Spectra and G–spectra

Eric Peterson

September 23, 2015

http://math.harvard.edu/~ecp/latex/talks/intro-to-spectra.pdf

Cell structures

Definition

A *cell structure* on a pointed space X is an inductive presentation by iteratively attaching n–disks:

$$\begin{tikzcd} \bigvee S^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d, dotted] \\ \bigvee D^n \arrow[r, dotted] & X^{(n)}. \end{tikzcd}$$

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Theorem ("Stability")

$$H^n(X; A) \cong H^{n+1}(\Sigma X; A),$$
$$\Sigma H^*(X; A) \cong H^*(\Sigma X; A).$$

Suspension: Freudenthal's theorem

Calculation: π_* of a suspension

n	1	2	3	4	5	6	7	8	...
$\pi_n S^1$	$\mathbb{Z}$	0	0	0	0	0	0	0	...
$\pi_{n+1} \Sigma S^1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	...

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Theorem (Freudenthal)

- X: s–connected space ($\pi_{*\leq s} X = 0$)
- Y: t–connective space ($\pi_{*<t} Y = 0$)

Then

$$F(Y, X) \to F(\Sigma Y, \Sigma X)$$

is a $(2s - t)$–equivalence.

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Corollary

The 2 matters: $\pi_n F(\Sigma^m Y, \Sigma^m X)$ is independent of $m \gg n$.

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Call "$\Sigma^\infty X$" the suspension spectrum of X.

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$$\begin{tikzcd} & \mathrm{SuspensionSpectra} \arrow[dr, "\Omega^\infty"] & \\ \mathrm{Spaces} \arrow[ur, "\Sigma^\infty"] \arrow[rr, "Q"] & & \mathrm{Spaces}. \end{tikzcd}$$

The Eilenberg–Mac Lane spectrum

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$$[\Sigma Y, Q\Sigma^* X] = [Y, \Omega Q \Sigma^* X] = [Y, Q\Sigma^{*-1} X.]$$

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Bad news: not all stable invariants

$K(A, n)$ represents a stable functor too:

$$[Y, K(A, n)] = H^n(Y; A).$$

$K(A, n+1)$ deloops $K(A, n)$, but $K(A, n) \neq QX$ for any X.

The Eilenberg–Mac Lane spectrum

$$\pi_* \Sigma^\infty K(A, n) = \begin{cases} A & \text{if } * = n, \\ 0 & \text{if } * \leq 2n,\ * \neq n, \\ \text{mystery groups} & \text{if } * > 2n. \end{cases}$$

So, "$\operatorname{colim}_n \Sigma^{-n} \Sigma^\infty K(A, n)$" has the right homotopy groups.

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Definition (Boardman, more or less)

A *spectrum* is an ind-diagram of things like $\Sigma^{-n}\Sigma^\infty X$.
The Eilenberg–Mac Lane spectrum is presented by the ind-system

$$HA := \{\Sigma^{-n}\Sigma^\infty K(A, n)\}.$$

Smash product, representability

Theorem (Boardman)

The smash product $\wedge$ lifts from spaces to spectra:

$$\{\Sigma^{n_\alpha}\Sigma^\infty X_\alpha\} \wedge \{\Sigma^{m_\beta}\Sigma^\infty Y_\beta\} =: \{\Sigma^{n_\alpha+m_\beta}\Sigma^\infty(X_\alpha \wedge Y_\beta)\}.$$

It has an adjoint, the function spectrum: $[Z \wedge Y, X] \simeq [Z, X^Y]$.

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Theorem

$$X \mapsto \pi_*(HA \wedge \Sigma^\infty X) \quad \text{and} \quad X \mapsto \pi_{-*}(HA^{\Sigma^\infty X})$$

satisfy the axioms of ordinary (co)homology with A coefficients.

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Theorem (Brown, Atiyah)

For $E_*(-)$ and $E^*(-)$ generalized (co)homology theories, there is a spectrum E such that

$$\widetilde{E}_*(X) \cong \pi_*(E \wedge \Sigma^\infty X) \quad \text{and} \quad \widetilde{E}^*(X) = \pi_{-*}(E^{\Sigma^\infty X}).$$

Gluing homology theories

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Example: Quotient sequences

The quotient sequence $\mathbb{S} \xrightarrow{2} \mathbb{S} \to \mathbb{S}/2$ induces an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2\mathbb{S} & \longrightarrow & \pi_2\mathbb{S}/2 & \longrightarrow & \pi_1\mathbb{S} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0. \end{array}$$

Spectra guarantee that these problems have consistent solutions.

