Pre-Talbot ANSS

Michael Andrews
Department of Mathematics
MIT

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1 The image of J

We have an unbased map

$$SO = \mathrm{colim}_n SO(n) \longrightarrow \mathrm{colim}_n \Omega^n S^n = QS^0$$

mapping into the 1-component of QS^0. The map induced by

$$SO \longrightarrow QS^0 \xrightarrow{-1} QS^0$$

on homotopy groups is called the J-homomorphism.

From now on let's work at an odd prime p where the image of J has a relatively simple description:

$$(\mathrm{Im}J)_n = \begin{cases} 0 & \text{if } n+1 \not\equiv 0 \pmod{2(p-1)} \\ \mathbb{Z}/p^{k+1} & \text{if } n+1 = 2(p-1)sp^k,\ s \nmid p \end{cases}$$

Let V be the Moore space S^0/p. We have a self map

$$v_1 : \Sigma^{2(p-1)} V \longrightarrow V.$$

The element

$$\alpha_{sp^k/1} : S^{2(p-1)sp^k} \longrightarrow \Sigma^{2(p-1)sp^k} V \xrightarrow{v_1^{sp^k}} V \longrightarrow S^1$$

is of order p and lies in the image of J. Trickery with vanishing lines shows that these elements lie in highest possible Adams filtration and there is a beautiful story concerning how the image of J shows up in the classical Adams SS. We will turn our attention to how the image of J is detected in the Adams-Novikov SS.

2 A guess as to how $\alpha_{sp^k/1}$ is detected

Recall that we are working at an odd prime p and that we have constructed a spectrum BP which is a retract of $MU_{(p)}$. We will see that $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$ and so the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow V$$

gives rise the a SES of BP_*BP-comodules

$$0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*/p \longrightarrow 0.$$

Assuming that the naming convention is sensible we might guess that $v_1 \in BP_*/p$ is a BP_*BP-comodule primitive defining an element $v_1 \in \mathrm{Cotor}^0_{BP_*BP}(BP_*/p, BP_*)$ detecting an element which we also call v_1:

$$S^{2(p-1)} \longrightarrow \Sigma^{2(p-1)}V \xrightarrow{v_1} V.$$

The map $V \longrightarrow S^1$ is BP-null but it gives rise to a connecting homomorphism; we should expect that the element $\alpha_{sp^k/1}$ is detected by the image of $v_1^{sp^k}$ under

$$\delta : \mathrm{Cotor}^0_{BP_*BP}(BP_*/p, BP_*) \longrightarrow \mathrm{Cotor}^1_{BP_*BP}(BP_*, BP_*).$$

Explictly we are guessing that $\alpha_{sp^k/1}$ is detected by

$$\left[\frac{dv_1^{sp^k}}{p}\right] \in \mathrm{Cotor}^1_{BP_*BP}(BP_*, BP_*),$$

where d denotes the coboundary map in the cobar construction $\Omega(BP_*, BP_*BP)$. We know there exists an element $\alpha_{sp^k/k+1}$ such that $p^k\alpha_{sp^k/k+1} = \alpha_{sp^k/1}$. We might guess that this is detected by an element

$$\left[\frac{dv_1^{sp^k}}{p^{k+1}}\right] \in \mathrm{Cotor}^1_{BP_*BP}(BP_*, BP_*).$$

So far it unclear that these elements are well-defined since we don't know how to compute d.

3 Hazewinkel's generators for (BP_*, BP_*BP)

Theorem (Hazewinkel): The Hurewicz homomorphim $\pi_*(BP) \longrightarrow H_*(BP)$ is an injection. There exist generators $v_1, v_2, v_3, \ldots$ for

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

and $m_1, m_2, m_3, \ldots$ for

$$H_*(BP) = \mathbb{Z}_{(p)}[m_1, m_2, m_3, \ldots]$$

such that $|v_i| = |m_i| = 2(p^i - 1)$ and under the inclusion we have

$$pm_{k+1} = \sum_{i+j=k} m_i v_{j+1}^{p^i}.$$

where by convention $m_0 = 1$.

