1. Emanuele Dotto, 09/16

1.1. **Why we care.** Start with a ring A and look at the category P_A of finitely generated projective moduels. Then $(|P_A|, \oplus)$ (where $|P_A|$ means classifying space of the category) is a coherently commutative (i.e. E_∞) monoid. Then $K(A)$ is the group completion of this. There are many explicit constructions. Here's one: $K(A) = BGL(A)^+ \times \mathbb{Z}$.

This can be extended to Waldhausen categories (categories with a notion of weak equivalence and cofibrations). Because of the group completion above, $\pi_0 K(A)$ is the (algebraic) group completion of the monoid of finitely-generated projective A-modules (up to isomorphism), under direct sums. The idea is that it's sometimes easier to understand the group completion of something than the starting monoid.

Examples 1.1.

- If you take K-theory of the category of pointed sets, you get the sphere spectrum. (This is Barratt-Priddy-Quillen.)
- If you take K-theory of the category of vector bundles over X, you get topological K-theory $K(X)$.
- "$K(\mathbb{Z})$ knows about arithmetic" – it's related to conjectures in number theory.
- $K(\Sigma^\infty \Omega M_+) \simeq \Sigma^\infty M_+ \vee Wh(M)$ (where the latter is the Whitehead spectrum, a geometric gadget related to the stable h-cobordisms of M). This equivalence is a big theorem of Waldhausen-Jahren-Rognes. Weiss-Williams showed that you can use this to get information about diffeomorphism groups.

One of the first computations was the algebraic K-theory of finite fields.

Theorem 1.2 (Quillen)**.**

$$\pi_i K(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/(q^j - 1)\mathbb{Z} & i = 2j - 1 \\ 0 & \textit{else.} \end{cases}$$

Proof idea. The main point is to define a homology isomorphism $\theta : BGL(\mathbb{F}_q) \to \mathrm{hofib}(\psi^q - 1)$ such that $\ker \pi_i \theta = E(\mathbb{F}_q)$, elementary matrices. (Here ψ is a map $BU \to BU$.) □

Another important K-theory is $K(\mathbb{Z})$; it's not complete, because of number theory. This calculation spans over 50 years; information is very scattered, but Weibel has a book about it, and another good reference is lecture notes by Soulé.

$K_0(\mathbb{Z})$ is easy: a finitely-generated projective $\mathbb{Z}$-module is just a free $\mathbb{Z}$-module, and those are classified by dimension.

- $K_0(\mathbb{Z}) = \mathbb{Z}$
- $K_1(\mathbb{Z}) = \mathbb{Z}/2$ (?)
- $K_2(\mathbb{Z}) = \mathbb{Z}/2$

- $K_3(\mathbb{Z}) = \mathbb{Z}/48$ (much harder, due to Lee-Szczarba)
- $K_4(\mathbb{Z}) = 0$ (Rognes, about 2000, very hard)
- $K_5(\mathbb{Z}) = \mathbb{Z}$ (Soulé and others)

Conjecture 1.3. $K_{4m}(\mathbb{Z}) = 0$ for $m \geq 1$ (but K_8 is not known).

Theorem 1.4 (Kurihara, $\sim$2000)**.** *The above conjecture is equivalent to the Vandiver conjecture.*

Everything else is known.

Theorem 1.5 (Quillen)**.** $K_n(\mathbb{Z})$ *is finitely generated.*

Theorem 1.6 (Borel)**.**

$$K_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \textit{finite} & n \equiv 1 \pmod 4 \\ \textit{finite} & \textit{otherwise.} \end{cases}$$

Proof. For $N \gg q$:

- $H^q(SL_N(\mathbb{Z}); \mathbb{R}) \cong H^q_{\text{cts}}(SL_N(\mathbb{R}); \mathbb{R})$.
- $H^*_{\text{cts}}(SL_N(\mathbb{R})) = \Lambda(e_5, e_9, \ldots, e_{4\lfloor \frac{N-1}{2} \rfloor})$
- $K_q(\mathbb{Z}) \otimes \mathbb{R} \cong \pi_q BSL(\mathbb{Z})^+ \otimes \mathbb{R}$

$\square$

The rest of the calculation of $K(\mathbb{Z})$ follows from:

Theorem 1.7 (Soulé/Dwyer-Friedlander, Voevodsky-Rost)**.**

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_p \xrightarrow{\cong} H^1_{\acute{e}t}(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(n))$$
$$K_{2n-2}(\mathbb{Z}) \otimes \mathbb{Z}_p \xrightarrow{\cong} H^2_{\acute{e}t}(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(n))$$

for $n \geq 2$ and p an odd prime.

