{l^k}(\bar{\mathbb{Q}})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$M_B = (2\pi i)\mathbb{Q},$$

where the Hodge decomposition is the pure Hodge type $(-1,-1)$ and ϕ_∞ acts as -1. We have

$$M_{\mathrm{dR}} = \mathbb{Q}$$

with $F^{-1} = \mathbb{Q}, F^0 = 0$. The Betti to de Rham comparison isomorphism is as follows. We think of $M_B \subset M_B \otimes \mathbb{C}$ as $(2\pi i)\mathbb{Q} \subset \mathbb{C}$. Then under the comparison isomorphism, M_{dR} is just the standard copy of $\mathbb{Q}$ included in $\mathbb{C}$. In other words, there is a way to choose a $\mathbb{Q}$-basis 1_B for M_B and 1_{dR} for M_{dR}, such that under the comparison isomorphism

$$I : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{C},$$

$1_B \otimes 1$ goes to $1_{\mathrm{dR}} \otimes 2\pi i$.

In general we have the relation

$$h^i(X)(n) = h^i(X) \otimes \mathbb{Q}(1)^{\otimes n}.$$

2.2. **L-functions.** Consider a motive M as in (2.1). Let p be a prime number. Choosing an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, we get the inertia and decomposition subgroups at p:

$$I_p \subset D_p \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

The quotient $D_p/I_p \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is topologically generated by the arithmetic Frobenius $\phi : x \mapsto x^p$. Fix another prime number $l \neq p$. Consider

$$P_p^l(M,T) := \det(1 - \phi^{-1}T, M_l^{I_p}).$$

This can be checked to be independent of the choice of ι_p.

Conjecture 2.2.1. *$P_p^l(M,T)$ is a polynomial in T with coefficients in $\mathbb{Z}$, and independent of l.*

This conjecture is known when X has good reduction at p, which means that there is a smooth projective model $\mathscr{X}$ of X over $\mathbb{Z}_p$. In this case I_p acts trivially on M_l and the conjecture follows from the proof of Weil's conjectures by Deligne. In particular, $P_p^l(M,T)$ is independent of l for almost all p. In the following we will assume this conjecture and write

$$P_p(M,T) := P_p^l(M,T).$$

If p is a prime at which X is not known to be of good reduction, we will say that p is a bad prime.

Let $L_p(M,s) = P_p(M,p^{-s})^{-1}, s \in \mathbb{C}$. The L-function associated to M is defined as

$$L(M,s) = \prod_p L_p(M,s). \tag{2.2}$$

There are conjectures about the factors L_p at bad primes p (purity conjectures) that will guarantee that the above infinite product converges absolutely and without zeros for $\Re s > 1 + w/2$.

Example 2.2.2. $L(\mathbb{Q}(1),s) = \zeta(s+1)$.

Example 2.2.3. L-functions of zero dimensional motives recover the classical notion of Artin L-functions (including Dedekind zeta functions discussed above and Dirichlet L-functions). For instance for F a number field we have

$$\zeta_F(s) = L(h^0(\operatorname{Spec} F)(0), s).$$

Generalizing properties of the Riemann and Dedekind zeta functions, the following conjecture about motivic L-functions is standard.

Conjecture 2.2.4. *(1) $L(M,s)$ has a meromorphic extension to $\mathbb{C}$, holomorphic outside $1 + w/2$. If w is odd, then $L(M,s)$ is entire.*

(2) $L(M,s)$ satisfies a functional equation (of a precise form) relating s and $w + 1 - s$.

We can think of the stripe

$$w/2 < \Re s < 1 + w/2$$

as the *critical stripe* of $L(M,s)$. Its central line $\Re s = (1 + w)/2$ is the line of symmetry for the functional equation. To the right of the critical stripe, the Euler product (2.2) converges absolutely without zeros (conjecturally). We will call this region the *convergence region*. We call $s = (1 + w)/2$ the *central point* and $s = 1+w/2$ the *near central point*. Thus the near central point is conjecturally the only possible pole for $L(M,s)$.

Conjecture 2.2.4 is regarded as far from approachable in general. For instance, for elliptic curves E over $\mathbb{Q}$ the conjecture for $L(h^1(E),s)$ is known only after the famous Modularity Theorem, which implies Fermat's Last Theorem. As for now the only known cases of the conjecture, apart from zero dimensional L-functions, are related to either abelian varieties or Shimura varieties.

3. Beilinson's conjectures on special values of L-functions

Consider a motive $M = h^i(X)(n)$ as in (2.1). The integers $i, n, w = i - 2n$ will appear in the following discussion.

3.1. **Elementary reduction.** We would like to study the behavior of $L(M,s)$ at an integer $s \in \mathbb{Z}$. Using the elementary relation

$$L_p(M, s+m) = L_p(M(m), s), m \in \mathbb{Z},$$

where $M(m) := M \otimes \mathbb{Q}(1)^{\otimes m} = h^i(X)(n+m)$, we reduce to studying $s = 0$. Moreover, using the conjectural functional equation we reduce to the case where $s = 0$ lies at or to the right of the central point $(1 + w)/2$. i.e. $w < 0$.

