5. Jay Shah: Axiomatic approach to algebraic K-theory

Question: what is the universal property of K_0? Suppose $\mathcal{C}$ is a category with notion of exact sequence. Then $K_0(\mathcal{C}) = \mathbb{Z}\{X : X \in \operatorname{ob}\mathcal{C}\}/\sim$ where $[X] \sim [X'] + [X'']$ if $X' \to X \to X''$ is an exact sequence of abelian groups.

This satisfies the universal property

$$\operatorname{Hom}_{Ab}(K_0\mathcal{C}, A) \cong \{\varphi : \operatorname{ob}\mathcal{C} \to A : \varphi(X) = \varphi(X') + \varphi(X'')\}.$$

Consider this in the setting of categories with cofibrations. Then ob and K_0 are both functors $\mathrm{Cat}_{cofib} \to Set$. K_0 satisfies:

(1) Let $E(\mathcal{C})$ be the category of exact sequences in $\mathcal{C}$. Then there are two functors $E(\mathcal{C}) \to \mathcal{C}$, one sending $(X' \to X \to X'')$ to X', and the other sending it to X''. Then $K_0(E(\mathcal{C})) \to K_0(\mathcal{C}) \times K_0(\mathcal{C})$ is an isomorphism. ("K_0 splits exact sequences.")
(2) Consider the monoid structure induced by the functor $E(\mathcal{C}) \to \mathcal{C}$ sending $(X' \to X \to X"')$ to X. Then this induces the structure of an abelian group on $K_0(\mathcal{C})$.

K_0 is universal w.r.t. functors satisfying the above two properties, with natural transformations from objects.

This gives a good motivation for considering algebraic K-theory. Pass from the objects of $\mathcal{C}$ to the moduli space $i\mathcal{C}$ of objects of $\mathcal{C}$. Consider this as an ∞-category, which you can think of as a category with mapping spaces. I had a map $\operatorname{ob}\mathcal{C} \to K_0\mathcal{C}$. Now, I want a map $i\mathcal{C} \to K\mathcal{C}$.

I'm going to be working with Waldhausen ∞-categories: an ∞-category $\mathcal{C}$ with a zero object and a subclass of maps (satisfying some properties), which I call the cofibrations. Denote the special maps by $\hookrightarrow$; they satisfy:

- closed under composition
- contain the equivalences
- closed under composition, and contain $0 \hookrightarrow X$ for all X
- closed under pushouts: if $X \hookrightarrow Y$ is a cofibration, and $X \to Z$ is another map, you can form the pushout diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

  and moreover $Z \to W$ is a cofibration.

Remark 5.1. Let $\mathcal{C}$ be a Waldhausen ∞-category. Then $0 \hookrightarrow X$ are cofibrations, and thus I can pushout $0 \hookrightarrow X$ and $0 \hookrightarrow Y$ to obtain the coproduct $X \vee Y$. So in particular, $\mathcal{C}$ always has coproducts.

If all maps in $\mathcal{C}$ are cofibrations, then $\mathcal{C}$ admits pushouts. So all finite colimits.

For example, if $\mathcal{C}$ is the derived category of a ring or ring spectrum, and every map is a cofibration, then $K(\mathcal{C}) = K(D^c(R))$.

I can consider pointed compact objects in local systems $\mathcal{C} = \mathrm{Fun}(X, \mathrm{Top})_*^\omega$; this admits finite colimits, and so the subcategory of cofibrations is maximal. Then $K(\mathcal{C})$ is called the A-theory of X.

If $\mathcal{C}$ is finite pointed sets and cofibrations are the monomorphisms, then $K(\mathcal{C}) = QS^0$ by Barratt-Priddy-Quillen.

Now I want to talk about the category Wald_∞ of Waldhausen ∞-categories. I've told you the objects; the morphisms are exact functors, as below:

Definition 5.2. $F : \mathcal{C} \to \mathcal{D}$ is *exact* if:

- F preserves the zero object:
- F preserves cofibrations;
- F preserves pushouts where one leg is cofibrant.

Wald_∞ has the following features:

- limits and filtered colimits are computed as in Cat.
- Wald_∞ admits direct sums: make $\mathcal{C} \to \mathcal{C} \times \mathcal{D} \leftarrow \mathcal{D}$ using the fact that these have zero objects.
- Has a zero object
- Compactly generated ∞-category: any can be written as the direct limit of compact subobjects.

What does $K : \mathrm{Wald}_\infty \to \mathrm{Top}$ satisfy?

- Need to send the zero object to a point
- Need to preserve filtered colimits
- I designed K_0 to split exact sequences, so this should "split cofiber sequences", and it should be "grouplike" (analogues of properties (1) and (2) in our earlier discussion of K_0).

Instead of cofiber sequences, I should be thinking of 1-step filtered objects.

Definition 5.3. Let $F_1\mathcal{C}$ be a Waldhausen category with objects $X \hookrightarrow Y$ in $\mathcal{C}$, and morphisms = commutative squares

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \hookrightarrow & Y' \end{array}$$

To give this the structure of a Waldhausen category, I need to specify the cofibrations: such a diagram is a cofibration if $X \hookrightarrow X'$ is a cofibration, and the map from the pushout $X' \cup_X Y' \to Y'$ is a cofibration.

The point is that I can write an exact functor $F_1 : F_1\mathcal{C} \to \mathcal{C}$; it sends $(X \hookrightarrow Y) \to Y/X$ (here Y/X means the cofiber). I also have an exact functor $I_{1,0} : F_1\mathcal{C} \to \mathcal{C}$ sending $(X \hookrightarrow Y) \mapsto X$. You can also consider the functor $F_1\mathcal{C} \overset{I_{1,1}}{\to} \mathcal{C}$ sending $(X \hookrightarrow Y) \to Y$.