Moreover: Using the AHSS we see that the Hurewicz homomorphism $\pi_*(BP \wedge BP) \longrightarrow H_*(BP \wedge BP)$ takes the form

$$\pi_*(BP)[t_1, t_2, t_3, \ldots] \longrightarrow H_*(BP)[t_1, t_2, t_3, \ldots]$$

where $|t_i| = 2(p^i - 1)$. In particular, $\pi_*(BP \wedge BP)$ and $H_*(BP \wedge BP)$ are flat (as left-modules) over $\pi_*(BP)$ and $H_*(BP)$, respectively. By convention, $t_0 = 1$.

Important: The Hopf algebroid $(\pi_*(BP), \pi_*(BP \wedge BP))$ over $\mathbb{Z}_{(p)}$ is determined by the Hopf algebroid $(H_*(BP), H_*(BP \wedge BP))$ over $\mathbb{Z}_{(p)}$. The $H_*(BP)$-bimodule structure of $H_*(BP \wedge BP)$ is given by:

1. The left $H_*(BP)$-module structure on $H_*(BP \wedge BP)$ is the obvious one on $H_*(BP)[t_1, t_2, t_3, \ldots]$, i.e. $\eta_L(m_k) = m_k$.

2. The right $H_*(BP)$-module structure on $H_*(BP \wedge BP)$ is described by $\eta_R(m_k) = \sum_{i+j=k} m_i t_j^{p^i}$.

Note: $\eta_R(m_1) = m_1 t_0^p + m_0 t_1 = m_1 + t_1$ so that $\eta_R(v_1) = \eta_R(pm_1) = pm_1 + pt_1 = v_1 + pt_1$. Also, we have a commuting diagram

$$\begin{tikzcd}
H_*(BP) \otimes H_*(BP) \arrow[r, "(\eta_L, \eta_R)"] \arrow[d, "\eta_L \otimes \eta_R"'] & H_*(BP \wedge BP) \arrow[d, "\Delta"] \\
H_*(BP \wedge BP) \otimes H_*(BP \wedge BP) \arrow[r] & H_*(BP \wedge BP) \otimes_{H_*(BP)} H_*(BP \wedge BP)
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \otimes m_1 - m_1 \otimes 1 \arrow[r, mapsto] \arrow[d, mapsto] & (m_1 + t_1) - m_1 = t_1 \arrow[d, mapsto] \\
1 \otimes (m_1 + t_1) - m_1 \otimes 1 \arrow[r, mapsto] & 1 \otimes m_1 + 1 \otimes t_1 - m_1 \otimes 1 = t_1 \otimes 1 + 1 \otimes t_1
\end{tikzcd}$$

so that $t_1 \in BP_*(BP)$ is a coalgebra primitive.

4 $dv_1^{sp^k}/p^{k+1}$

Recall that in the unreduced cobar construction we have

$$\begin{tikzcd}
\Omega^0(BP_*; BP_*BP) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(BP_*; BP_*BP) \arrow[d, equal] \\
BP_* \arrow[r, "\psi_R - (-\otimes 1)"] \arrow[d, equal] & BP_* \otimes_{BP_*} BP_*BP \arrow[d, equal] \\
BP_* \arrow[r, "\eta_R - \eta_L"] & BP_*BP
\end{tikzcd}$$

Recall that $\eta_L(v_1) = v_1$ and $\eta_R(v_1) = v_1 + pt_1$. Thus in $\Omega(BP_*; BP_*BP)$

$$d(v_1^{sp^k}) = \eta_R(v_1^{sp^k}) - \eta_L(v_1^{sp^k}) = \sum_{\substack{i+j=sp^k \\ i>0}} p^i \binom{sp^k}{i} v_1^j [t_1^i].$$

Whenever $i > 0$, $p^{k+1} | p^i \binom{sp^k}{i}$ and so $\frac{dv_1^{sp^k}}{p^{k+1}}$ is a well-defined cocyle in $\Omega^1(BP_*, BP_*BP)$, determining an element

$$\overline{\alpha}_{sp^k/k+1} \in \mathrm{Cotor}^1_{BP_*BP}(BP_*, BP_*).$$

5 The chromatic spectral sequence

We wish to show the elements $\overline{\alpha}_{sp^k/k+1}$ generate the 1-line of the ANSS. For this we'll use the chromatic SS. We consider the following in which each "down, up-right" is a SES.