When $w = -1$, $s = 0$ is the central point. When $w = -2$, $s = 0$ is the near central point. When $w < -2$, $s = 0$ lies in the convergence region. We remark that in this last case $L(M, 0)$ can in principle be computed using (2.2), without assuming any conjectures, although the uniqueness of the factors L_p at bad p might be unknown. In contrast, in the $w = -1$ case one has to prove meromorphic extension of $L(M, s)$ before making sense of the central value $L^*(M, 0)$.

3.2. **The regulator map.** Assume from now on $w < 0$. Beilinson defines a regulator map of the form

$$r : C_1 \to C_2, \tag{3.1}$$

where

$$C_1 = H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$$
$$C_2 = H^{i+1}_{\mathcal{M}_\infty}(X_{\mathbb{R}}, n).$$

Detailed discussion of C_1, C_2, as well as r is postponed to later sections. Here let me just briefly indicate the nature of these objects.

C_1 is a $\mathbb{Q}$-vector space, conjecturally of finite dimension. Here $H^*_{\mathcal{M}}(X, \mathbb{Q}(n))$ is the so-called *motivic cohomology* of X with coefficients in $\mathbb{Q}(n)$. We remind the reader that the theory of motivic cohomology (with $\mathbb{Z}(n)$ coefficient, so to speak,) is already in some sense established thanks to the work of Voevodsky et. al. (To be discussed briefly in §4.4.) Nevertheless, it was expected long before this that the $\mathbb{Q}(n)$ coefficient motivic cohomology should be more concretely given in terms of algebraic K-theory as follows (meaning of the notation will be explained in §5)

$$H^p_{\mathcal{M}}(X, \mathbb{Q}(q)) = (K_{2q-p}(X) \otimes \mathbb{Q})^{(q)}. \tag{3.2}$$

Beilinson took (3.2) as definition when he considered the regulator map in the 1980's. It is now known that Voevodsky's theory of motivic cohomology indeed gives the correct $\mathbb{Q}(n)$ coefficient cohomology as (3.2). The extra subscript $\mathbb{Z}$ in the definition of C_1 indicates a modification that has something to do with choosing a model of X over $\mathbb{Z}$.

C_2 is called the *absolute Hodge cohomology* of X (with $\mathbb{R}(n)$ coefficient), and is a much more elementary object. It is a finite dimensional $\mathbb{R}$-vector space which can be defined and computed elementarily in terms of $M_B \otimes \mathbb{R} \in \mathcal{M}_\infty$. Moreover, the $\mathbb{Q}$-structure on M_B together with the $\mathbb{Q}$-structure on M_{dR} (and the knowledge of the comparison isomorphism $I : M_B \otimes \mathbb{C} \to M_{\mathrm{dR}} \otimes \mathbb{C}$) equips the top exterior product space $\det C_2$ of C_2 with a $\mathbb{Q}$-structure. In case C_2 is zero as an $\mathbb{R}$-vector space, this $\mathbb{Q}$-structure means that C_2 still remembers an element of $\mathbb{R}^\times/\mathbb{Q}^\times$, denoted by $c^+(M)$, which will be referred to as *Deligne's period* if $w \leq -2$. When $w = -1$, there is also a definition of Deligne's period $c^+(M) \in \mathbb{R}^\times/\mathbb{Q}^\times$, which has little to do with C_2. (e.g. C_2 could be non-zero.) For the definition of Deligne's period see Definition 4.3.1. More details about C_2 will be presented in §4.3.

The regulator map r is $\mathbb{Q}$-linear and induces an $\mathbb{R}$-linear map $C_1 \otimes \mathbb{R} \to C_2$.

3.3. **The conjectures.**

Conjecture 3.3.1 (Beilinson)**.** *Assume $w < -2$. Then*

(1) $r \otimes \mathbb{R} : C_1 \otimes_{\mathbb{Q}} \mathbb{R} \to C_2$ is an isomorphism.

(2) With respect to the $\mathbb{Q}$-structures on $\det C_1$ and $\det C_2$, we view $\det r$ as an element of $\mathbb{R}^\times/\mathbb{Q}^\times$. We have $0 \neq L(M, 0) \equiv \det r \mod \mathbb{Q}^\times$.

Let's make several remarks.

Remark 3.3.2. If $C_2 = 0$ (and $w < -2$), then the conjecture predicts that $C_1 = 0$ and

$$L(M, 0) \equiv c^+(M) \mod \mathbb{Q}^\times.$$

This case is often referred to as *critical*. In this case the introduction of $c^+(M)$ and the conjecture about the L-value was due to Deligne. For example, for the Riemann zeta function, the positive even integers are critical and the positive odd integers ≥ 3 are non-critical. c.f. Example 4.3.3

Remark 3.3.3. The functional equation translates between $s = 0$ and $s = w+1$, the latter lying to the left of the central point. The conjectural form of the functional equation and the conjectural behavior of the L-function would imply that $\dim_{\mathbb{R}} C_2$ is equal to the order of vanishing of $L(M, s)$ at $s = w + 1$.

