K-THEORY OF FINITE FIELDS

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ABSTRACT. These are some notes from a talk I gave at Juvitop. I basically outlined the argument Quillen gave in his original paper [1] to compute the algebraic K-groups of a finite field. If you happen to actually read this and have any comments or questions or find any mistakes, then please feel free to email me.

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1. INTRODUCTION

R will be a ring q a prime number. Recall one definition of the algebraic K-groups of a ring is

$$K_i(R) := \pi_i(BGL(R)^+)$$

where the plus construction is taken with respect to the elementary matrices $E(R) \subset GL(R) = \pi_1(BGL(R))$. We also have that $BGL(R)^+$ is simple, which follows from the fact that it is an H-space (this was basically done in the last talk, or at least alluded to). We'll take this claim as given. Our goal is to define a map

$$\theta : BGL(\mathbb{F}_q) \to F\psi^q$$

for some space $F\psi^q$ which will be constructed as the homotopy fixed points of a map $BU \to BU$ representing the Adams operation ψ^q on complex K-theory, such that θ_* is an isomorphism on homology. We will then show

Proposition 1.0.1. $F\psi^q$ is simple with $\pi_{2j}(F\psi^q) = 0$ and $\pi_{2j-1}(F\psi^q) = \mathbb{Z}/(q^j - 1)$.

This will give us the calculation.

2. THE MAIN THEOREM

Let's pretend that I've done all that and see where it takes us. The Hurewicz homomorphism gives us

$$
\begin{array}{ccc}
\pi_1(BGL(\mathbb{F}_q)) & \longrightarrow & \pi_1(F\psi^q) \\
\downarrow & & \downarrow{\scriptstyle \cong} \\
H_1(BGL(\mathbb{F}_q)) & \overset{\cong}{\longrightarrow} & H_1(F\psi^q)
\end{array}
$$

where the horizontal maps are induced by θ. The map on the right is an isomorphism because $\pi_1(F\psi^q)$ is abelian. The map on the left is just the quotient map $GL(\mathbb{F}_q) \to GL(\mathbb{F}_q)/E(\mathbb{F}_q)$ and so is surjective, hence the top map is too, both with kernel $E(\mathbb{F}_q)$. By the universal property of the plus construction we have a map $\theta' : BGL(\mathbb{F}_q)^+ \to F\psi^q$ such that $\theta = \theta'i$ where $i : BGL(\mathbb{F}_q) \to BGL(\mathbb{F}_q)^+$. Now i and θ are both isomorphisms on homology, so that θ' is as well. Both $BGL(\mathbb{F}_q)^+$ and $F\psi^q$ are simple, so by Whitehead's theorem we have that θ' is a homotopy equivalence. Then with

$$K_i(\mathbb{F}_q) = \pi_i(BGL(\mathbb{F}_q)^+) = \pi_i(F\psi^q)$$

we immediately have

Theorem 2.0.1. $K_{2j}(\mathbb{F}_q) = 0$ and $K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}/(q^j - 1)$.

And we're done. Well, not quite. I still owe you some more information. In particular I should probably

(1) define $F\psi^q$ and determine some of its basic properties, including those given in 1.0.1, as there's nothing more basic than its homotopy,
(2) construct the map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$,
(3) and prove θ_* is an isomorphism on H_*.

I'm not actually going to do 3. It involves lots of algebra and computations with spectral sequences and what not, so for the purpose of this talk I'll leave it the master himself [1]. But I'll do the other two, which are just some homotopy theory and representation theory, respectively.

3. The Space $F\psi^q$

3.1. **Adams operations on** BU. On complex K-theory, we have Adams operations

$$\psi^q : \widetilde{K}(B) \to \widetilde{K}(B)$$

for all q and B compact. So by Yoneda, we'd like to view the natural transformations ψ^q as maps $BU \to BU$ (recall $\widetilde{K}(-)$ is representable by pointed maps $[-, BU]$), but BU is not compact. Fortunately, we can get around this.

We can give BU the structure of a cell complex with cells X^n in only even dimension; i.e., such that $X^{2n} = X^{2n+1}$. Our goal is to show $\widetilde{K}^{-1}(X^n) = 0$ and apply the Milnor exact sequence. We'll use induction for the first. For the base case, we have

$$\widetilde{K}^{-1}(X^1) = \widetilde{K}^{-1}(X^0) = \widetilde{K}^{-1}(*) = 0$$