$$\begin{tikzcd}
BP_* \arrow[d] & & BP_*/p^\infty \arrow[d] & & BP_*/(p^\infty, v_1^\infty) \arrow[d] & & \cdots \\
p^{-1}BP_* \arrow[urr] & & v_1^{-1}BP_*/p^\infty \arrow[urr] & & v_2^{-1}BP_*/(p^\infty, v_1^\infty) \arrow[urr] & &
\end{tikzcd}$$

[It is clear that we can construct the diagram above as BP_*-modules. It is less clear that all the objects have the structure of a right BP_*BP-comodule. We'll assume this for now.]

We let $M_k = v_k^{-1}BP_*/(p^\infty, v_1, \ldots, v_{k-1}^\infty)$, where we now have the convention that $v_0 = p$. Applying $\mathrm{Cotor}_{BP_*BP}(-, BP_*)$ we get an exact couple.

$$\begin{tikzcd}
\cdots & \mathrm{Cotor}^{t,u}_{BP_*BP}(BP_*/(p^\infty, \ldots, v_{s-1}^\infty), BP_*) \arrow[l] \arrow[d] & \mathrm{Cotor}^{t-1,u}_{BP_*BP}(BP_*/(p^\infty, \ldots, v_s^\infty), BP_*) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \mathrm{Cotor}^{t,u}_{BP_*BP}(M_s, BP_*) \arrow[ur, dotted] & \mathrm{Cotor}^{t-1,u}_{BP_*BP}(M_{s+1}, BP_*) \arrow[ur, dotted] &
\end{tikzcd}$$

Here the dashed line raises the degree of s relative to what is indicated. We get a SS with

$$E_1^{s,t,u} = \mathrm{Cotor}^{t,u}_{BP_*BP}(M_s, BP_*) \overset{s}{\Longrightarrow} \mathrm{Cotor}^{s+t,u}_{BP_*BP}(BP_*, BP_*), \qquad d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,t-r+1,u}.$$

We wish to use this SS to compute the 0 and 1-line of the E_2-page of the ANSS. If M is a right BP_*BP-comodule write $H^s(M)$ for $\mathrm{Cotor}^{s,*}_{BP_*BP}(M, BP_*)$. The above SS takes the form

$$H^t(M_s) \overset{s}{\Longrightarrow} H^{s+t}(BP_*).$$

6 $H^*(M_0)$

$BP \wedge H\mathbb{Q}$ is a wedge of $H\mathbb{Q}$'s. Thus $H\mathbb{Q} \longrightarrow H\mathbb{Q} \wedge BP$ splits and by definition $H\mathbb{Q}$ is BP-injective. We conclude that

$$H^*(M_0) = \mathrm{Cotor}_{BP_*BP}(p^{-1}BP_*, BP_*) = \mathrm{Cotor}_{BP_*BP}(BP_*(S\mathbb{Q}), BP_*) = E_2(H\mathbb{Q}; BP)$$

is concentrated in degree 0 where it is equal to $\mathbb{Q}$ (and the u grading is 0).

7 Some useful results

7.1 Primitives

We have already used that the following diagrams commute

$$\begin{tikzcd}
BP_* \arrow[r, "x \longmapsto x \otimes 1"] \arrow[d, equal] & BP_* \otimes_{BP_*} BP_*BP \arrow[d, "\cong"] \\
BP_* \arrow[r, "\eta_L"] & BP_*BP
\end{tikzcd}
\qquad
\begin{tikzcd}
BP_* \arrow[r, "\psi_R"] \arrow[d, equal] & BP_* \otimes_{BP_*} BP_*BP \\
BP_* \arrow[r, "\eta_R"] & BP_*BP \arrow[u, "\cong"']
\end{tikzcd}$$

This tells us that BP_*BP-comodule primitives are elements of $\ker(\eta_R - \eta_L)$.