Beilinson also formulates the conjecture for $w = -1$ and $w = -2$. For $w = -2$, $s = 0$ is the near central point and may be a pole of $L(M, s)$. The order of the pole is related to algebraic cycles on X by Tate's conjecture. The relevant space is

$$N^{n-1}(X) := \mathrm{CH}^{n-1}(X)/\,\mathrm{CH}^{n-1}(X)_0,$$

where the Chow group $\mathrm{CH}^j(X)$ is the group of codimension j algebraic cycles with $\mathbb{Q}$ coefficients on X modulo rational equivalence and $\mathrm{CH}^j(X)_0$ is the subgroup of homologically trivial cycles. Tate conjectures that

$$-\operatorname{ord}_{s=0} L(M, s) = \dim_{\mathbb{Q}} N^{n-1}.$$

The regulator map needed in this case is built from r as above together with the cycle class map

$$r' : N^{n-1}(X) \to H^{2n-1}_{\mathcal{M}_\infty}(X_{\mathbb{R}}, n) = C_2.$$

Conjecture 3.3.4 (Beilinson)**.** *Assume $w = -2$. Then*

1. *$(r \oplus r') \otimes \mathbb{R} : C_1 \otimes \mathbb{R} \oplus N^{n-1}(X) \otimes \mathbb{R} \to C_2$ is an isomorphism.*
2. *$L^*(M, 0) \equiv \det(r \oplus r') \mod \mathbb{Q}^\times$.*

For $w = -1$, we are dealing with the central point $s = 0$. In this case we have Deligne's period $c^+(M)$ defined, (c.f. Definition 4.3.1 and Remark 4.3.2), which has little to do with C_2.

Conjecture 3.3.5 (Beilinson)**.** *Assume $w = -1$. Then*

1. *There is a non-degenerate height pairing*

$$h : \mathrm{CH}^n(X)_0 \otimes \mathrm{CH}^{\dim X+1-n}(X)_0 \to \mathbb{R}$$

2. *$\operatorname{ord}_{s=0} L(M, s) = \dim_{\mathbb{Q}} \mathrm{CH}^n(X)_0$.*
3. *$L^*(M, 0) \equiv c^+(M) \det(h) \mod \mathbb{Q}^\times$.*

There are also ways to formulate the conjectures uniformly for all $w < 0$. Moreover, the conjecture of Bloch-Kato about special values of L-functions supplement Beilinson's conjectures in that they determine the L-values completely, rather than just up to $\mathbb{Q}^\times$. c.f. [Kin03].

3.4. **Known cases.** The conjectures for Dedekind zeta functions and Dirichlet L-functions are proved by Borel and Beilinson. In higher dimensions only partial results are known, all related to elliptic curves or Shimura varieties. c.f. [Nek94] §8.

4. Motivic cohomology and absolute cohomologies

The object $H^*_{\mathcal{M}_\infty}(X,n)$ is an example of an *absolute cohomology theory*, and the motivic cohomology $H^*_{\mathcal{M}}(X,\mathbb{Q}(n))$ is conjecturally the universal construction of absolute cohomology. Thus the regulator map (3.1) can be regarded as a realization map. We discuss the formal aspects of this philosophy as well as give an explicit description of $H^*_{\mathcal{M}_\infty}(X,n)$ in this section.

4.1. **Two philosophies.** Compare the following two philosophies:

(1) **Motives**. (Grothendieck.) Let $\mathcal{V}$ be a suitable subcategory of the category of algebraic varieties over a field k. We want to construct a universal cohomology theory on $\mathcal{V}$. In particular we want to construct a category $\mathcal{M}$ of motives, as the target of that universal cohomology theory. For $X \in \mathcal{V}$, we can take its universal cohomology and get objects $h^i(X)(n)$ of $\mathcal{M}$. (Here (n) is the n-th Tate twist.) Any reasonable concrete cohomology theory will factor through a realization functor from $\mathcal{M}$ to the target category of that cohomology.

(2) **Motivic complexes**. (Beilinson.) We want to do better than Philosophy (1). For $X \in \mathcal{V}$, we would like to produce an object $\underline{R\Gamma}(X,n)$ of the derived category $D(\mathcal{M})$ of $\mathcal{M}$, supposing the latter makes sense, for each $n \in \mathbb{Z}$. The cohomology of $\underline{R\Gamma}(X,n)$ will recover $h^i(X)(n) \in \mathcal{M}$ as before. We think of $X \mapsto \underline{R\Gamma}(X,n)$ as the universal way of forming cochain complexes.

Here is one important thing that Philosophy (2) allows us to do. The category $\mathcal{M}$ should be a tensor category and contains the unit object $\mathbf{1}$. Let $A := \mathrm{End}_{\mathcal{M}}(\mathbf{1})$. Consider the functor

$$\Gamma_{\mathcal{M}} := \mathrm{Hom}_{\mathcal{M}}(\mathbf{1},\cdot) : \mathcal{M} \to A - mod.$$

Take its derived functor

$$R\Gamma_{\mathcal{M}} : D(M) \to D^+(A - mod).$$

For $X \in \mathcal{V}$, we can consider the "motivic cochain complex"

$$R\Gamma_{\mathcal{M}}(X,n) := R\Gamma_{\mathcal{M}}(\underline{R\Gamma}(X,n)) \in D^+(A - mod).$$

We denote its cohomology by

$$H^i_{\mathcal{M}}(X,n) := H^i(R\Gamma_{\mathcal{M}}(X,n)) \in A - mod. \tag{4.1}$$

