ALGEBRAIC K-THEORY: DEFINITIONS & PROPERTIES (TALK NOTES)

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1. Brief history of the lower K-groups

Reference: Grayson, "Quillen's Work in Algebraic K-Theory"

1.1. **The Grothendieck group.**

Grothendieck's work in the 60s on generalizing Riemann-Roch led him to consider vector bundles (coherent sheaves) on an algebraic variety X. These form a commutative monoid under $\oplus$, and we may **group complete** to obtain:

$$K(X) = \langle [E],\, E \in \mathrm{Vect}(X) \mid [E] = [E'] + [E''] \text{ for each SES } 0 \to E' \to E \to E'' \to 0 \rangle$$

Let $X = \operatorname{Spec} R$ be affine. Write $K_0(R) = K(X)$.

Serre: (vector bundles on $\operatorname{Spec} R$) $\longleftrightarrow$ (f.g. projective R-modules)

Examples.

(a) F a field (or a PID).

Every projective F-module is free, so $K_0(F) \xrightarrow{\dim} \mathbb{Z}$ is an isomorphism.

(b) $\mathcal{O}_K$, the ring of integers of a number field K.

It's a Dedekind domain, so every f.g. projective module has the form $\mathcal{O}_K^{n-1} \oplus I$ for some ideal I that is uniquely determined in the ideal class group. So $K_0(\mathcal{O}_K) \cong \mathbb{Z} \oplus \mathrm{Cl}(K)$.

1.2. **Topological K-theory.**

- Motivated by Grothendieck, Atiyah and Hirzebruch studied the situation where X is a finite simplicial complex.
- Using Σ and *Bott periodicity*, they constructed a generalized cohomology theory $K^*(X)$ with $K^0(X) = K(X)$.
- Adams introduced cohomology operations ψ^k on $K(X)$ to study vector fields on spheres. The ψ^k's are characterized by $\psi^k[L] = [L^{\otimes k}]$ for line bundles L. Quillen later uses these to compute the cohomology of $GL(\mathbb{F}_q)$.

1.3. K_1 **and** K_2**.** Can something similar to topological K-theory be done in the algebraic setting?

Definition (Bass)**.** $K_1(R) = GL(R)/[GL(R), GL(R)]$

Idea, motivated by Σ: a vector bundle over ΣX is determined by gluing data along the equator $X \subset \Sigma X$, i.e., by $GL(E)$ where E is a trivial bundle. Homotopic maps give isomorphic bundles, so we should quotient out by "$GL(E)^\circ \supseteq E(R)$", the subgroup generated by elementary matrices.

Lemma (Whitehead)**.** *$[GL(R), GL(R)] = [E(R), E(R)] = E(R)$.*

Let's compute some K-groups.

Examples.

(a) F a field (or a commutative local ring).

$E(F) = SL(F)$, so $K_1(F) \cong F^\times$.

(b) R a commutative ring.

Have *determinant map* $\det : K_1(R) \to R^\times$. Have splitting $R^\times = GL_1(R) \hookrightarrow GL(R) \twoheadrightarrow K_1(R)$. Thus $K_1(R) \cong R^\times \oplus SK_1(R)$ where $SK_1(R) = \ker \det$.

If $R = \mathcal{O}_K$, then $K_1(\mathcal{O}_K) \cong \mathcal{O}_K^\times \oplus SK_1(\mathcal{O}_K)$. By *Dirichlet's unit theorem*, $\mathcal{O}_K^\times \cong \mu(K) \oplus \mathbb{Z}^{r-1}$; by *Bass-Milnor-Serre*, $SK_1(\mathcal{O}_K) = 0$. So $K_1(\mathcal{O}_K) \cong \mu(K) \oplus \mathbb{Z}^{r-1}$.

Why is this group K_1?

Proposition (Localization)**.** *R a Dedekind domain, F its fraction field. Have exact sequence:*

$$\bigoplus_{\mathfrak{p}} K_1(R/\mathfrak{p}) \to K_1(R) \to K_1(F) \to \bigoplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(F) \to 0$$

Looks like the end of some long exact sequence.

Later:

Definition (Milnor)**.** $K_2(R) = \ker(\mathrm{St}\, R \to E(R))$

The *Steinberg group* $\mathrm{St}\, R$ is the abstract group modelled on elementary matrices with just the "obvious" relations, so $K_2(R)$ is the group of "non-obvious" relations amongst elementary matrices.

Bass-Tate added $K_2(F)$ to the above localization sequence; Bass further added $K_2(R)$.

Remark. Can compute K_2 of a field using *Matsumoto's theorem*, e.g., $K_2(\mathbb{F}_q) = 0$.

2. Quillen's +-construction

Somehow, need input from <u>homotopy theory</u> to access higher K-groups.

Quillen's motivation: work on *Adams' conjecture*, discussion with <u>Sullivan</u>. Our motivation: *topological K-theory*. Recall:

- Complex K-theory represented by spectrum $KU = \{(\mathbb{Z}\times)BU, \Omega BU, \ldots\}$ and $U \simeq BGL(\mathbb{C})$
- Real K-theory represented by spectrum $KO = \{(\mathbb{Z}\times)BO, \Omega BO, \ldots\}$ and $O \simeq BGL(\mathbb{R})$.