7.2 Landweber and Morava

$p \in BP_*$ is a BP_*BP-comodule primitive and so we can form the BP_*BP-comodule BP_*/p. We have seen that $(\eta_R - \eta_L)(v_1) = pt_1 \in BP_*BP$ and so $v_1 \in BP_*/p$ is a BP_*BP-comodule primitive. Part of a theorem due to Landweber and Morava says that

$$H^0(BP_*) = \mathbb{Z}_{(p)} \qquad \text{and} \qquad H^0(BP_*/p) = \mathbb{F}_p[v_1].$$

Thus

$$H^0(v_1^{-1}BP_*/p) = \mathbb{F}_p[v_1, v_1^{-1}].$$

7.3 $t_1 \in BP_*BP$

Recall that $t_1 \in BP_*BP$ is a coalgebra primitive. Thus, it defines an element $[t_1] \in H^1(M)$ for any right BP_*BP-comodule.

Theorem: $[t_1] \neq 0$ in $H^1(v_1^{-1}BP_*/p)$.

8 $H^0(M_1)$

The SES of BP_*BP-comodules

$$\begin{array}{ccccccccc} 0 \longrightarrow & v_1^{-1}BP_*/p & \xrightarrow{i} & v_1^{-1}BP_*/p^\infty & \xrightarrow{p} & v_1^{-1}BP_*/p^\infty & \longrightarrow 0 \\ & a & \longmapsto & \frac{a}{p} & & & \end{array}$$

gives a LES

$$0 \longrightarrow H^0(v_1^{-1}BP_*/p) \xrightarrow{i} H^0(M_1) \xrightarrow{p} H^0(M_1) \xrightarrow{\delta} H^1(v_1^{-1}BP_*/p) \longrightarrow \dots.$$

Given a nonzero element $x \in H^0(M_1)$ there exists a $k \geq 0$ such that $p^k x \neq 0$ and $p^{k+1}x = 0$. Then $p^k x$ is in the image of i. Thus we can calculate $H^0(M_1)$ by taking elements in the image of i and analysing how p-divisible they are; we know we cannot divide an element by p if the image under δ is nonzero. We need a lemma.

Lemma: $x = \frac{v_1^{sp^k}}{p^{k+1}} \in \Omega(M_1; BP_*BP)$ is a cocycle. If $p \nmid s$ then $\delta[x]$ is nonzero.

Proof: Note that in the statement of the lemma s is allowed to be negative. Recall that

$$d(v_1^{sp^k}) = \eta_R(v_1^{sp^k}) - \eta_L(v_1^{sp^k}) = \sum_{\substack{i+j=sp^k \\ i>0}} p^i \binom{sp^k}{i} v_1^j [t_1^i]$$

in $\Omega(BP_*; BP_*BP)$. $p^{k+1} | p^i \binom{sp^k}{i}$ for $i > 0$ and so in $\Omega(p^{-1}BP_*; BP_*BP)$, when we calculate

$$d\left(\frac{v_1^{sp^k}}{p^{k+1}}\right)$$

we actually obtain an element of $\Omega(BP_*, BP_*BP)$. [Dividing by p commutes with the differential since p is primitive in BP_*.] Thus, in $\Omega(BP_*/p^\infty; BP_*BP)$ and $\Omega(M_1; BP_*BP)$

$$d\left(\frac{v_1^{sp^k}}{p^{k+1}}\right) = 0.$$

Since we are assuming that p is odd, we have $p^{k+2}|p^i\binom{sp^k}{i}$, whenever $i \geq 2$. Thus when we calculate

$$d\left(\frac{v_1^{sp^k}}{p^{k+2}}\right)$$

in $\Omega(p^{-1}BP_*; BP_*BP)$ we obtain

$$\frac{sv_1^{sp^k-1}}{p}[t_1]$$

plus an element of $\Omega(BP_*; BP_*BP)$. We conclude that $\delta[x]$ is $sv_1^{sp^k-1}[t_1]$. v_1 acts isomorphically on $H^*(v_1^{-1}BP_*/p)$. Since $s \nmid p$ and $[t_1] \neq 0$ we conclude $\delta[x] \neq 0$. We've proved the lemma for $s > 0$.