Depending on whether we worked with the $\mathbb{Z}$-linear or $\mathbb{Q}$-linear version of $\mathcal{M}$, (so that $A = \mathbb{Z}$ or $\mathbb{Q}$ resp.) we also denote $H^i_{\mathcal{M}}(X,n)$ by $H^i_{\mathcal{M}}(X,\mathbb{Z}(n))$ or $H^i_{\mathcal{M}}(X,\mathbb{Q}(n))$, respectively. These are the so-called *motivic cohomology* of X. By construction, there should be a spectral sequence

$$\mathrm{Ext}^p_{\mathcal{M}}(\mathbf{1}, H^q(\underline{R\Gamma}(X,n))) = \mathrm{Ext}^p_{\mathcal{M}}(\mathbf{1}, h^q(X)(n)) \Rightarrow H^{p+q}_{\mathcal{M}}(X,n). \tag{4.2}$$

We will refer to this spectral sequence as the *geometric to absolute spectral sequence*. Beilinson speculates that it degenerates for smooth projective X. Moreover, Beilinson expects that $\mathrm{Ext}^i_{\mathcal{M}}$ vanishes for i greater than the Kronecker dimension of the ground field k (= transcendence degree over $\mathbb{Q}$ +1 if $\mathrm{char}\, k = 0$ or transcendence degree over $\mathbb{F}_p$ if $\mathrm{char}\, k = p$.) In particular if $k = \mathbb{Q}$ and X is smooth projective over $\mathbb{Q}$, we would have a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{M}}(\mathbf{1}, h^i(X)(n)) \to H^{i+1}_{\mathcal{M}}(X,n) \to \mathrm{Hom}_{\mathcal{M}}(\mathbf{1}, h^{i+1}(X)(n)) \to 0, \tag{4.3}$$

where the third group vanishes when $w = 1 - 2n \leq -2$ for weight reasons.

N.B. Beilinson knows that for the above speculations about the motivic complexes and motivic cohmology to work, the category $\mathcal{M}$ has to be some category of *mixed motives* (i.e. allowing non-smooth varieties), whose associated category of semi-simple objects would be the category of pure motives. This is in sharp contrast to Grothendieck's philosophy about motives, where one is totally allowed to restrict attention to smooth projective varieties and consider only *pure motives*. Unfortunately, unlike the situation with pure motives, where a proof of the standard conjectures about algebraic cycles would automatically lead to a good theory, as for now people have not found a good framework to formulate the theory of mixed motives.[1]

4.2. **Realizations: absolute cohomology theories.** We can project the previous story under a realization functor. Let g be the name of a reasonable (e.g. Weil) cohomology theory, e.g. $H^*_g = H^*_B, H^*_{\mathrm{dR}}$, etc. Let $\mathcal{M}_g$ be the target category of g. We would have a realization functor $\mathcal{M} \to \mathcal{M}_g$. Thus we replace the *conjectural* functors

$$\mathcal{V} \ni X \mapsto h^i(X)(n) \in \mathcal{M}$$

by the *concrete* functors:

$$\mathcal{V} \ni X \mapsto H^i_g(X)(n) \in \mathcal{M}_g.$$

Then we may seek to carry out a construction that would be the realization of Philosophy (2) w.r.t. to the realization $\mathcal{M} \to \mathcal{M}_g$. That is to say, we would like to construct, for each $X \in \mathcal{V}$, an object

$$\underline{R\Gamma}(X, n) \in D(\mathcal{M}_g)$$

in some natural way, such that its cohomology recovers $H^i_g(X)(n) \in \mathcal{M}_g$. Suppose this is done, then we can proceed analogously as before. We define

$$A_g := \mathrm{End}_{\mathcal{M}_g}(\mathbf{1}),$$

$$\Gamma_{\mathcal{M}_g} := \mathrm{Hom}_{\mathcal{M}_g}(\mathbf{1}, \cdot) : \mathcal{M}_g \to A_g - mod,$$

and define the corresponding *absolute cochain complex* and *absolute cohomology* by

$$R\Gamma_{\mathcal{M}_g}(X, n) := R\Gamma_{\mathcal{M}_g}(\underline{R\Gamma}(X, n)) \in D^b(A_g - mod)$$

$$H^i_{\mathcal{M}_g}(X, n) := H^i(R\Gamma_{\mathcal{M}_g}(X, n)) \in A_g - mod.$$

We also call the original cohomology theory $H^*_g(X)(n)$ a *geometric* cohomology. The motivation for this terminology will be clear from Example 4.2.1 below. Thus one might say that Philosophy (1) of motives is seeking the universal geometric cohomology, whereas the motivic cohomology produced by Philosophy (2) is the universal absolute cohomology.

We would get realization maps for absolute cohomologies:

$$H^i_{\mathcal{M}}(X, n) \to H^i_{\mathcal{M}_g}(X, n). \tag{4.4}$$

Beilinson's regulator map (3.1) can be understood in terms of (4.4). We also have the realization of the geometric to absolute spectral sequence (4.2):

$$\mathrm{Ext}^p_{\mathcal{M}_g}(\mathbf{1}, H^q_g(X)(n)) \Rightarrow H^{p+q}_{\mathcal{M}_g}(X, n).$$

One should expect this to degenerate for nice X (e.g. smooth and projective).

[1]Please correct me if I am wrong.