For a ring R, we should look at the classifying space <u>$BGL(R)$</u> and its homotopy groups.

<u>Problems:</u>

- $\pi_1 BGL(R) = GL(R)$, not $K_1(R)$
- $\pi_i BGL(R) = 0$ for $i > 1$.

Therefore, need to modify $BGL(R)$. <u>Kervaire</u> introduced a construction to kill a *perfect* normal subgroup of π_1 without changing H_*.

Proposition (<u>+ construction</u>)**.** *Let X be a connected pointed CW complex, $N \trianglelefteq \pi_1 X$ perfect. There exists a connected CW complex X^+ and a map $i : X \to X^+$ such that*

(i) $\pi_1(i)$ is the quotient map $\pi_1 X \to \pi_1 X^+ \cong \pi_1 X/N$.
(ii) $H_*(i)$ is an isomorphism with any system of local coefficients.

Furthermore, this satisfies a universal property: if Y is some other connected space and $f : X \to Y$ is such that $N \leq \ker f_*$, then there exists $f^+ : X^+ \to Y$, unique up to homotopy, such that

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, dashed, "f^+"] \\
& X^+
\end{tikzcd}
$$

Therefore, + is functorial up to homotopy.

Sketch of a construction.

(1) Attach 2-cells to kill $N \trianglelefteq \pi_1 X$.

(2) Attach 3-cells to correct "noise" in H_*.

The assumption that N is perfect means we can attach 3-cells "precisely", i.e., without affecting other H_* groups. □

Example. Σ_n symmetric group on n letters, $A_n \leq \Sigma_n$ alternating group, which is normal and perfect if $n \geq 5$. Take colimits and form $B\Sigma_\infty^+$ with respect to A_∞. The theorem of Barratt-Priddy-Quillen says $\pi_* B\Sigma_\infty^+ \cong \pi_* \mathbb{S}$.

Back to K-theory: $E(R) \triangleleft GL(R)$ is perfect, so form $BGL(R)^+$ with respect to $E(R)$.

Definition. $K_n(R) = \pi_n BGL(R)^+$ for $n \geq 1$.

By construction, $K_1 R = \pi_1 BGL(R)^+ = GL(R)/E(R)$. Can also see this definition agrees with classical K_2 using *group cohomology*.

Problem: $BGL(R)^+$ is path connected. Ad hoc fix: define $K(R) = K_0 R \times BGL(R)^+$, the **$K$-theory space of** R.

Theorem (Quillen, Gersten-Wagoner)**.** *$\Omega BGL(SR)^+ \simeq K_0 R \times BGL(R)^+$, where SR is the "suspension" of R.*

Thus obtain a <u>delooping</u> of $K(R)$, i.e., a **Ω-spectrum** $\{K_0(S^n R) \times BGL(S^n R)^+\}$ whose n-th homotopy group is $K_n R$.

2.1. **Products.**

Proposition. *$BGL(R)^+$ is a commutative <u>H-group</u>.*

The operation is induced by a map $\oplus : GL(R) \times GL(R) \to G(R)$ interweaving rows and columns:

$$BGL(R)^+ \times BGL(R)^+ \simeq B(GL(R) \times GL(R))^+ \xrightarrow{B\oplus^+} BGL(R)^+.$$

Theorem (Loday)**.** *There exists a bilinear and associative* <u>*product*</u> $K_pR \otimes K_qR \to K_{p+q}(R \otimes S)$ *that is natural in* R *and* S*.*

If R *is commutative, then* $K_pR \otimes K_qR \to K_{p+q}R$ *is graded-commutative.*

This product is induced by the tensor product of matrices, appropriately stabilized.

2.2. **Group completion.**

Definition. A *group completion* of a homotopy associative, homotopy commutative H-space X is a map $f : X \to Y$ where Y is a homotopy associative, homotopy commutative H-space such that

(i) $f_* : \pi_0 X \to \pi_0 Y$ is the group completion of the commutative monoid $\pi_0 X$

(ii) The map $(\pi_0 X)^{-1} H_* X \to H_* Y$ induced by f is an isomorphism.

Let $(\mathcal{S}, \oplus)$ be a *symmetric monoidal category* acting on $\mathcal{X}$ (denote this action by $\oplus$ also). Define $\mathcal{S}^{-1}\mathcal{X}$ to be the category whose objects are $(s, x) \in \mathrm{ob}(\mathcal{S} \times \mathcal{X})$ and whose morphisms $(s, x) \to (t, y)$ are pairs of equivalence classes of maps $u \oplus s \to t$, $u \oplus x \to y$ for some $u \in \mathrm{ob}\,\mathcal{S}$.