A slightly more general series of techniques is given by trace methods. The idea: map K-theory to a more treatable object: THH or TC. This construction is due to Böbstedt-Hshiang-Madsen. These methods were introduced for figuring out the K-theoretic Novikov conjecture, which says that the assembly map

$$BG_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[G])$$

is rationally split injective assuming some finiteness conditions on G. (This is due to BHM.)

Waldhausen showed that there is a map $\Sigma^\infty_+ X \to K(\Sigma^\infty \Omega X_+) =: A(X)$ that has a splitting.

If $I \subset A$ is an ideal, let $\widetilde{K}(A) := \mathrm{hofib}(K(A) \to K(A/I))$. For example, if $I = (x)$ and $A = k[x]/x^n$ for a perfect field, then

$$\widetilde{K}(k[x]/x^n) = \begin{cases} W_{nk}(k)/V_n(W_q(h)) & i = 2q-1 \\ 0 & \text{otherwise.} \end{cases}$$

(Here V_n is the Verschiebung and W is Witt vectors.) This is due to Hesselholt-Madsen.

You can get information about $K(S)$ in terms of $K(\mathbb{Z})$.

1.2. **The trace.** Topological Hochschild homology is:

$$THH(A) = A \widehat{\otimes} S^1$$

Think of this as configurations of points in S^1 with labels in A. If A is not commutative you can still do this. There is an S^1 action on $THH(A)$. You can talk about the fixed points of this action. Then topological cyclic homology $TC(A)$ is defined using those fised points.

The trace is a map $K(A) \to TC(A)$. You can extend the THH construction to categories. The map $K(A) \to TC(A)$ roughly takes $c \mapsto$ the configuration of points labelled by $\mathbb{1}_c$.

Theorem 1.8 (Dundas-Goodwillie-McCarthy)**.** *If $B \to A$ is surjective and nilpotent fiber in π_0 then*

$$\begin{tikzcd} K(B) \arrow[r] \arrow[d] & TC(B) \arrow[d] \\ K(A) \arrow[r] & TC(A) \end{tikzcd}$$

is homotopy cartesian (after p-completion).

One example of such a map is the Hurewicz map, which gives rise to a cartesian square

$$\begin{tikzcd} K(S) \arrow[r] \arrow[d] & TC(S) \arrow[d] \\ K(\mathbb{Z}) \arrow[r] & TC(\mathbb{Z}) \end{tikzcd}$$

$K(\mathbb{Z})$ is more-or-less understood. $TC(S)$ is known, by work of BHM, and $TC(\mathbb{Z})$ is known, due to Böbstedt-Madsen and Rognes. Blumberg-Mandell used this to show that $K(S)^\wedge_p \to TC(S)^\wedge_p \times K(\mathbb{Z})^\wedge_p$ is split injective (in π_*) for all odd p.

The proof was scattered over 25 years. It's the main application of Goodwillie calculus.

Proof idea.

- Dundas: reduce to simplicial rings.
- Goodwillie: reduce to rings. If you have a simplicial ring, you can either take the K-theory in the category of simplicial rings, or you can take K-theory levelwise (much easier). In general, these are not at all the same thing, but it works out for the homotopy fiber $K(B) \to K(A)$.
- Reduce inductively to split square-zero extensions $A \ltimes M \to A$.

- Extend $K(A \rtimes M)$ to a functor: make the Dold-Thom construction $M(-)$ (bimodule of configurations of points in your space with labels in M). So you get a functor $K(A \rtimes M(-)) : \mathrm{Top}_* \to \mathrm{Sp}$. Do the same for TC.
- Use calculus: show that $D_1 K \simeq D_1 TC$, by showing $D_1 K \simeq THH(A, M(S^1 \wedge -)) \simeq D_1 TC$ (actually for the last equivalence you have to p-complete. This is due to Dundas-McCarthy and Hesselholt). By calculus, $\widetilde{K} \simeq \widetilde{TC}$.

$\square$