Example 4.2.1. We illustrate the ideas with the $\Lambda = \mathbb{Z}/N\mathbb{Z}$ coefficient étale cohomology

$$X \mapsto H^*_{\text{ét}}(X_{k^{sep}}, \mathbb{Z}/N\mathbb{Z}),$$

where $N \in \mathbb{Z}$ is coprime to $\operatorname{char} k$. Here the target category $\mathcal{M}_g$ is the category of $\Lambda[\mathrm{Gal}(k^{sep}/k)]$-modules. The terminology "geometric cohomology" is justified in this case, as $H^*_{\text{ét}}(X_{k^{sep}}, \Lambda)$ is the cohomology of the geometric fiber $X_{k^{sep}}$. We have

$$\mathbf{1} = \Lambda$$

$$A_g = \Lambda$$

$$\Gamma_{\mathcal{M}_g}(\cdot) = (\cdot)^{\mathrm{Gal}(k^{sep}/k)}.$$

The object $\underline{R\Gamma}(X, n) \in D(\mathcal{M}_g)$ is $Rf_*(\underline{\Lambda})$, where $f : X \to \operatorname{Spec} k$ is the structure map of X. The corresponding absolute cohomology is

$$H^i_{\mathcal{M}_g}(X, n) = H^i_{\text{ét}}(X, \Lambda(n)).$$

The geometric to absolute spectral sequence in this case is the Hochschild-Serre spectral sequence.

4.3. **The absolute Hodge cohomology.** The Betti version of the above story is worked out by Beilinson. We consider the category $\mathcal{V}$ of separated schemes of finite type over $\mathbb{R}$. We will use the $\mathbb{R}$-coefficient Betti cohomology, so the target category is $\mathcal{M}_\infty$ defined in Definition 2.1.1. For each $X \in \mathcal{V}$, Beilinson constructs

$$\underline{R\Gamma}(X, n) \in D^b(\mathcal{M}_\infty),$$

whose cohomology objects are $H^*_B(X(\mathbb{C}), \mathbb{R}(n)) \in M_\infty$. Proceeding as before, we obtain the *absolute Hodge cohomology*

$$H^*_{\mathcal{M}_\infty}(X, n).$$

N.B. One could also work with $\mathcal{MH}^+_{\mathbb{Q}}$ or $\mathcal{MH}^+_{\mathbb{Z}}$ instead of $\mathcal{M}_\infty = \mathcal{MH}^+_{\mathbb{R}}$ and obtain the corresponding absolute cohomology groups, abbreviated as $H^*_{\mathcal{H}}(X, \mathbb{Q}(n))$ and $H^*_{\mathcal{H}}(X, \mathbb{Z}(n))$ respectively. We have

$$H^*_{\mathcal{H}}(X, \mathbb{Q}(n)) = H^*_{\mathcal{H}}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

but

$$H^*_{\mathcal{M}_\infty}(X, n) \neq H^*_{\mathcal{H}}(X, \mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{R}$$

in general.

The geometric to absolute spectral sequence

$$\mathrm{Ext}^p_{\mathcal{M}_\infty}(\mathbf{1}, H^q_B(X(\mathbb{C}), \mathbb{R}(n))) \Rightarrow H^{p+q}_{\mathcal{M}_\infty}(X, n)$$

degenerates when X is smooth and projective. Moreover, for $N \in \mathcal{M}_\infty$, we have

$$\mathrm{Ext}^i_{\mathcal{M}_\infty}(\mathbf{1}, N) = \begin{cases} W_0 N^+ \cap F^0 N_{\mathbb{C}}\ , & i = 0 \\ W_0 N^+ \backslash W_0 N^{\mathrm{dR}} / F^0(W_0 N^{\mathrm{dR}})\ , & i = 1\ , \\ 0\ , & i \geq 2 \end{cases} \tag{4.5}$$

where

$$N^+ := N^{\phi_\infty}$$

$$N^{\mathrm{dR}} := (N_{\mathbb{C}})^{\phi_\infty \otimes c}.$$

Therefore, for smooth and projective X over $\mathbb{R}$, denoting

$$N := H^i_B(X(\mathbb{C}), \mathbb{R}(n)),$$

$$N' := H_B^{i+1}(X(\mathbb{C}), \mathbb{R}(n)),$$
$$w := i - 2n,$$

we have a short exact sequence analogous to (4.3):

$$0 \to \mathrm{Ext}^1_{\mathcal{M}_\infty}(\mathbf{1}, N) \to H^{i+1}_{\mathcal{M}_\infty}(X, n) \to \mathrm{Hom}_{\mathcal{M}_\infty}(\mathbf{1}, N') \to 0. \tag{4.6}$$

Note that N and N' are pure of weight w and $w+1$ respectively, so using 4.5 and the vanishing of the first or the third group above for weight reasons, we have

$$H^{i+1}_{\mathcal{M}_\infty}(X, n) = \begin{cases} \mathrm{Ext}^1(\mathbf{1}, N) = N^+ \backslash N^{\mathrm{dR}}/F^0(N^{\mathrm{dR}}) \ , & w \le -2 \\ \mathrm{Hom}(\mathbf{1}, N') = (N')^+ \cap F^0(N'_{\mathbb{C}}) \ , & w = -1 \\ 0 \ , & w \ge 0. \end{cases}$$