Then $\mathcal{S}$ acts on $\mathcal{S}^{-1}\mathcal{X}$ by $s \cdot (t, x) = (t, s \oplus x)$, and this action is *invertible.*

Theorem (Quillen)**.** *Let* $\mathcal{S}$ *be a symmetric monoidal category in which "translations are faithful". Then* $B(\mathcal{S}^{-1}\mathcal{S})$ *is the group completion of* $B\mathcal{S}$*. We care about the case* $\mathcal{S} = \mathrm{iso}\,\mathbf{P}(R)$*, the category of* f.g. projective R-modules *with isomorphisms as morphisms.*

Proof outline. Condition on π_0 is easy to verify. For condition on H_*, construct a spectral sequence $E^2_{pq} = H_p(\mathcal{S}^{-1}(*); (\pi_0\mathcal{S})^{-1} H_q \mathcal{S}) \Rightarrow H_{p+q}\mathcal{S}^{-1}\mathcal{S}$ which degenerates to the desired isomorphism. $\square$

Proposition. $\mathcal{S} = \mathrm{iso}\,\mathbf{P}(R)$*. Then* $B(\mathcal{S}^{-1}\mathcal{S}) \simeq K_0 R \times BGL(R)^+$*.*

Proposition follows from two lemmas:

Lemma. *Let* $\mathcal{S} = \operatorname{iso}\mathbf{F}(R)$*, where* $\mathbf{F}(R)$ *is the category of f.g. free* R*-modules, so* $B\mathcal{S} \simeq \coprod_{n\geq 0} BGL_n(R)$*. Then* $B(\mathcal{S}^{-1}\mathcal{S}) \simeq \mathbb{Z} \times BGL(R)^+$*.*

Proof. We'll just show that the basepoint $(0,0)$ component $B(\mathcal{S}^{-1}\mathcal{S})_0$ of $B(\mathcal{S}^{-1}\mathcal{S})$ is homotopy equivalent to $BGL(R)^+$.

Step 1. Construct a map $\phi : BGL(R) \to B(\mathcal{S}^{-1}\mathcal{S})_0$.

Define $\phi_n : BGL_nR \to B\operatorname{Aut}(R^n, R^n) \hookrightarrow B(\mathcal{S}^{-1}\mathcal{S})_0$ to be the map induced by $(g \in GL_nR) \mapsto ((g,1) \in \operatorname{Aut}(R^n, R^n))$. Have homotopy commutative diagram

$$\begin{tikzcd} BGL_nR \arrow[rr] \arrow[dr, "\phi_n"'] & & BGL_{n+1}R \arrow[dl, "\phi_{n+1}"] \\ & B(\mathcal{S}^{-1}\mathcal{S})_0 & \end{tikzcd}$$

Thus, get map $\phi : BGL(R) = \operatorname{hocolim}_n BGL_{n+1}R \to B(\mathcal{S}^{-1}\mathcal{S})_0$

Step 2. Show ϕ is acyclic (induces H_*-isos).

$B(\mathcal{S}^{-1}\mathcal{S})$ is the *group completion* of $B\mathcal{S}$. Let e be the class of $[R]$ in $\pi_0\mathcal{S}$. Then

$$\begin{aligned} H_*B(\mathcal{S}^{-1}\mathcal{S}) \cong H_*B\mathcal{S}[\frac{1}{e}] &\cong \operatorname{colim}(H_*B\mathcal{S} \xrightarrow{(\oplus R)_*} H_*B\mathcal{S} \xrightarrow{(\oplus R)_*} \cdots) \\ &\cong H_* \operatorname{hocolim}(B\mathcal{S} \xrightarrow{\oplus R} B\mathcal{S} \xrightarrow{\oplus R} \cdots) \\ &\cong H_*BGL(R). \end{aligned}$$

(Recall that $B\mathcal{S}$ is a H-space with operation $\oplus$.)

Step 3. B_0, $BGL(R)^+$ are connected CW H-spaces with same H_* and π_1 (because

$$\pi_1 B(\mathcal{S}^{-1}\mathcal{S})_0 \cong H_1B(\mathcal{S}^{-1}\mathcal{S}) \cong H_1BGL(R) = GL(R)/E(R)).$$

So by the relative Hurewicz theorem, Whitehead's theorem, etc., $BGL(R)^+ \simeq B(\mathcal{S}^{-1}\mathcal{S})_0$. $\square$

Lemma. *Let* $F : \mathcal{S} \to \mathcal{T}$ *be cofinal functors between symmetric monoidal categories (i.e., for each* $t \in \mathcal{T}$*, there exists* $s \in \mathcal{S}$ *and* $t' \in \mathcal{T}$ *such that* $t \oplus t' \cong F(s)$*) such that* $\operatorname{Aut}_{\mathcal{S}}(s) = \operatorname{Aut}_{\mathcal{T}}(F(s))$ *for all* s*. Then* $B(\mathcal{S}^{-1}\mathcal{S}) \to B(\mathcal{T}^{-1}\mathcal{T})$ *induces equivalences of basepoint components.*

Applying this lemma to $\mathbf{F}(R) \hookrightarrow \mathbf{P}(R)$, we get

$$B(\mathcal{S}^{-1}\mathcal{S}) \simeq \pi_0 B\mathcal{S}^{-1}\mathcal{S} \times BGL(R)^+ \cong K_0R \times BGL(R)^+.$$

3. The Q construction

Reference: Quillen, "Higher Algebraic K-Theory, Vol. I"

Turns out it was difficult to prove the basic theorems of algebraic K-theory using the $+$ construction. We might expect this from the ad hoc addition of $K_0(R)$, which is divorced from homotopy theory. In 1972, Quillen came up with a new definition of K_*. The natural setting for this construction is exact categories.