As you might expect from the notation, this will extend to a simplicial object.

Definition 5.4. A reduced finitary functor $\mathrm{Wald}_\infty \overset{\Phi}{\to} \mathrm{Top}$ is *additive* if:

(1) $\Phi(F_1\mathcal{C}) \overset{(F_1,I_{1,0})}{\to} \Phi(\mathcal{C}) \times \Phi(\mathcal{C})$ is an equivalence
(2) $\Phi(F_1\mathcal{C}) \overset{(I_{1,0},I_{1,1})}{\to} \Phi(\mathcal{C}) \times \Phi(\mathcal{C})$ is an equivalence.

Definition 5.5. Algebraic K-theory is an additive functor $\mathrm{Wald}_\infty \to \mathrm{Top}$, which receives a natural transformation from i (the moduli space of objects functor) such that for any additive functor Φ, $\mathrm{Nat}(K, \Phi) \to \mathrm{Nat}(i, \Phi)$ is an equivalence.

Remark 5.6. i is corepresentable by Fin_*. (Proof: exercise.) So $\mathrm{Nat}(i, \Phi) \simeq \Phi(\mathrm{Fin}_*)$.

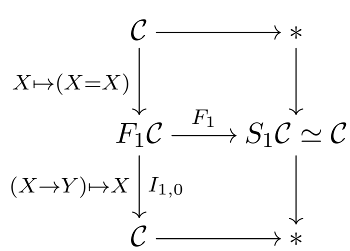
Corollary 5.7. *If $\Phi = K$, then the space of global operations on K-theory* $\mathrm{Nat}(K, K) \simeq K(\mathrm{Fin}_*) \simeq QS^0$ *by BPQ.*

Note: the domain matters – Blumberg, Gepner, and Tabuada calculate K-theory on stable ∞-categories, and their global operations is $K(S)$.

You have to define THH on Wald_∞ as an additive functor. (You usually don't think of THH as landing in spaces.) Clark says you can do this. In this case, $THH(\mathrm{Fin}_*) \simeq THH(S) \simeq S$.

Construction of K-theory. I'm looking at additive functors $F^{add}(\mathrm{Wald}, \mathrm{Top}) \hookrightarrow \mathrm{Fin}_*^{filt}(\mathrm{Wald}, \mathrm{Top})$ and trying to prove the existence of a left adjoint. This should remind you of a similar situation: $\mathrm{Exc}(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (where Exc means 1-excisive functors). This has a left adjoint P_1 that can be explicitly constructed: $P_1(F) = \mathrm{colim}_{n\to\infty} \Omega^n F \Sigma^n$.

I want to connect additivity to the property of being excisive.



Apply Φ to this diagram. Note that the lefthand composition is the identity (this is a retract diagram).

$$\begin{array}{ccc}
\Phi\mathcal{C} & \longrightarrow & * \\
\downarrow & & \downarrow \\
\Phi F_1\mathcal{C} & \longrightarrow & \Phi(\mathcal{C}) \\
\downarrow & & \downarrow \\
\Phi(\mathcal{C}) & \longrightarrow & *
\end{array}$$

The bottom square is a pullback square by additivity; so the top square is too. Additive functors should turn pushout squares into pullback squares.

Extend to simplicial objects. $F_m\mathcal{C}$ has objects $X_0 \hookrightarrow \ldots \hookrightarrow X_m$; $S_m\mathcal{C}$ has objects $0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_m$. You can come up with some obvious face maps to get simplicial objects $F_\bullet\mathcal{C}$, $S_\bullet\mathcal{C}$. If I apply i to $S_\bullet$, I get Waldhausen's $S_\bullet$ construction.

$$\begin{array}{ccc}
\mathcal{C} & \longrightarrow & * \\
{\scriptstyle X\mapsto(X=X)}\downarrow & & \downarrow \\
F_m\mathcal{C} & \xrightarrow{F_m} & S_m\mathcal{C} \simeq \mathcal{C} \\
{\scriptstyle (X\to Y)\mapsto X}\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & *
\end{array}$$

I have a levelwise pullback square if I consider simplicial diagrams:

$$\begin{array}{ccc}
\Phi\mathcal{C} & \longrightarrow & * \\
\downarrow & & \downarrow \\
\Phi F_\bullet\mathcal{C} & \longrightarrow & \Phi S_\bullet\mathcal{C}
\end{array}$$

$F_\bullet$ admits a contracting homotopy, gotten by taking $X_1 \to \cdots \to X_n$ to $0 \to X_1 \to \cdots \to X_n$ (this fails to be a contracting homotopy on $S_\bullet$ in the same way that the argument showing EG is contractible doesn't work on BG). So I should expect the bottom left to be trivial. The problem is that something is wrong with geometric realizations in Wald_∞. But in some enlargement of Wald_∞ I can expect this to be trivial. The idea is to force the $S_\bullet$ construction to actually be the suspension.

So make the enlargement $\mathrm{Wald}_\infty \hookrightarrow D\,\mathrm{Wald}_\infty$ by formally making geometric realization. Now I want to make a localization, called $D_{\mathit{fiss}}\,\mathrm{Wald}_\infty$, where the $S_\bullet$ is forced to be the suspension. So the analogy works perfectly: I can form the adjoint to the inclusion of additive functors into filtered functors in the same way as P_1 is defined.

$$P_1\Phi \simeq \mathrm{colim}_{n\to\infty}\,\Omega^n \circ \Phi \circ S_\bullet^n.$$

Magic: you might think this colimit is hard to compute. But, you' be wrong: if $\Phi = i$, then this is $\Omega \circ I \circ S_\bullet$. This is exactly Waldhausen's construction. The higher P_n's are all trivial for i.