Now we consider a motive $M = h^i(X)(n)$ as in (2.1) with X projective smooth over $\mathbb{Q}$. Assume $w \le -2$. Then the object C_2 considered in the regulator map (3.1) is given by

$$C_2 = H^{i+1}_{\mathcal{M}_\infty}(X_{\mathbb{R}}, n) = (M_B \otimes \mathbb{R})^+ \backslash (M_B \otimes \mathbb{R})^{\mathrm{dR}}/F^0((M_B \otimes \mathbb{R})^{\mathrm{dR}}).$$

But the Betti to de Rham comparison isomorphism induces an isomorphism

$$(M_B \otimes \mathbb{R})^{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{R},$$

and F^0 of LHS corresponds to the subspace $(F^0 M_{\mathrm{dR}}) \otimes \mathbb{R}$ of RHS. We rewrite

$$C_2 = (M_B^+ \otimes \mathbb{R}) \backslash M_{\mathrm{dR}} \otimes \mathbb{R}/(F^0 M_{\mathrm{dR}}) \otimes \mathbb{R}.$$

This exhibits a $\mathbb{Q}$ structure on $\det C_2$ that comes from the $\mathbb{Q}$ structures on M_B^+, M_{dR}, and $F^0 M_{\mathrm{dR}}$.

At this point we give the definition of Deligne's period $c^+(M)$ of M.

Definition 4.3.1. Suppose the weight w of M is negative. Consider the following map induced from the Betti to de Rham comparison

$$\alpha_M : M_B^+ \otimes \mathbb{R} \to M_{\mathrm{dR}} \otimes \mathbb{R}/(F^0 M_{\mathrm{dR}}) \otimes \mathbb{R}.$$

α_M. It is always injective. We call M *critical* if α_M is an isomorphism. In this case we define *Deligne's period* $c^+(M) \in \mathbb{R}^\times/\mathbb{Q}^\times$ to be the determinant of α_M, w.r.t. the $\mathbb{Q}$ structures on M_B^+, M_{dR}, and $F^0 M_{\mathrm{dR}}$.

Remark 4.3.2. When $w \le -2$, from the above discussion we see that M is critical if and only if $C_2 = 0$. When this is the case $c^+(M)$ coincides with the $\mathbb{Q}$ structure on $\det C_2$. (c.f. the discussion about C_2 in §3.2.) When $w = -1$, for weight reason M is critical. In this case $c^+(M)$ appears in Conjecture 3.3.5.

Example 4.3.3. $M = h^0(\mathrm{Spec}\,\mathbb{Q})(n) = \mathbb{Q}(n), n \in \mathbb{Z}_{\ge 1}$. In this case C_2 is the target of the regulator map that is conjectured to be related to $\zeta(n)$. For the realizations and Betti to de Rham comparison isomorphism for M see Example 2.1.3 (with things raised to the n-th tensor power). We have $C_2 = \mathbb{R}$ when n is odd and $C_2 = 0$ when n is even. We have thus seen that M is critical (c.f. Remark 3.3.2) exactly when n is even. In this case the $\mathbb{Q}$ structure on $\det C_2$, namely Deligne's period $c^+(M)$, is easily seen to be $\pi^n \in \mathbb{R}^\times/\mathbb{Q}^\times$, as it is the determinant of the comparison isomorphism

$$M_B^+ \otimes \mathbb{R} = M_B \otimes \mathbb{R} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{R}.$$

The conjecture $\zeta(n) \equiv c^+(M) \mod \mathbb{Q}^\times$ follows from Euler's formula (1.5).

4.4. **Digression: Motivic cohomology partially achieved.** We digress to record the current status of motivic cohomology for the reader's information. This subsection is not needed in the subsequent and may be skipped.

Beilinson conjectures that the $\mathbb{Z}(n)$ coefficient motivic cohomology $H^*_{\mathcal{M}}(X, \mathbb{Z}(n))$ could be computed as the hypercohomology of a complex of Zariski sheaves on X. More precisely, he conjectures

Conjecture 4.4.1. *Let k be a field. For each $n \geq 0$ there exists a complex of sheaves $\mathbb{Z}(n)$, on the Zariski site $\mathcal{V}$ of smooth quasi-projective varieties over k, satisfying the following.*

(1) $\mathbb{Z}(0) = \mathbb{Z}$ (concentrated in degree zero), $\mathbb{Z}(1) = \mathcal{O}^*[-1]$.
(2) $H^n(\operatorname{Spec} F, \mathbb{Z}(n))$ is the n-th Milnor K-group of F for any field F of finite type over k.
(3) $H^{2n}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (Chow group with $\mathbb{Z}$ coefficients) for any $X \in \mathcal{V}$.
(4) $H^p(X, \mathbb{Z}(n)) = 0,\ p < 0, X \in \mathcal{V}$.
(5) For $X \in \mathcal{V}$, there is a E_2-spectral sequence
$$H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$
(6) Comparison with mod l absolute étale cohomology. More precisely, we have
$$\mathbb{Z}(n) \otimes^L \mathbb{Z}/l\mathbb{Z} \cong \tau_{\leq n} R\pi_* \mu_l^{\otimes n},$$
for l a prime different from $\operatorname{char} k$, where π is the forgetful functor from the étale site to the Zariski site and $\tau_{\leq n}$ is the usual truncation functor on the derived category.
(7) For $X \in \mathcal{V}$, we have $H^i(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$, where the RHS is defined by (3.2).

