On the topological Hochschild homology of bu. I.

J.E. McClure and R.E. Staffeldt
Department of Mathematics
University of Kentucky
Lexington, KY 40506
USA

Preliminary Version, August 10, 1989

1 Introduction

The purpose of this paper and its sequel is to determine the homotopy groups of the spectrum $THH(\ell)$. Here p is an odd prime, ℓ is the Adams summand of p-local connective K-theory (see for example [25]) and THH is the topological Hochschild homology construction introduced by Bökstedt in [3]. In the present paper we will determine the mod p homotopy groups of $THH(\ell)$ and also the integral homotopy groups of $THH(L)$ (where L denotes the periodic Adams summand). In the sequel we will investigate the integral homotopy groups of $THH(\ell)$ using our present results as a starting point.

The THH construction appears to be of basic importance in algebraic K-theory because it combines two useful properties: it can be used to construct good approximations to the algebraic K-theory functor, and it is very accessible to calculation. We shall review what is known about the first property in a moment; the second property was demonstrated by Bökstedt's calculation, in his paper [4], of the homotopy groups of $THH(H\mathbf{Z}/p)$ and $THH(H\mathbf{Z})$ (here $H\mathbf{Z}/p$ and $H\mathbf{Z}$ denote the evident Eilenberg-Mac Lane spectra). It is natural to ask about $THH(R)$ for other popular ring spectra R, and our work is a first step in this direction. We pay special attention to the connective case because this is the case which is likely to be relevant in applications (see Subsection 1.4 below).

The calculation which we present in this paper is a homotopy-theoretic one which uses the Adams spectral sequence (hereafter abbreviated ASS). This calculation has several interesting features; in particular it is a pleasing example of an ASS calculation in which, although there are infinitely many differentials, it is still possible to get the complete answer.

Here is a summary of the contents of the paper. In Section 2 we review the facts we need to know about ordinary Hochschild homology. In Section 3 we do the same for topological Hochschild homology. In Section 4 we calculate the mod p homology of $THH(\ell)$ and use it to find the E_2 term of the ASS converging to $\pi_*(THH(\ell); \mathbf{Z}/p)$. This section also contains a quick calculation, which was pointed out to us by Larry Smith and Andy Baker, of the homotopy groups of $THH(BP)$, where BP is the Brown-Peterson summand of complex cobordism. In Section 5 we calculate the mod p K-theory of $THH(\ell)$ and use it to determine the "v_1-inverted" homotopy of $THH(\ell)$. In Section 6 we work backwards from this result to determine the behavior of the v_1-inverted ASS for $THH(\ell)$. In Section 7 we show that the behavior of the v_1-inverted ASS completely determines that of the ASS itself, thereby completing the calculation of $\pi_*(THH(\ell); \mathbf{Z}/p)$. In Section 8, which depends only on Sections 2, 3, and 5, we calculate $\pi_* THH(L)$. In Section 9 we confess that our definition of the spectrum ℓ is not the usual one; on the other hand we show that it agrees with the usual one up to p-adic completion. Our definition has the advantage that it provides an E_∞ structure for ℓ; this implies that ℓ has an A_∞ structure, which is necessary in order for $THH(\ell)$ to be defined, and it also provides extra structure for $THH(\ell)$ which will be used in the sequel to determine differentials and extensions in the ASS converging to $\pi_* THH(\ell)$.

Acknowledgements: We would like to thank everyone who has discussed this subject with us, especially Andy Baker, Marcel Bökstedt, Nick Kuhn, and Friedhelm Waldhausen.

In the remainder of the introduction we shall give a short summary of some things which are known or suspected about algebraic K-theory; these provide motivation for the THH construction, but none of what follows will actually be used in our work.