Now assume $\widetilde{K}^{-1}(X^{2n}) = \widetilde{K}^{-1}(X^{2n+1}) = 0$. We have $X^{2n+2}/X^{2n} \cong \bigvee S^{2n+2}$ so by the exact sequence of a cofibration in K-theory we have that

$$0 = \prod \widetilde{K}^{-1}(S^{2n+2}) = \widetilde{K}^{-1}\left(\bigvee S^{2n+2}\right) \to \widetilde{K}^{-1}(X^{2n+2}) \to \widetilde{K}^{-1}(X^{2n}) = 0$$

is exact, hence $\widetilde{K}^{-1}(X^{2n+2}) = 0$. The Milnor exact sequence for K-theory can be written as

$$0 \to \varprojlim{}^1 \widetilde{K}^{-1}(X^n) \to [BU, BU] \to \varprojlim[X^n, BU] \to 0$$

The first term is now 0 so the second map is an isomorphism. We already have Adams operations ψ^q defined on $[X^n, BU] = \widetilde{K}(X^m)$ so we take the limit and get map $BU \to BU$, which we also call ψ^q, representing the Adams operation.

3.2. **Defining** $F\psi^q$. For a space X, the fixed points of a map $g : X \to X$ can be seen as the pullback of the diagonal $X \to X^2$ along $(1, g) : X \to X^2$. However, this isn't very homotopy-theoretic. Instead let's replace the diagonal $BU \to BU^2$ with a homotopy-equivalent fibration $BU \xrightarrow{\cong} BU^I \to BU^2$, giving us a map $\Delta : BU^I \to BU$ that sends a path to its endpoints. Let $F\psi^q$ be the pullback of Δ along $(1, \psi^q) : BU \to BU^2$. To be explicit, elements of $F\psi^q$ are pairs (x, p) of points $x \in BU$ and paths p from x to $\psi^q(x)$. If we choose basepoints then the vertical maps in the following pullback diagram are fibrations with fiber ΩBU

$$
\begin{array}{ccc}
F\psi^q & \longrightarrow & BU^I \\
\varphi \downarrow & & \downarrow \Delta \\
BU & \xrightarrow{(1, \psi^q)} & BU^2
\end{array}
$$

In a similar way as before, we can define a map $d : BU^2 \to BU$ representing the difference operation on complex K-theory, so that (a, b) get sent to $a - b$. We can use d to extend the previous diagram as

$$
\begin{array}{ccccc}
F\psi^q & \longrightarrow & BU^I & \xrightarrow{m} & PBU \\
\varphi \downarrow & & \downarrow \Delta & & \downarrow n \\
BU & \xrightarrow{(1, \psi^q)} & BU^2 & \xrightarrow{d} & BU
\end{array}
$$

where PBU is the path space ending at some fixed point $b \in BU$, $n(p) = p(0)$, and $m(p)$ is the path $t \mapsto d(p(t), p(1))$ joining $d(p(0), p(1))$ to b (note we can choose d such that $d(x, x) = b$ for all $x \in BU$). You can then easily check that $F\psi^q$ is the pullback of n along $d(1, \psi^q) = 1 - \psi^q$, and that

Proposition 3.2.1. $F\psi^q$ is the homotopy fiber of $1 - \psi^q$.

3.3. **The homotopy of $F\psi^q$.** We can now prove 1.0.1. By 3.2.1 we have the LES

$$
\cdots \to \pi_j(BU) \xrightarrow{(1 - \psi^q)^*} \pi_j(BU) \to \pi_{j-1}(F\psi^q) \to \pi_{j-1}(BU) \to \cdots
$$

where $\pi_j(BU) = \widetilde{K}(S^j)$ is $\mathbb{Z}$ if j is even and 0 if j is odd. Also ψ^q acts by multiplication by q on $\widetilde{K}(S^2)$, hence by multiplication by q^n on $\widetilde{K}(S^{2n})$. So $(1 - \psi^q)^*$ is injective and these proeprties, combined with exactness, easily give us the homotopy groups of $F\psi^q$.

In general for a fibration the action of π_1 of the fiber on its higher homotopy comes from π_1 of the total space acting on the higher homotopy of the fiber. In our case $\pi_1(BU) = 0$ so we have that $F\psi^q$ is simple.

3.4. **Random lemma.** We'll prove this now, and use it later to construct the map θ. It turns out to be quite important.

Lemma 3.4.1. If X is a space such that $[X, U] = 0$ (recall $U \cong \Omega BU$), then

$$
\varphi_* : [X, F\psi^q] \to [X, BU]^{\psi^q}
$$

is an isomorphism.