Definition. An *exact category* is an additive category $\mathcal{M}$ equipped with a family $\mathcal{E}$ of "exact sequences" satisfying. . .

Think: an abelian category with usual exact sequences; even more concretely: $\mathbf{P}(R)$, the category of f.g. projective R-modules.

If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence, then we call $M' \to M$ an *admissible monomorphism* ($\rightarrowtail$) and $M \to M''$ an *admissible epimorphism* ($\twoheadrightarrow$).

Let $\mathcal{M}$ be a (small) exact category.

Construction (Q construction)**.** Define a new category $Q\mathcal{M}$ whose objects are the same as $\mathcal{M}$ and whose morphisms $M \to M'$ are equivalence classes of diagrams $M \twoheadleftarrow N \rightarrowtail M'$ (two diagrams are equivalent if there is an isomorphism of diagrams inducing the identity on M and M').

Composition $(M' \twoheadleftarrow N' \rightarrowtail M'') \circ (M \twoheadleftarrow N \rightarrowtail M')$ is given by

$$\begin{tikzcd}
N \times_{M'} N' \arrow[r, tail] \arrow[d, two heads] & N' \arrow[r, tail] \arrow[d, two heads] & M'' \\
N \arrow[r, tail] \arrow[d, two heads] & M' & \\
M & &
\end{tikzcd}$$

Given an admissible mono $M \overset{i}{\rightarrowtail} M'$ in $\mathcal{M}$, get a morphism $i_! : M \overset{\mathrm{id}}{\twoheadleftarrow} M \overset{i}{\rightarrowtail} M'$ in $Q\mathcal{M}$, called injective.

Given an admissible epi $M' \overset{j}{\twoheadrightarrow} M$ in $\mathcal{M}$, get a morphism $j^! : M \overset{j}{\twoheadleftarrow} M' \overset{\mathrm{id}}{\rightarrowtail} M'$ in $Q\mathcal{M}$, called <u>surjective</u>.

Proposition. *$\mathcal{M} \rightsquigarrow Q\mathcal{M}$ is <u>universal</u> for assignments $\mathcal{M} \rightsquigarrow \mathcal{C}$, where $\mathcal{C}$ has "injective", "surjective" maps satisfying .... That is, given $\mathcal{C}$, exists functor $Q\mathcal{M} \to \mathcal{C}$ such that....*

<u>Key theorem:</u>

Theorem (Quillen)**.** *$\pi_1(BQ\mathcal{M}, 0) \cong K_0\mathcal{M}$.*

Remark. Quillen's original proof uses coverings, morphism-inverting functors, etc.

Proof of theorem. Want to give a presentation of $\pi_1 BQ\mathcal{M}$; look at 2-skeleton of $BQ\mathcal{M}$.

Let $T = \{i_{M!} : 0 \to M\}$. This is a maximal tree in the 1-skeleton of $BQ\mathcal{M}$. Collapse T, and see that $\pi_1 BQ\mathcal{M}$ is generated by morphisms $[f]$ in $\mathcal{Q}M$ modulo relations $[i_{M!}] = 1$, $[f \circ g] = [f][g]$.

- If $i : M \rightarrowtail M'$ is any admissible mono, then $[i_!] = 1$. Proof: can express $i_!$ into terms of $i_{M!}$ and $i_{M'!}$.
- If $j : M' \twoheadrightarrow M$ is any admissible epi, then $[j^!]$ can be expressed in terms of $[j_M^!]$ and $[j_{M'}^!]$. (Here, j_M is the map $M \twoheadrightarrow 0$.) Proof: same as above.
- Every morphism in $Q\mathcal{M}$ factors as $i_! \circ j^!$ tautologically, so $\pi_1 BQ\mathcal{M}$ is generated by $\{[i_!], [j^!]\}$, i.e., just by $[j_M^!]$ because of previous two points.

Now suppose $M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M''$ is an exact sequence in $\mathcal{M}$. <u>Claim</u>: $j^! \circ i_{M''!} = i_! \circ j_{M'}^!$. Proof:

$$\begin{tikzcd}
M' \arrow[r, tail, "i"] \arrow[d, two heads, "j_{M'}"'] & M \arrow[r, tail] \arrow[d, two heads, "j"] & M \\
0 \arrow[r, tail, "i_{M''}"'] \arrow[d, two heads] & M'' & \\
0 & &
\end{tikzcd}$$

Therefore, $[j^!] = [j_{M'}^!]$. So,

$$[j_M^!] = [(j_{M''} \circ j)^!] = [j^! \circ j_{M''}^!] = [j^!][j_{M''}^!] = [j_{M'}^!][j_{M''}^!].$$

This is the **additivity relation**.