1.1 The Dennis Trace Map

The simplest way in which Hochschild homology is related to algebraic K-theory is via the Dennis trace map, which is a natural transformation

$$\tau : K_* S \to \mathbf{HH}_* S;$$

here S is a discrete ring and $\mathbf{HH}_* S$ denotes ordinary Hochschild homology. (See [26, pages 106–114] or [27, Section II.1] for a discussion of τ). Unfortunately, the map τ usually loses too much information to be useful for purposes of calculation. It is, however, possible to improve τ by factoring it through one of the variants of cyclic homology: that is, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{HC}^-_* S \arrow[dr, "\pi"] & \\
K_* S \arrow[u, "\alpha"] \arrow[r, "\tau"] & \mathbf{HH}_* S
\end{tikzcd}$$

(see [27, page 364] for the definitions of $\mathbf{HC}^-$ and π and [27, Section 2.3] for the definition of α). The following basic theorem, due to Goodwillie [27, Theorem II.3.4], says that the map α can be used to calculate *rationalized* relative algebraic K-theory in certain situations.

Theorem 1.1 *If $S_1 \to S_2$ is a surjection with nilpotent kernel then*

$$\alpha \otimes \mathbf{Q} : K_*(S_1 \to S_2) \otimes \mathbf{Q} \to \mathbf{HC}_*^-(S_1 \to S_2) \otimes \mathbf{Q}$$

is an isomorphism.

See [27, pages 365 and 373] for the definitions of $\mathbf{HC}^-$ and α in the relative situation.

The most important application of Theorem 1.1 is to Waldhausen's functor $A(X)$. For this, one needs to generalize Theorem 1.1 to apply to simplicial rings S. This can be done (see [27]), and in this generality the hypothesis of Theorem 1.1 is replaced by the much less stringent hypothesis that the map

$$\pi_0 S_1 \to \pi_0 S_2$$

be a surjection with nilpotent kernel (see [27]). Now given a space X, it is easy to construct a simplicial ring whose K-theory agrees rationally with $A(X)$, and thus Theorem 1.1 can be applied to calculate $A(X \to Y) \otimes \mathbf{Q}$ whenever $X \to Y$ is a 2-connected map (see [27, pages 348–349]).

1.2 Algebraic K-theory of Ring Spectra

The reason for introducing topological Hochschild homology is to try to formulate and prove an analog of Theorem 1.1 which holds *integrally* and not just rationally. One can get a hint as to how to do this by recalling that one of the basic principles of Waldhausen's work on algebraic K-theory is that the K-functor should be applied not just to rings but to ring *spectra* (also called "brave new rings"). Waldhausen gave a sketch of how to do this in [28], and a precise construction was given by May in [29] (also see [30]). For technical reasons one must restrict to A_∞ ring spectra, but in practice this is not an inconvenience. We shall refer to this functor as Waldhausen K-theory and denote it by K^W; when R is an A_∞ ring spectrum, $K^W(R)$ is a spectrum whose homotopy groups will be denoted by $K_*^W(R)$. The functor K_*^W generalizes both K_* and $A(X)$, for when R is the Eilenberg-Mac Lane spectrum HS associated to a discrete ring S one has the equation

$$K_*^W HS = K_* S; \tag{1}$$

and when R is the sphere spectrum S^0, or more generally the suspension spectrum $\Sigma^\infty(\Omega X)_+$, one has

$$K^W(S^0) = A(*)$$

and

$$K^W(\Sigma^\infty(\Omega X)_+) = A(X)$$

(here $(\Omega X)_+$ denotes the space obtained by adding a disjoint basepoint to the loop space of X).

1.3 Topological Hochschild Homology

In view of what has been said so far, it is natural to try to approximate $K_*^W R$ by means of a Hochschild homology construction which can be applied to A_∞ ring spectra R. This is what topological Hochschild homology $THH(R)$ is. It is clear enough in principle how one should construct $THH(R)$ (see Section 3), although the technical details are quite complicated (see [3] and [10]). $THH(R)$ is a spectrum and we shall denote its homotopy $\pi_* THH(R)$ by $\mathbf{THH}_*(R)$.

There is a natural transformation

$$\tau' : K_*^W R \to \mathbf{THH}_* R$$

which is analogous to the Dennis trace map. (See [3, Section 2] for the construction of τ').