Ring spectra

Ring-valued cohomology theories induce multiplication maps on representing spectra. In homotopy theory, associativity is a structure rather than a property:

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Leads to quasicategories and A_∞–rings ("coherently associative"). It pays off: A_∞–rings have a good theory of modules, ...

Generalized cellular chains

Theorem (Atiyah–Hirzebruch)

Let E be a generalized homology theory and X a cellular space.

$$E^1_{p,q} = C^{\mathrm{cell}}_p(X; E_q) \Rightarrow E_{p+q}X.$$

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For $E = HA$, there is a sense in which $HA \wedge \Sigma^\infty X \simeq C_*(X; A)$.

$$E \wedge \Sigma^\infty X \leftrightsquigarrow \text{“}E\text{–chains on }X\text{”}.$$

In good cases, this is “base change” from $\mathbb{S}$ to E.

Intermission

Basics of equivariant homotopy theory

Where spaces had points, G–spaces have orbits:

$$G/H \xrightarrow{\text{equivariant}} X.$$

Different choices of $H \leq G$ stratify the space:

$$G/H \mapsto F_G(G/H_+, X) = X^H.$$

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Definitions

$$\underline{\pi}_n(X) : G/H \mapsto [G/H_+ \wedge S^n, X]_G = \pi_n X^H$$

A weak equivalence of G–spaces is a G–map which is a $\underline{\pi}_*$–iso. That is, for each H

$$\pi_* X^H \xrightarrow{\simeq} \pi_* Y^H.$$

Equivariant obstruction theory

Definition

A *G–cell structure* on a pointed G–space X is a presentation by iteratively attaching n–disks of the form $G/H_+ \wedge D^n$ along images of $G/H_+ \wedge S^{n-1}$.

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$$\underline{C}^n(X; \underline{M}) : G/H \mapsto \mathrm{Hom}(H_n((X^H)^n, (X^H)^{n-1}), \underline{M}(G/H)).$$

Satisfies the "obvious" Eilenberg–Steenrod axioms.

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Warning

This works, but it's not great. No Poincaré duality, for instance.

Bredon cohomology

Question

Sphere could also mean $S^V := V^+$ for V a G–representation.
Spheres grade cohomology theories: $S^n \leftrightsquigarrow H^n$.
When can a representation be put in for $*$ in $\underline{H}^*(X; \underline{M})$?

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Answer

Exactly when $\underline{M}$ is a *Mackey functor*:

for any G–map	$f : G/H \to G/K$
we choose a "transfer map"	$t(f) : \underline{M}(G/H) \to \underline{M}(G/K)$

satisfying a "double coset formula" reminiscent of character theory.
(The definition is set up so that $G/H \mapsto \mathrm{Rep}(H)$ fits.)

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These are great: Poincaré duality and everything else you could hope for.

Definitions, redux

Define suspension G–spectra by

$$[\Sigma^\infty_G Y, \Sigma^\infty_G X]_G = [Y, \operatorname{colim}_V \Omega^V \Sigma^V X]_G.$$

Equivariant Freudenthal says this colimit is degenerate. G–spectra are ind-systems of S^V–desuspensions of suspension G–spectra.

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Theorem, redux

For $\underline{E}_\star(-)$ and $\underline{E}^\star(-)$ generalized Bredon (co)homology theories (i.e., $\star = V$ is allowed), there is a G–spectrum E such that

$$\underline{\widetilde{E}}_\star(X) \cong \underline{\pi}_\star(E \wedge \Sigma^\infty_G X) \quad \text{and} \quad \underline{\widetilde{E}}^\star(X) = \underline{\pi}_{-\star}(E^{\Sigma^\infty_G X}).$$

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Theorem, redux

For any Mackey functor $\underline{M}$, there is an Eilenberg–Mac Lane G–spectrum $H\underline{M}$ presenting Bredon cohomology $\underline{H}^\star(-; M)$.

Stable fixed points

We built G–spaces so that they carry fixed point data: "X^H". This splits into three notions of fixed points for G–spectra:

- Geometric:
$$\Phi^H(\Sigma_G^\infty X) = \Sigma^\infty X^H,$$
$$\Phi^H(\operatorname{colim}_\alpha\{X_\alpha\}) = \operatorname{colim}_\alpha\{\Phi^H X_\alpha\},$$
$$\Phi^H(X \wedge Y) = \Phi^H(X) \wedge \Phi^H(Y).$$
- Categorical: $[E, X^H] = [E, X]_H$, $\quad \underline{\pi}_n(X) : G/H \mapsto \pi_n X^H$.
- Homotopical: $X^{hH} = F_H(EH_+, X)$.