Conversely, every relation $[f][g] = [f \circ g]$ can be rewritten in terms of the additivity relation. Given $g : M \twoheadleftarrow N \rightarrowtail M'$ and $f : M' \twoheadleftarrow N' \rightarrowtail M''$), their composite is given by

$$
\begin{tikzcd}
N \times_{M'} N' \arrow[r, tail] \arrow[d, two heads] & N' \arrow[r, tail] \arrow[d, two heads] & M'' \\
N \arrow[r, tail] \arrow[d, two heads] & M' & \\
M & &
\end{tikzcd}
$$

Let $K = \ker(N' \twoheadrightarrow M') \cong \ker(N \times_M N' \twoheadrightarrow N)$, let $K' = \ker(N \times_M N' \twoheadrightarrow M)$ and $K'' = \ker(N \twoheadrightarrow M)$. Then $K \rightarrowtail K' \twoheadrightarrow K''$ is exact, so $[j^!_{K'}] = [j^!_K][j^!_{K''}]$. Then

$$[f \circ g] = [j^!_{N \times_{M'} N}][j^!_M]^{-1} = [j^!_{K'}] = [j^!_K][j^!_{K''}] = ([j^!_{N'}][j^!_{M'}]^{-1})([j^!_N][j^!_M]^{-1}) = [f][g].$$

Therefore,

$$\pi_1 BQ\mathcal{M} = \langle [j^!_M],\, M \in \operatorname{ob}\mathcal{M} \mid [j^!_M] = [j^!_{M'}][j^!_{M''}] \text{ for each SES } M' \rightarrowtail M \twoheadrightarrow M'' \rangle \cong K_0\mathcal{M}.$$

$\square$

Definition. $K\mathcal{M} = \Omega BQ\mathcal{M}$. $K_i\mathcal{M} = \pi_i K\mathcal{M} = \pi_{i+1} BQ\mathcal{M}$.

Special cases:

- $\mathcal{M} = \mathbf{P}(R)$ category of f.g. projective R-modules. Set $K_i(R) := K_i\mathbf{P}(R)$.
- $\mathcal{M} = \mathbf{M}(R)$ category of all f.g. R-modules. Set $G_i(R) := K_i\mathbf{M}(R)$.

Proposition (Elementary properties)**.**

(a) An exact functor induces a homomorphism of K-groups. For example

(i) $R \to S$ ring map. Get forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$. Get $K_iS \to K_iR$.

(ii) S a flat R-module, so $- \otimes_R S$ is exact. Get $K_iR \to K_iS$.

(b) $K_i\mathcal{M}^{\mathrm{op}} \cong K_i\mathcal{M}$.

(c) $K_i(\mathcal{M} \times \mathcal{M}') \cong K_i\mathcal{M} \times K_i\mathcal{M}'$.

(d) K_i commutes with filtered colimits.

Plus equals Q.

Theorem. *$\mathcal{M}$ a split exact category, $\mathcal{S} = \operatorname{iso}\mathcal{M}$. Then $\mathcal{S}^{-1}\mathcal{S} \simeq Q\mathcal{M}$.*

Corollary ($\underline{+ = Q}$)**.** *$K_0R \times BGL(R)^+ \simeq BQ\mathbf{P}(R)$.*

Proof. We've proved that if $\mathcal{M} = \mathbf{P}(R)$, then $K_0R \times BGL(R)^+ \simeq B(\mathcal{S}^{-1}\mathcal{S})$. By the theorem, $B(\mathcal{S}^{-1}\mathcal{S}) \simeq BQ\mathbf{P}(R)$. □

Proof of theorem.

Step 1. Define a category $\mathcal{E}$ of exact sequences in $\mathcal{M}$ whose morphisms $(A \rightarrowtail B \twoheadrightarrow C) \to (A' \rightarrowtail B' \twoheadrightarrow C')$ are equivalence classes of commutative diagrams

$$\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & C \\
A' \arrow[u, tail] \arrow[r, tail] \arrow[d, equal] & B \arrow[u, equal] \arrow[r, two heads] \arrow[d, tail] & C'' \arrow[u, two heads] \arrow[d, tail] \\
A' \arrow[r, tail] & B \arrow[r, two heads] & C'
\end{tikzcd}$$

(Two such diagrams are equivalent if they are isomorphic via isomorphisms that are the identity except possibly at C''.)

Step 2. Define an action of $\mathcal{S}$ on $\mathcal{E}$: i.e., $\mathcal{S} \times \mathcal{E} \to \mathcal{E}$ by

$$(s, A \rightarrowtail B \twoheadrightarrow C) \quad \mapsto \quad (s \oplus A \rightarrowtail s \oplus B \twoheadrightarrow C).$$

Thus we can get $\mathcal{S}^{-1}\mathcal{E}$.