In the special case $R = HS$ it is important to note that the analog of equation (1) does *not* hold for $\mathbf{THH}_*$; that is, it is not true that $\mathbf{THH}_*(HS)$ agrees with $\mathbf{HH}_*(S)$ for a discrete ring S. Instead, there is a commutative diagram which shows that τ' gives a second way of lifting the Dennis trace map:

$$\begin{tikzcd} K_* S \arrow[r, "\tau"] \arrow[d, "\tau'"'] & \mathbf{HH}_* S \\ \mathbf{THH}_* HS \arrow[ur, "\phi"'] & \end{tikzcd}$$

(See Remark 3.5 for a hint about the construction of the map ϕ). In the special case $S = \mathbf{Z}$, Bökstedt has shown that τ' is nonzero in infinitely many dimensions; more precisely, what he shows is that for each prime p the localization of τ' at p is an epimorphism in dimension $2p-1$ (see [5]). Note that this cannot be true for τ for the trivial reason that $\mathbf{HH}_*\mathbf{Z}$ is zero in all positive dimensions.

In the cases $R = S^0$ and $R = \Sigma^\infty(\Omega X)_+$ mentioned above one can give explicit descriptions of $THH(R)$:

$$THH(S^0) = S^0$$

and

$$THH(\Sigma^\infty(\Omega X)_+) = \Sigma^\infty(\Lambda X)_+,$$

where Λ denotes the free loop space; the first equation is obvious from the definition in Section 3 and the second follows from that definition and [31, Theorem ?].

Probably the most important fact about τ' is that it can be identified with the map from K^W to its first Goodwillie derivative; more precisely we mean the derivative "at $X = S^0$" of the functor

$$X \mapsto K^W(R \wedge (\Omega X)_+)$$

from pointed spaces to spectra ([32]; also see [33] for the definition of the derivative and the proof of this fact in the special case $R = S^0$). This fact is significant in two ways: it implies that THH is a "first order" approximation to K^W in much the same way that stable homotopy is a first order approximation to unstable homotopy, and it can be used to obtain a "higher order" approximation, as we explain in the next subsection.

1.4 Topological Cyclic and Epicyclic Homology

The next step is to consider functors which combine the desirable properties of $\mathbf{HC}^-$ and THH. For example, one can define topological *cyclic* homology THC^- by observing that the spectrum $THH(R)$ has a natural cyclic structure and therefore has an S^1 action (at least if everything works as in the category of spaces— cf. [31]), and letting

$$\mathbf{THC}_*^- R = \pi_* THH(R)^{hS^1},$$

where hS^1 denotes the homotopy fixed-point spectrum (cf. Remark 3.5). Unfortunately it is known that this functor cannot satisfy an integral version of Theorem 1.1 (see [34, Section 7]). On the other hand, there is considerable evidence for the following conjecture

Conjecture 1.2 *It is possible to construct a functor THE, related to THH and THC^-, and a natural transformation*

$$\tau'' : K^W(R) \to THE(R)$$

which induces an equivalence of derivatives.

The notation THE stands for "topological epicyclic homology"; see [34, Section 6]. It is possible that THE can be taken to be the functor defined in [35].

If the conjecture is true then the calculus of functors will imply the following integral version of Theorem 1.1: the induced map

$$\tau'' : K_*^W(R_1 \to R_2) \to \mathbf{THE}_*(R_1 \to R_2)$$

is an isomorphism for any map $R_1 \to R_2$ of A_∞ ring spectra such that

$$\pi_i R_1 \to \pi_i R_2$$

is an isomorphism for $i \leq 0$.

This explains the statement we made earlier that connective spectra are of particular importance for the potential applications.

We conclude with one further remark about the potential applications of THH. In [23], Waldhausen has proposed an interesting program for studying the relative K^W theory of the map

$$S^0 \to H\mathbf{Z}$$

by means of the intermediate spectra $K^W(L_n(S^0))$ and $K^W(L_n(S^0)_c)$, where $L_n(S^0)$ denotes the L_n-localization of the sphere (see [36]) and $L_n(S^0)_c$ is the associated connective spectrum. When $n = 1$ the spectrum $L_1(S^0)_c$ is the connective image-of-J spectrum j. It seems likely that the results and methods of our work are a good way to obtain information about $THH(j)$.

