SUVITOP 08/10/14

Adams spectral sequence

Motivation: Studying stable htpy classes $[X,Y]$

$d : [X,Y] \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$ $\quad \mathcal{A}$ = Steenrod algebra (mod p)

$\mathrm{Ext}^1_{\mathcal{A}}$ = measures whether short exact seq. split

$f : S^{2n-1} \to S^n \quad \Rightarrow \quad 0 \to H^*(S^{2n}) \to H^*(C_f) \to H^*(S^n) \to 0$

Q: Does $\mathcal{A}$ act trivially on $H^*(C_f)$

Thm (Adams '60): Sq^n is nontrivial iff $n = 1, 2, 4, 8$.

(b) Serre's method of computing htpy grps

$G = \pi_n X$, X $(n-1)$-connected

$X' \to X \to K(G,n)$ + Serre SS

Adams' idea: let's kill all the htpy grps detected by mod p cohomology.

Thm (Adams): Let X, Y be connective spectra of finite type. There exists a SS w/ $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X), H^*(Y))$ converging to $[Y, X^\wedge_p]$.

① Construction of the ASS

$H = H\mathbb{F}_p$ ring spectrum

Prop: $\mathcal{A} \simeq H^* H\mathbb{F}_p = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$

Pf: Computation w/ SSS. □

Def: A generalized Eilenberg-Mac Lane spectrum is a finite wedge of suspensions of $H\mathbb{F}_p$.

For any finite type spectrum Y $\quad d : [Y, K] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(H^*Y, H^*K)$ if K is a gen EM space.

$\forall\ Y \xrightarrow{f} X \xrightarrow{j} K$ $\quad K$ gen. EM w/ onto j^* $\quad jf = 0 \iff f^* = 0$

Adams resolution

Def: Let X be a spectrum. An <u>Adams resolution</u> of X is

$$\begin{tikzcd} X \arrow[d, "j_0"] & X^1 \arrow[l] \arrow[d, "j_1"] & X^2 \arrow[l] \arrow[d, "j_2"] & \cdots \arrow[l] \\ K^0 \arrow[r] & K^1 \arrow[r] & K^2 \arrow[r] & \end{tikzcd}$$

such that
- $X^{n+1} \to X^n \to K^n$ are cofiber sequences
- K^n are gen. EM spectra
- $j_n^* : H^* K^n \to H^* X^n$ is onto.

Lemma: Adams resolution exist and can be chosen functially in X.

Prf: Suppose we've up to X^s, let $K^s = H \wedge X^s$, $j^s : \underset{X^s}{\overset{\shortparallel}{S \wedge X^s}} \to H \wedge S^s$ and let X^{s+1} be the fiber j^s.

By Künneth formula $H^* K^s = A^* \otimes_{\mathbb{F}_p} H^* X^s$, so it is free. $\square$

Prp: $P^s = H^*(\Sigma^s K^s)$ $\quad \delta^s = (\Sigma^{s-1} \delta_{s-1})^* \circ (\Sigma^s j^s)^*$

Then

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \to H^*(X) \to 0$$

is a free resolution of $H^*(X)$ as graded A_*-modules.

Prf: Look at the diagram. $\square$

Exact couples & E_1-page

Applying $[\Sigma^* Y, -]$ to the Adams resolution we get

$$\begin{tikzcd} {[\Sigma^* Y, X]} \arrow[d] & {[\Sigma^* Y, X^1]} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ {[\Sigma^* Y, K^0]} \arrow[ur] & {[\Sigma^* Y, K^1]} \arrow[ur] & \end{tikzcd}$$

This is an exact couple giving a spectral sequence w/ E_1 page

$$E_1^{s,t} = [\Sigma^t Y, K^s]$$

<u>Prop</u>: $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y))$

<u>Pf</u>: $[\Sigma^t Y, K^s] = \mathrm{Hom}_{\mathcal{A}}(H^* K^s, H^{*+t} Y)$

& the first differential does what you expect. $\square$

<u>Adams filtration & convergence</u>

<u>Def</u>: Given an Adams resolution the Adams filtration is

$$F^s = \mathrm{Im}([Y, X^s]_* \to [Y, X]_*) \subseteq [Y, X]_*.$$

<u>Lemma</u>: The Adams filtration does not depend on the Adams resolution.

In fact $f \in F^s \iff f$ factors as composition of $(s-1)$ maps which are 0 in cohomology.

<u>Pf</u>: $\Rightarrow$) $X^s \to X^{s-1}$ is 0 in cohomology.

$\Leftarrow$)

$$\begin{tikzcd}
\Sigma^t Y \arrow[r] & U_s \arrow[r] \arrow[d] & X^{s-1} \\
& U_{s-1} \arrow[r] \arrow[d] & X^{s-1} \\
& \vdots \arrow[d] & \vdots \arrow[d] \\
& U_0 \arrow[r, equal] & X \arrow[r] & K_0
\end{tikzcd}$$

And I can inductively lift $\square$

The spectral sequence convergence depends on $\varprojlim X^s$ which is 0 after p-completion. $\square$

<u>Multiplicative structure</u>

<u>Def</u>: A pairing of SS is $\{\phi_r : E_r' \otimes E_r'' \to E_r\}_{r \geq 0}$ s.t.

$$d^r(\phi_r(x \otimes y)) = \phi_r(d^r x \otimes y) + (-1)^{|x|} \phi_r(x \otimes d^r y)$$

The ASS has <u>two pairings</u>

- one $\otimes$ $\rightsquigarrow$ smash product
- composition (Yoneda product)

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2. Examples

(a) $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$ Minimal resolution

Take $P_0 = \mathcal{A} \longrightarrow \mathbb{F}_2$

P_0 : $|\, g_{0,0} \,|\, Sq^1 g_{0,0} \,|\, Sq^2 g_{0,0} \;\; Sq^3 g_{0,0}$

P_1 : $|\, g_{0,1} \,|\, Sq^1 g_{0,1}$

Etc ... (complicated) In the first line there's an element $h_i = g_{1,2^i}$

<u>Thm (Adams '66)</u>: $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) = 0$ for $0 < t-s < 2s-\varepsilon$

for $\varepsilon = \begin{cases} 1 & s \equiv 0,1 \ (4) \\ 2 & s \equiv 2 \ (4) \\ 3 & s \equiv 3 \ (4) \end{cases}$

$\Rightarrow E_2^{s,t} = E_\infty^{s,t}$ $t \leq 11$

<u>Thm</u>: $\pi_i \mathbb{S}_2^\wedge = \begin{cases} \mathbb{Z}/2 & i = 1,2,6 \\ \mathbb{Z}/8 & i = 3 \\ 0 & i = 4,5 \\ \mathbb{Z}/16 & i = 7 \end{cases}$

<u>The Adams-Novikov SS</u>

<u>IDEA</u>: Nothing here was special about H. We can use any flat spectrum E