Step 3. The functor $q : \mathcal{E} \to Q\mathcal{M}$ defined by $(A \rightarrowtail B \twoheadrightarrow C) \mapsto C$ induces a functor $\tilde{q} : \mathcal{S}^{-1}\mathcal{E} \to Q\mathcal{M}$.

$\mathcal{S}^{-1}\mathcal{E}$ is fibered over $Q\mathcal{M}$ by $\tilde{q}$, i.e., $\tilde{q}^{-1}C \hookrightarrow C \downarrow \tilde{q}$ is a homotopy equivalence for any object C in $Q\mathcal{M}$.

Step 4. Use Quillen's Theorem B to get a fibration sequence

$$B(\mathcal{S}^{-1}\mathcal{S}) \simeq B(\tilde{q}^{-1}C) \to B(\mathcal{S}^{-1}\mathcal{E}) \xrightarrow{B\tilde{q}} BQ\mathcal{M}.$$

The first arrow is induced from $(A \in \mathcal{S}) \mapsto (A \rightarrowtail A \oplus C \twoheadrightarrow C \in q^{-1}C)$.

We get a long exact sequence in homotopy groups.

Step 5. Show that $B(\mathcal{S}^{-1}\mathcal{E})$ is contractible.

- $\mathcal{E}$ is contractible. Using Quillen's Theorem A, can show $\mathcal{E} \simeq iQ\mathcal{M}$, the category $Q\mathcal{M}$ with just the admissible monomorphisms. The category $iQ\mathcal{M}$ has an initial object, and thus is contractible.
- $\mathcal{S}$ acts by homotopy equivalences on $\mathcal{E}$, so $B\mathcal{E} \to B(\mathcal{S}^{-1}\mathcal{E})$ is a homotopy equivalence.

Combining this with the LES from step 4, we have a weak equivalence which can be promoted to a homotopy equivalence using Whitehead's theorem. $\square$

4. Basic properties

The proofs of the basic properties rely on applying Quillen's Theorem A and Theorem B to cleverly constructed categories. Theorem A gives conditions for when a functor $F : \mathcal{C} \to \mathcal{C}'$ is a homotopy equivalence. Theorem B gives a relationship between $\mathcal{C}$, $\mathcal{C}'$ and $Y \downarrow F$ (Y is an object of $\mathcal{C}'$) in terms of a long exact sequence.

Theorem (Quillen's Theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. If the category $Y \downarrow F$ (or $F \downarrow Y$) is contractible for every object Y of $\mathcal{C}'$, then the functor F is a homotopy equivalence.*

Theorem (Quillen's Theorem B)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor such that for every arrow $Y \to Y'$ in $\mathcal{C}'$, the induced functor $Y' \downarrow F \to Y \downarrow F$ is a homotopy equivalence. Then for any object Y of $\mathcal{C}'$, the square*

$$\begin{tikzcd} Y \downarrow F \arrow[r, "j"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ Y \downarrow \mathcal{C}' \arrow[r, "j'"'] & \mathcal{C}' \end{tikzcd}$$

is homotopy cartesian, where $j(X, v) = X$, $F'(X, v) = (FX, v)$, and $j'(Y', v) = Y'$.

Consequently for any X in $F^{-1}(Y)$ (since $Y \downarrow \mathcal{C}'$ has an initial object and is contractible), we have an exact sequence

$$\cdots \to \pi_{i+1}(B\mathcal{C}', Y) \to \pi_i(B(Y \downarrow F), \bar{X}) \xrightarrow{j_*} \pi_i(B\mathcal{C}, X) \xrightarrow{F_*} \pi_i(B\mathcal{C}', Y) \to \cdots$$

where $\bar{X} = (X, \mathrm{id}_Y)$.

4.1. **Additivity.**

Theorem (Additivity). *$\mathcal{M}, \mathcal{M}'$ exact categories, $0 \to F' \to F \to F'' \to 0$ a SES of exact functors $\mathcal{M} \to \mathcal{M}'$. Then $F_* = F'_* + F''_*$.*

Proof sketch. Reduce to universal example: let $\mathcal{E}$ be the category of SES in $\mathcal{M}$, with usual morphisms unlike before. Let $s, q, t : \mathcal{E} \to \mathcal{M}$ be the sub, quotient, and total object functors. Make $\mathcal{E}$ into an exact category by declaring s, q, t to be exact. It is enough to show that $t_* = s_* + q_*$.

Define $f : \mathcal{M} \times \mathcal{M} \to \mathcal{E}$ by $(M', M'') \mapsto (0 \to M' \to M' \oplus M'' \to M'' \to 0)$. Then $tf = (s \oplus q)f$, and

$$t_* f_* = (s_* + q_*) f_* = s_* f_* + q_* f_*.$$

Want to cancel f_*. Note f_* is a section of $(s \oplus q)_*$; so want to show that $s \oplus q$ is a homotopy equivalence. By Theorem A, enough to show that $\mathcal{C} = (s, q) \downarrow (M, N)$ is contractible for all objects M, N in $\mathcal{M}$. The objects of $\mathcal{C}$ are triples (E, u, v) where $E \in \operatorname{ob} \mathcal{E}$, and $u : sE \to M$, $v : qE \to N$ are maps in $Q\mathcal{M}$.