Suslin-Voevodsky have constructed a candidate for $\mathbb{Z}(n)$, and thanks to the work of many people, most notably Voevodsky, all the properties (1) - (7) except (4) have been verified. For a discussion of this see [Fri08] §6.3.

5. Construction of the regulator map

In this section we sketch the construction of the regulator map (3.1). Let $M = h^i(X)(n)$ as in (2.1), with $w = i - 2n < 0$.

We have already seen the definition and a concrete description of $C_2 = H^{i+1}_{\mathcal{M}_\infty}(X, n)$ in §4.3. Let's describe C_1. By definition, we have
$$C_1 = H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}},$$
where
$$H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) := [K_{2n-(i+1)}(X) \otimes \mathbb{Q}]^{(n)}.$$
Here the superscript (n) means the weight n subspace for the Adams operators. Let me briefly recall what this means. For a ring A, there is a family of operations $\left\{\psi^k\right\}_{k\geq 1}$ acting on each K-group $K_j(A)$ that can be defined for example using Quillen's plus construction. On K_0, each ψ^k is a group homomorphism and sends $[P]$ to $[P^{\otimes k}]$ for each projective R module P of rank one. We define
$$[K_j(A) \otimes \mathbb{Q}]^{(m)} := \left\{x \in K_j(A) \otimes \mathbb{Q} | \psi^k(x) = k^m x,\ \forall k \geq 1\right\}.$$
Then it is a fact that we have the decomposition
$$K_j(A) \otimes \mathbb{Q} = \bigoplus_{m\geq 0} [K_j(A) \otimes \mathbb{Q}]^{(m)}.$$

Now for our variety X over $\mathbb{Q}$, there is the so called Jouanolou's trick that finds an affine scheme $\operatorname{Spec} A$ together with a morphism $\pi : \operatorname{Spec} A \to X$ that makes $\operatorname{Spec} A$ a vector bundle over X. The morphism π induces isomorphism on the K-groups, and we define the weight subspaces of $K_j(X) \otimes \mathbb{Q}$ in terms of those for A.

Now we explain the subscript $\mathbb{Z}$ in the definition of C_1. That it is well defined is only conjectural. Choose a proper flat model $\mathcal{X}$ over $\mathbb{Z}$ for X, which always exists, and define

$$[K_j(X) \otimes \mathbb{Q}]^{(m)}_{\mathbb{Z}} := \operatorname{im}(K'_j(\mathcal{X}) \otimes \mathbb{Q}),$$

where image means first mapping into $K_j(X) \otimes \mathbb{Q}$ and then project to the weight m subspace. Beilinson conjectures that this definition is independent of the choice of $\mathcal{X}$. In fact, two choices of proper regular models $\mathcal{X}$ would always give rise to the same result, but unfortunately such models are not known to exist in general. Conjecturally, we would also have

$$H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) = H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$$

for $n > \max(i, \dim X) + 1$.

Example 5.0.2. $H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}, \mathbb{Q}(1))_{\mathbb{Z}} = 0, H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}, \mathbb{Q}(1)) = \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.

The essential ingredient of the construction of the regulator map (3.1) is the construction of the (higher) Chern class maps: For Y a scheme over $\mathbb{R}$ satisfying certain conditions, we construct maps

$$c_{j,m} : K_j(Y) \to H^{2m-j}_{\mathcal{M}_\infty}(Y, m). \tag{5.1}$$

After that, we can then define the (higher) Chern character maps as

$$ch_j := \sum_{m \geq 0} \frac{(-1)^{m-1}}{(m-1)!} c_{j,m} : K_j(Y) \otimes \mathbb{Q} \to \bigoplus_{m \geq 0} H^{2m-j}_{\mathcal{M}_\infty}(Y, m) \tag{5.2}$$

for $j > 0$ and define

$$ch_0 : K_0(Y) \otimes \mathbb{Q} \to \bigoplus_{m \geq 0} H^{2m}_{\mathcal{M}_\infty}(Y, m)$$

to be the usual Chern character. It can be checked that the Chern character maps ch_j map the weight m subspace $[K_j(Y) \otimes \mathbb{Q}]^{(m)}$ to $H^{2m-j}_{\mathcal{M}_\infty}(Y, m)$.

Example 5.0.3. ch_1 maps $[K_1(\mathbb{R}) \otimes \mathbb{Q}]^{(1)} \cong \mathbb{R}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ to $H^1_{\mathcal{M}_\infty}(\operatorname{Spec} \mathbb{R}, 1) \cong \mathbb{R}$. It turns out to be given by the usual logarithm $\mathbb{R}^\times \to \mathbb{R}$.

Finally the regulator (3.1) is defined as the following composition:

$$C_1 = [K_{2n-i-1}(X) \otimes \mathbb{Q}]^{(n)} \to [K_{2n-i-1}(X_{\mathbb{R}}) \otimes \mathbb{Q}]^{(n)} \xrightarrow{ch_{2n-i-1}} H^{i+1}_{\mathcal{M}_\infty}(X_{\mathbb{R}}, n) = C_2.$$

In the following we will sketch the construction of the Chern class maps (5.1), and omit checking that the resulting Chern characters behave correctly w.r.t. the weight subspaces. There turns out to be an axiomatic approach producing Chern class maps from higher K-groups of a reasonable scheme to cohomology groups of that scheme for any fixed cohomology theory that satisfies certain axioms. When Y is smooth and quasi-projective, and the cohomology theory in question is the absolute Hodge cohomology $H^*_{\mathcal{M}_\infty}$, we can proceed as follows. We follow [Nek94] §5, but also see [Sch88] §4. ([Sch88] uses the so-called Deligne cohomology instead of the absolute Hodge cohomology $H^*_{\mathcal{M}_\infty}$. The relation between them is discussed for example in [Nek94] §7.)