2 A Brief Review of Hochschild Homology

In this section we recall the facts we need about ordinary (algebraic) Hochschild homology. Our basic reference for this subject is [8, Chapters IX and X].

If S is a graded algebra over a ground field k, its Hochschild homology $\mathbf{HH}_*(S)$ is defined to be the homology of the *Hochschild complex* [8, page 175]

$$\begin{array}{c} \vdots \\ \downarrow \\ S \otimes S \otimes S \\ \downarrow \\ S \otimes S \\ \downarrow \\ S, \end{array} \tag{2}$$

in which the differential is given by the formula

$$\begin{aligned} d(t_0 \otimes \cdots \otimes t_n) \quad &= \quad \sum_{i=0}^{n-1} (-1)^i t_0 \otimes \cdots \otimes t_i t_{i+1} \otimes \cdots \otimes t_n \\ &\quad + (-1)^n (-1)^{|t_n|(|t_0| + \cdots + |t_{n-1}|)} t_n t_0 \otimes t_1 \otimes \cdots \otimes t_{n-1}. \end{aligned}$$

As one might expect, $\mathbf{HH}_*(S)$ can also be described in terms of Tor; it is

$$\mathrm{Tor}^{S \otimes S^{\mathrm{op}}}(S, S),$$

where the first factor of $S \otimes S^{\mathrm{op}}$ acts on S by multiplication on the left and the second factor by multiplication on the right [8, page 169]. The reader may perhaps wonder why one uses this definition for the homology of S instead of the "obvious" definition $\mathrm{Tor}^S(k, k)$. For our purposes, the answer is that the latter is the appropriate definition for the category of *augmented* algebras, but we need to work more generally; the functor $\mathbf{HH}_*(S)$ is closely related to $\mathrm{Tor}^S(k, k)$, but it is defined for arbitrary algebras S.

There is an evident natural map

$$\iota : S \to \mathbf{HH}_0(S),$$

which is an isomorphism when S is commutative. There is also a "suspension" map

$$\sigma : S \to \mathbf{HH}_1(S)$$

which takes $t \in S$ to the class of $1 \otimes t$. If S is commutative there is a product

$$\mathbf{HH}_i(S) \otimes \mathbf{HH}_j(S) \to \mathbf{HH}_{i+j}(S),$$

which gives $\mathbf{HH}_*(S)$ the structure of a commutative graded S-algebra (see [8, page 217]); moreover ι is a ring homomorphism and σ is a derivation:

$$\sigma(st) = s\sigma(t) + (-1)^{|s||t|} t\sigma(s). \tag{3}$$

It will not surprise the reader to find that there are times when we actually need to compute the ring $\mathbf{HH}_*(S)$. The following result is sufficient for our purposes.

Proposition 2.1 *If S has the form*

$$\mathbf{Z}/p\,[x_1, x_2, \ldots] \otimes \Lambda(y_1, y_2, \ldots),$$

then $\mathbf{HH}_*(S)$ has the form

$$S \otimes \Lambda(\sigma(x_1), \sigma(x_2), \ldots) \otimes \Gamma(\sigma(y_1), \sigma(y_2), \ldots),$$

where the inclusion of the first factor is the natural map

$$\iota : S \xrightarrow{\cong} \mathbf{HH}_0(S)$$

and σ is the suspension map

$$S \to \mathbf{HH}_1(S).$$

Proof. Let

$$\varphi : S \to S \otimes S$$

be the ring map which takes x_i and y_j to

$$x_i \otimes 1 - 1 \otimes x_i$$

and

$$y_j \otimes 1 - 1 \otimes y_j,$$

respectively. By [8, Theorem X.6.1], φ induces an isomorphism

$$\mathrm{Tor}^S(S_\varphi, \mathbf{Z}/p) \to \mathrm{Tor}^{S\otimes S}(S, S) = \mathbf{HH}_*(S),$$

where S_φ denotes the S-module structure on S obtained by pulling back its $S \otimes S$-module structure along φ. In our case S is commutative, so that S_φ has the trivial S-module structure, and we conclude that there is an isomorphism

$$S \otimes \mathrm{Tor}^S(\mathbf{Z}/p, \mathbf{Z}/p) \cong \mathbf{HH}_*(S). \tag{4}$$