We can replace $\mathcal{C}$ by its homotopy equivalent full subcategory $\mathcal{C}''$ consisting of (E, u, v) where u is surjective and v is injective. But $\mathcal{C}''$ has an initial object $(0, j^!_M, i_{N!})$, so it's contractible. □

Corollary.

(a) $F : \mathcal{M} \to \mathcal{M}'$ exact. $0 = F_0 \subset \cdots \subset F_n = F$ a filtration with $F_{p-1}X \rightarrowtail F_pX$ an admissible mono for each X. If F_p/F_{p-1} is exact, then $F_* = \sum_p (F_p/F_{p-1})_*$.

(b) $0 \to F_0 \to \cdots \to F_n \to 0$ exact. Then $\sum_p (-1)^p (F_p)_* = 0$.

Remark (Eilenberg-Mazur swindle). Why not consider all (not necessarily f.g.) modules? If there exists an exact endofunctor $\infty : \mathcal{M} \to \mathcal{M}$ such that $\infty \simeq 1 \oplus \infty$, then $\infty_* = 1_* + \infty_* \Rightarrow 1_* = 0$, i.e., $K\mathcal{M}$ is contractible. In case of all modules, can set $\infty(M) = M \oplus M \oplus \cdots$.

4.2. **Resolution.** Let $\mathcal{P}$ be a full subcategory of a small exact category $\mathcal{M}$ that is closed under extensions. Suppose we can resolve objects in $\mathcal{M}$ by objects in $\mathcal{P}$. What can we say about $K_i\mathcal{P}$ and $K_i\mathcal{M}$?

Theorem (Resolution). *Assume $\mathcal{P}$ is closed under extensions in $\mathcal{M}$ and further that* *(i) for every SES $M' \rightarrowtail M \twoheadrightarrow M''$, if M, M'' are in $\mathcal{P}$, then so is M'.*

(ii) given $j : M \twoheadrightarrow P$, there exist $j' : P' \twoheadrightarrow P$ and $f : P' \to M$ such that $j \circ f = j'$ (e.g., if $\mathcal{M}$ "has enough projectives").

Let $\mathcal{P}_n$ be the full subcategory of $\mathcal{M}$ consisting of those objects with projective dimension $\leq n$, and put $\mathcal{P}_\infty = \operatorname{colim}_n \mathcal{P}_n$. Then $K_i\mathcal{P} \xrightarrow{\cong} K_i\mathcal{P}_1 \xrightarrow{\cong} \cdots \xrightarrow{\cong} K_i\mathcal{P}_\infty$.

Corollary. *If R is regular noetherian, then $K_iR \xrightarrow{\cong} G_iR$.*

4.3. **Dévissage and localization.** Let $\mathcal{A}$ be a small abelian category, $\mathcal{B} \subset \mathcal{A}$ a nonempty full subcategory closed under subobjects, quotient objects, and finite products. What can we say about $K_i\mathcal{B}$ and $K_i\mathcal{A}$?

Theorem (Dévissage)**.** *Suppose every object M of $\mathcal{A}$ has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_j/M_{j-1} is in $\mathcal{B}$ for each j. Then $K_i\mathcal{B} \xrightarrow{\cong} K_i\mathcal{A}$.*

Setting $\mathcal{B} = \{$semisimple objects in $\mathcal{A}\}$, we obtain:

Corollary. *Suppose every object in $\mathcal{A}$ has finite length. Then $K_i\mathcal{A} \cong \bigoplus_j K_iD_j$ where $\{X_j\}$ are isoclasses of simple objects of $\mathcal{A}$ and $D_j = \operatorname{End}(X_j)^{\mathrm{op}}$.*

Theorem (Localization)**.** *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$ and $\mathcal{A}/\mathcal{B}$ be the associated quotient abelian category (same objects as $\mathcal{A}$ but morphisms $A_1 \to A_2$ are elements of $\operatorname{colim}_{A_1' \subseteq A_1,\, A_2' \subseteq A_2} \operatorname{Hom}_{\mathcal{A}}(A_1', A_2/A_2')$).*

Then there exists a LES

$$\cdots \to K_{i+1}(\mathcal{A}/\mathcal{B}) \to K_i\mathcal{B} \to K_i\mathcal{A} \to K_i(\mathcal{A}/\mathcal{B}) \to \cdots .$$

Corollary. *R a Dedekind domain, F its fraction field. Have LES*

$$\cdots \to K_{i+1}F \to \bigoplus_{\mathfrak{p}} K_i(R/\mathfrak{p}) \to K_iR \to K_iF \to \cdots .$$

Proof.