Firstly, we use Jouanolou's trick mentioned above to reduce to the case $Y = \operatorname{Spec} A$ is affine. This reduction step is the only place we need the original Y to be smooth and quasi-projective. The following discussion holds for an arbitrary finitely generated $\mathbb{R}$-algebra A.

Given any element x of $K_j(A)$, we map it under the Hurewicz map to

$$H_i(\mathrm{GL}(A), \mathbb{R}) = \varinjlim_N H_i(B\,\mathrm{GL}_N(A), \mathbb{R}).$$

It then suffices to construct maps

$$c_{j,m} : H_j(B\,\mathrm{GL}_N(A), \mathbb{R}) \to H^{2m-j}_{\mathcal{M}_\infty}(\operatorname{Spec} A, m). \tag{5.3}$$

The functor $A \mapsto B\,\mathrm{GL}_N(A)$ from the category of $\mathbb{R}$-algebras to the category of simplicial sets is represented by a simplicial scheme $(B\,\mathrm{GL}_N)_\bullet$, which we abbreviate as $B_\bullet$. Graphically $B_\bullet$ is given by

$$\operatorname{Spec}\mathbb{R} \;\; \overset{\leftarrow}{\underset{\leftarrow}{}} \;\; \mathrm{GL}_N \;\; \overset{\xleftarrow{\mathrm{id}\times 1}}{\underset{\xleftarrow{1\times\mathrm{id}}}{\xleftarrow{\mu}}} \;\; \mathrm{GL}_N \times \mathrm{GL}_N \cdots\cdot \;,$$

where the GL_N's are viewed as group schemes over $\mathbb{R}$ and μ is the multiplication morphism.

Now we need three abstract ingredients involving simplicial schemes to produce a result that does not involve simplicial schemes.

The first abstract ingredient is the definition of $H^p_{\mathcal{M}_\infty}(S_\bullet, q)$ for a simplicial scheme $S_\bullet$ over $\mathbb{R}$, plus an isomorphism

$$\iota : H^{2m}_{\mathcal{M}_\infty}(B_\bullet, m) \xrightarrow{\sim} H^{2m}(B_\bullet(\mathbb{C}), \mathbb{Q}(m)) \otimes \mathbb{R}.$$

Note that $H^{2m}(B_\bullet(\mathbb{C}), \mathbb{Q}(m))$ is just the familiar object $H^{2m}(B\,\mathrm{GL}_N(\mathbb{C}), \mathbb{Q}(m))$ and we have

$$\bigoplus_{m\geq 0} H^{2m}(B\,\mathrm{GL}_N(\mathbb{C}), \mathbb{Q}(m)) = \mathbb{Q}[c_1, \cdots, c_N],$$

where $c_m \in H^{2m}(B\,\mathrm{GL}_N(\mathbb{C}), \mathbb{Q}(m))$ is the m-th Chern class of the universal bundle $\mathcal{E}$ over $B\,\mathrm{GL}_N(\mathbb{C})$.

The second abstract ingredient is the evaluation map

$$ev : \operatorname{Spec} A \times B_\bullet(A) \to B_\bullet,$$

as a morphism of simplicial schemes over $\mathbb{R}$. Whenever we have a scheme P and a simplicial set Q, we form their product $P \times Q$ to be a simplicial scheme. The evaluation map induces a map

$$ev^* : H^p_{\mathcal{M}_\infty}(B_\bullet, q) \to H^p_{\mathcal{M}_\infty}(\operatorname{Spec} A \times B_\bullet(A), q).$$

The third abstract ingredient is the Künneth formula and cap product for $H^*_{\mathcal{M}_\infty}$ in the following sense. Let P be a scheme over $\mathbb{R}$ and Q a simplicial set. As before we form their product $P\times Q$: this is a simplicial scheme. We have Künneth formula:

$$H^p_{\mathcal{M}_\infty}(P \times Q, q) \cong \bigoplus_j H^{p-j}_{\mathcal{M}_\infty}(P, q) \otimes H^j(Q, \mathbb{R}).$$

Correspondingly we have a cap product

$$\cap : H^{2m}_{\mathcal{M}_\infty}(P \times Q, m) \otimes H_j(Q, \mathbb{R}) \to H^{2m-j}_{\mathcal{M}_\infty}(P, m).$$

Putting these three ingredients together, we produce a pairing, now not involving simplicial schemes:

$$\langle \cdot, \cdot \rangle : [H^{2m}(B\,\mathrm{GL}_N(\mathbb{C}), \mathbb{Q}(m)) \otimes \mathbb{R}] \otimes H_j(B\,\mathrm{GL}_N(A), \mathbb{R}) \to H^{2m-j}_{\mathcal{M}_\infty}(\operatorname{Spec} A, m),$$

by

$$\langle x, y \rangle := (ev^* \iota^{-1}(x)) \cap y.$$

The map $\langle c_m, \cdot \rangle$ gives the desired map (5.3).

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