Proof. A map $X \to F\psi^q$ is the same thing as a map $f : X \to BU$ and a homotopy from f to $\psi^q f$, hence φ_* is surjective. Now suppose $g : X \to F\psi^q$ is such that $\varphi_*(g)$ is homotopic to the constant map via some homotopy $G : X \times I \to BU$. Backing up the fibration from 3.2.1 we have a lift of the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & F\psi^q \\
(\mathrm{id}, 0) \downarrow & & \downarrow \varphi \\
X \times I & \xrightarrow{G} & BU
\end{array}
$$

with the map on the right a fibration with fiber ΩBU. We conclude that the map g factors through this fiber and is thus 0 by assumption. □

4. CONSTRUCTING THE MAP θ

4.1. **Some representation theory.** Let $R(kG)$ be the representation ring over a field k of a finite group G, which we shall fix for now. We have a map $R(\mathbb{C}G) \to K^0(BG) = [BG, BU \times \mathbb{Z}]$ sending V to the mixing $EG \times_G V$. Composing with the projection gives a map to $[BG, BU]$. In pretty much the same way as for K^0, we can construct Adams operations on $R(\mathbb{C}G)$ such that for characters χ we have $\psi^q(\chi(g)) = \chi(g^q)$. From this it is easy to see that the operations on the representation ring are compatible with those of complex K-theory in the sense that the following diagram commutes for the map constructed above

$$
\begin{array}{ccc}
R(\mathbb{C}G) & \longrightarrow & K^0(BG) \\
\psi^q \downarrow & & \downarrow \psi^q \\
R(\mathbb{C}G) & \longrightarrow & K^0(BG)
\end{array}
$$

This gives us a map on ψ^q fixed points $R(\mathbb{C}G) \to [BG, BU]^{\psi^q}$ induced from the map above.

We also have another piece of information about the relationship between the complex representations and K-theory.

Theorem 4.1.1 (Atiyah-Segal completion theorem). *The map $R(\mathbb{C}G) \to K^0(BG)$ described above induces an isomorphism $\widehat{R(\mathbb{C}G)}_I \cong K^0(BG)$, where I is the augmentation ideal of the representation ring. Further, $K^1(BG) = 0$.*

Corollary 4.1.2. $[BG, BU]^{\psi^q} \cong [BG, BU^{\psi^q}] = [BG, F\psi^q]$

Proof. $0 = K^1(BG) = [BG, U]$. Now apply 3.4.1. □

Let's stop and see where we are. We have a map $R(\mathbb{C}G)^{\psi^q} \to [BG, F\psi^q]$, and our goal was to produce a map $BGL(\mathbb{F}_q) \to F\psi^q$. So we'll probably end up using $GL_n(\mathbb{F}_q)$ for our group G. Then our map tell us that for every complex representation of $GL_n(\mathbb{F}_q)$ fixed under ψ^q we get a map (unique up to homotopy) $BGL_n(\mathbb{F}_q) \to F\psi^q$. We could then take the colimit. But we don't want just any map θ, we want it to be an induce an isomorphism on homology! So in order to get a better handle on the map coming from a representation and the induced map on homology, we will try to relate complex representations (fixed under Adams operations) to representations over $\mathbb{F}_q$. This is the Brauer lift.

4.2. **Some more representation theory: the Brauer lift.** Let's fix an embedding $\rho : \overline{\mathbb{F}_q}^* \to \mathbb{C}^*$. Our group G is still finite, and we'll take E to be a finite-dimensional G-representation over $\overline{\mathbb{F}_q}$. Define the *Brauer character* χ_E of E by

$$
\chi_E(g) = \sum \rho(\lambda_i)
$$

where the λ_i are the eigenvalues of g acting on E.

Claim. χ_E is the character of a unique virtual complex representation ρE.

Turns out this is true, so we'll use it. This gives us a map $R(\overline{\mathbb{F}_q}G) \to R(\mathbb{C}G)$.

Now let $E \in R(\mathbb{F}_q G)$. Extend scalars and write

$$
\overline{E} = E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}
$$

We can also check that $\psi^q(\rho\overline{E}) = \rho\overline{E}$, as the λ_i are stable under Frobenius $x \mapsto x^q$ and $\psi^q(\chi)(g) = \chi(g^q)$. So we have a map

$$
R(\mathbb{F}_q G) \to R(\mathbb{C}G)^{\psi^q}
$$

which we call the Brauer lift. Combining this with what we did before we now have a map

$$R(\mathbb{F}_q G) \to [BG, F\psi^q]$$

so that for any finite-dimensional G-representation over $\mathbb{F}_q$ we get a homotopy class of maps $BG \to F\psi^q$. Taking $GL_n(\mathbb{F}_q)$ as our group, we let $\theta_n : BGL_n(\mathbb{F}_q) \to F\psi^q$ be the map corresponding to the standard representation $\mathbb{F}_q^n$ of $GL_n(\mathbb{F}_q)$. Define $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ as the colimit of the θ_n. It turns out this induces an isomorphism on homology.

References

1. D. Quillen, *On the cohomlogy and k-theory of the general linear groups over a finite field*, Annals of Mathematics (1972).

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