- Apply localization to (torsion R-modules) $\subset \mathbf{M}(R)$. Associated quotient category is $\mathbf{P}(F)$.
- Resolution theorem says $K_i\mathbf{M}(R) \cong K_iR$.
- Dévissage theorem says K_i(torsion R-modules) $\cong \bigoplus_{\mathfrak{p}} K_i(R/\mathfrak{p})$.

□

This is one of the things we originally wanted to generalize!

4.4. Fundamental theorem for rings.

Theorem. *R regular noetherian.*

(a) $K_i(R[t]) \cong K_i R$

(b) $K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R)$.

Theorem. *R any ring. Have exact sequence*

$$0 \to K_i(R) \to K_i(R[t]) \oplus K_i(R[t^{-1}]) \to K_i(R[t, t^{-1}]) \to K_{i-1} R \to 0.$$

Remark.

- The case $i = 1$ was proved by Bass.
- Can generalize these theorems to *schemes*.

5. Waldhausen K-theory

Reference: Waldhausen, "Algebraic K-Theory of Spaces"

- generalizes Quillen K-theory
- generalized further by Thomason-Trobaugh for complexes; Barwick, etc. for higher categories, ...

Definition. A *Waldhausen category* is a pointed category $\mathcal{C}$ together with subcategories $\mathrm{co}\,\mathcal{C}$ (called cofibrations) and $w\mathcal{C}$ (called weak equivalences) satisfying...

Example. Any exact category is Waldhausen by taking admissible monos to be cofibrations and isos to be weak equivalences.

Let $\mathcal{C}$ be a Waldhausen category. Let $[n] = \{0 \leq 1 \leq \cdots \leq n\}$. Consider functors $X : \mathrm{Ar}[n] \to \mathcal{C}$, $(i, j) \mapsto X(i, j)$ such that

(i) for $i \in \mathrm{ob}[n]$, $X(i, i)$ is a zero object

(ii) for $i \leq j \leq k$ in $[n]$, $X(i, j) \to X(i, k)$ is a cofibration and

$$\begin{tikzcd}
X(i,j) \arrow[r, tail] \arrow[d] & X(i,k) \arrow[d] \\
X(j,j) \arrow[r, tail] & X(j,k)
\end{tikzcd}$$

is a pushout square.

- $S_n\mathcal{C}$ is the category of these functors and their natural transformations.
- $wS_n\mathcal{C}$ is the subcategory of these functors with natural transformations $X \to X'$ such that $X(i,j) \xrightarrow{\sim} X'(i,j)$ is a weak equivalence for all $i \le j$.

i.e., an object in these categories looks like

$$\begin{tikzcd}
X(0,0) \arrow[r, tail] & X(0,1) \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & X(0,n) \arrow[d] \\
& X(1,1) \arrow[r, tail] & \cdots \arrow[r, tail] \arrow[d] & X(1,n) \arrow[d] \\
& & \cdot \arrow[r, tail] & \vdots \arrow[d] \\
& & & X(n,n)
\end{tikzcd}$$

and is determined by the sequence of cofibrations

$$X(0,1) \rightarrowtail \cdots \rightarrowtail X(0,n),$$

together with a choice of cofibers $X(i,j) = X(0,j)/X(0,i)$.

Letting n vary, we obtain simplicial categories

$$S.\mathcal{C} : [n] \mapsto S_n\mathcal{C}$$

$$wS.\mathcal{C} : [n] \mapsto wS_n\mathcal{C}.$$

Consider the geometric realization $|wS.\mathcal{C}|$. Have $|wS_0\mathcal{C}| \cong *$, $|wS_1\mathcal{C}| \cong |w\mathcal{C}|$. So the 1-skeleton of $|wS.\mathcal{C}|$ is obtained by attaching $|w\mathcal{C}| \times |\Delta^1|$ to $*$, and can be identified as the suspension $S^1 \wedge |w\mathcal{C}|$. The adjoint of the inclusion $S^1 \wedge |w\mathcal{C}| \hookrightarrow |wS.\mathcal{C}|$ gives a map $|w\mathcal{C}| \to \Omega|wS.\mathcal{C}|$.

Definition. The K**-theory space** of the Waldhausen category $\mathcal{C}$ is $\Omega|wS.\mathcal{C}|$.

Upshot. This gives a <u>spectrum</u>: the $S.$ construction extends to simplicial categories by naturality and can be applied to $S.\mathcal{C}$ to produce a bisimplical category $S.S.\mathcal{C}$. Iterating this, this gives a spectrum

$$n \mapsto |wS.^{(n)}\mathcal{C}| := |w\underbrace{S.\ldots S}_{n}.\mathcal{C}|$$

whose structure maps are the maps $|w\mathcal{C}| \to \Omega|wS.\mathcal{C}|$ defined above.

This is a Ω-spectrum beyond the first term, and so we may equivalently define the K-theory space of $\mathcal{C}$ as

$$\Omega^{\infty}|wS.^{(\infty)}\mathcal{C}| = \operatorname{colim}_n \Omega^n|wS.^{(n)}\mathcal{C}|.$$

Selected references

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