Setting $s = 1$ in the first computation and replacing k by $(k+1)$ we see that $v_1^{p^{k+1}}$ is primitive mod p^{k+2}. Thus

$$v_1^{p^{k+1}} : BP_*\langle p^{-k-2}\rangle/BP_* \longrightarrow BP_*\langle p^{-k-2}\rangle/BP_*$$

is a comodule map. This means that multiplication by $v_1^{p^{k+1}} = v_1^{p\cdot p^k}$ commutes with d on

$$v_1^{sp^k}/p^{k+1} \qquad \text{and} \qquad v_1^{sp^k}/p^{k+2}.$$

Thus we have proved the lemma for all $s \neq 0$. We're then done by the following lemma.

Lemma: $\frac{1}{p^k} \in \Omega(M_1; BP_*BP)$ is a cocycle for all k.

Conclusion: We know that the image of i is

$$\left\{\left[\frac{v_1^{sp^k}}{p}\right] : s \in \mathbb{Z} - p\mathbb{Z},\ k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\left[\frac{1}{p}\right]\right\}$$

and we have analysed how p-divisible these elements are. Up to a little thinking we have computed $H^0(M_1)$: it is generated as an abelian group by

$$\left\{\left[\frac{v_1^{sp^k}}{p^{k+1}}\right] : s \in \mathbb{Z} - p\mathbb{Z},\ k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\left[\frac{1}{p^k}\right] : k \in \mathbb{N}\right\}.$$

9 Computing the relevant part of the CSS

We have a SES of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow p^{-1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{\infty} \longrightarrow 0.$$

Localising gives

$$0 \longrightarrow \mathbb{Z}_{(p)} \longrightarrow \mathbb{Q} \xrightarrow{q} \mathbb{Z}/p^{\infty} \longrightarrow 0.$$

Since we know that $H^0(BP_*) = \mathbb{Z}_{(p)}$ we have a commuting diagram

$$\begin{tikzcd} H^0(M_0) \arrow[r, "d_1"] \arrow[d, equal] & H^0(M_1) \\ \mathbb{Q} \arrow[r, "q"] & \mathbb{Z}/p^{\infty} \arrow[u, "u\text{-grading-equal-to-0-part included}"'] \end{tikzcd}$$

What about $d_1 : H^0(M_1) \longrightarrow H^0(M_2)$. This is just the composite

$$H^0(v_1^{-1}BP_*/p^{\infty}) \longrightarrow H^0(BP_*/(p^{\infty}, v_1^{\infty})) \longrightarrow H^0(v_2^{-1}BP_*/(p^{\infty}, v_1^{\infty})).$$

We see immediately that

$$\left\{ \left[\frac{v_1^{sp^k}}{p^{k+1}} \right] : s \in \mathbb{N} - p\mathbb{N},\ k \in \mathbb{N} \cup \{0\} \right\} \cup \left\{ \left[\frac{1}{p^k} \right] : k \in \mathbb{N} \right\}$$

is killed by d_1. On the other hand the image of

$$\left\{ \left[\frac{v_1^{sp^k}}{p^{k+1}} \right] : -s \in \mathbb{N} - p\mathbb{N},\ k \in \mathbb{N} \cup \{0\} \right\}$$

under d_1 is nonzero (since $H^0(M)$ consists of primitives, in particular, a submodule of M, there is no quotienting to worry about). Thus

$$E_{\infty}^{1,0,*} = \left\{ \left[\frac{v_1^{sp^k}}{p^{k+1}} \right] : s \in \mathbb{N} - p\mathbb{N},\ k \in \mathbb{N} \cup \{0\} \right\},$$

which looks suspiciously familiar (see sections 1 and 2). The element in $H^1(BP_*)$ detected by

$$\left[\frac{v_1^{sp^k}}{p^{k+1}} \right]$$

is represented by

$$\frac{dv_1^{sp^k}}{p^{k+1}} = sv_1^{sp^k - 1}[t_1] + \sum_{\substack{i+j=sp^k \\ i>1}} p^{i-k-1} \binom{sp^k}{i} v_1^j [t_1^i] \in \Omega^1(BP_*; BP_*BP)$$

and so it is our friend $\overline{\alpha}_{sp^k/k+1}$.