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There is a map of fiber sequences

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & X^H \arrow[r] \arrow[d] & \Phi^H(X) \arrow[d] \\ X_{hH} \arrow[r] & X^{hH} \arrow[r] & X^{tH}. \end{tikzcd}$$

Generally, this is the best we can say.

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X_{hH} if $H = C_p$

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"homotopy orbits" "Tate spectrum"

Generally, this is the best we can say.

Advertisement: $KO \simeq KU^{hC_2}$

KU exists as a C_2–spectrum with action by complex conjugation.

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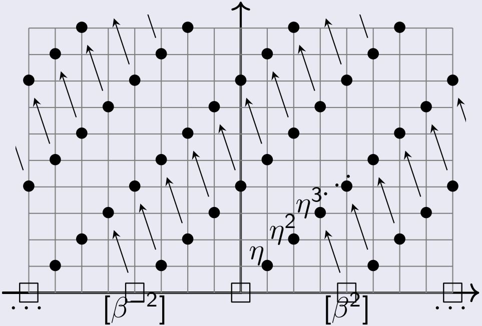
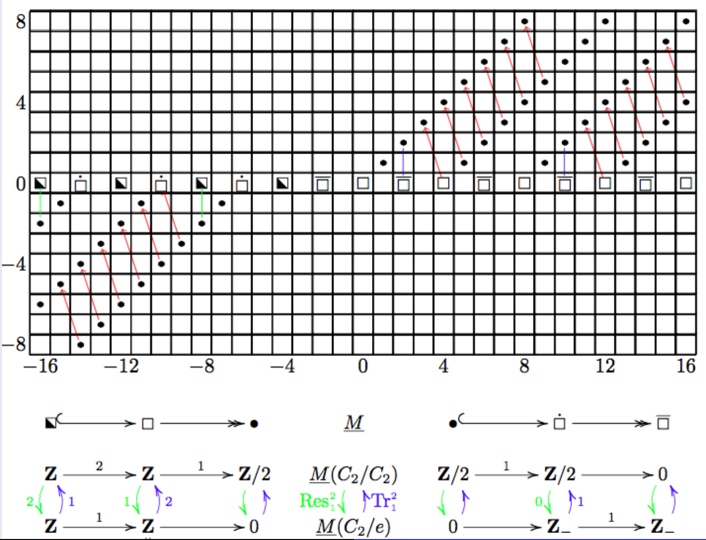
Homotopy fixed point spectral sequence: $H^*_{gp}(C_2; \pi_* KU) \Rightarrow \pi_* KO$



Slice spectral sequence (Dugger)

You can also get the homotopy groups as Mackey functors.



Advertisement: *THH*

Theorem (McCarthy)

Let $f : R \to S$ be a surjection of rings with nilpotent kernel. Then there is a pullback square

$$\begin{array}{ccc} K(R)^\wedge_p & \xrightarrow{\text{``trace''}} & TC(R)^\wedge_p \\ \downarrow & & \downarrow \\ K(S)^\wedge_p & \xrightarrow{\text{``trace''}} & TC(S)^\wedge_p, \end{array}$$

where

$$TC(R) = \operatorname{fib}\left(\lim_{n\to\infty} THH(R)^{C_{p^n}} \xrightarrow{R-\mathrm{id}} \lim_{n\to\infty} THH(R)^{C_{p^n}} \right)$$

and *THH* is the subject of this (and the Thursday) seminar.

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Theorem (McCarthy)

Let $f : R \to S$ be a surjection of rings with nilpotent kernel. Then there is a pullback square

$$\begin{array}{ccc} K(R)^\wedge_p & \xrightarrow{\text{“trace”}} & TC(R)^\wedge_p \\ \downarrow & & \downarrow \\ K(S)^\wedge_p & \xrightarrow{\text{“trace”}} & TC(S)^\wedge_p, \end{array}$$

where

$$TC(R) = \operatorname{fib}\left(\lim_{n\to\infty} THH(R)^{C_{p^n}} \xrightarrow{R-\mathrm{id}} \lim_{n\to\infty} THH(R)^{C_{p^n}} \right)$$

and *THH* is the subject of this (and the Thursday) seminar.

There are lots of theorems along these lines, relating equivariant structure on *THH* to sundry things in algebraic K–theory.