But it is well known that

$$\mathrm{Tor}^S(\mathbf{Z}/p, \mathbf{Z}/p) \cong \Lambda(\sigma'(x_1), \sigma'(x_2), \ldots) \otimes \Gamma(\sigma'(y_1), \sigma'(y_2), \ldots),$$

where Γ denotes a divided polynomial algebra and σ' is the suspension map

$$S \to \mathrm{Tor}_1^S(\mathbf{Z}/p, \mathbf{Z}/p);$$

and it is not hard to check that the isomorphism (4) takes $\sigma'(x_i)$ to $\sigma(x_i)$ and $\sigma'(y_j)$ to $\sigma(y_j)$. ♣

3 Introduction To Topological Hochschild Homology

In this section we turn to the topological version of Hochschild homology. Our references for the foundations are [3] and [10], and we refer to those sources for all technical details.

Roughly speaking, topological Hochschild homology is constructed by replacing the algebra S in the Hochschild complex (2) by a ring spectrum R. We will show how to carry this idea out when the multiplication in R is strictly associative (which is the case considered in [3, Section 1]), but in fact it can be done whenever R has an A_∞ structure (see [10]).

First we must reformulate the definition of $\mathbf{HH}_*(S)$. Let $HH_\bullet(S)$ be the simplicial abelian group

$$\begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ S \otimes S \otimes S \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ S \otimes S \\ \downarrow\uparrow\downarrow \\ S. \end{array}$$

Here the face maps ∂_i and degeneracy maps s_i are given by the formulas

$$\partial_i(t_0 \otimes \cdots \otimes t_n) = \begin{cases} t_0 \otimes \cdots \otimes t_i t_{i+1} \otimes \cdots \otimes t_n & \text{if } 0 \le i < n \\ \\ t_n t_0 \otimes t_1 \otimes \cdots \otimes t_{n-1} & \text{if } i = n, \end{cases}$$

and

$$s_i(t_0 \otimes \cdots \otimes t_n) = t_0 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n.$$

Clearly the Hochschild complex is the chain complex associated to this simplicial abelian group. But for any simplicial abelian group, the homology of its associated chain complex is the same as the homotopy of its geometric realization (see [15, Theorem 22.1]), so in our case we conclude

$$\mathbf{HH}_*(S) = \pi_*|HH_\bullet(S)|.$$

We now define the *topological Hochschild homology* spectrum $THH(R)$ associated to a ring spectrum R to be the geometric realization of the simplicial spectrum

$$THH_\bullet(R) = \begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ R \wedge R \wedge R \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ R \wedge R \\ \downarrow\uparrow\downarrow \\ R. \end{array}$$

The face map

$$\partial_i : \overbrace{R \wedge \cdots \wedge R}^{n+1} \to \overbrace{R \wedge \cdots \wedge R}^{n}$$

is defined by the following equation (suitably interpreted):

$$\partial_i(r_0 \wedge \cdots \wedge r_n) = \begin{cases} r_0 \wedge \cdots \wedge r_i r_{i+1} \wedge \cdots \wedge r_n & \text{if } 0 \le i < n \\ \\ r_n r_0 \wedge r_1 \wedge \cdots \wedge r_{n-1} & \text{if } i = n. \end{cases}$$

The degeneracy map

$$s_i : \overbrace{R \wedge \cdots \wedge R}^{n+1} \to \overbrace{R \wedge \cdots \wedge R}^{n+2}$$

is defined to be the composite

$$\overbrace{R \wedge \cdots \wedge R}^{n+1} \overset{\cong}{\to} \overbrace{R \wedge \cdots \wedge R}^{i+1} \wedge S^0 \wedge \overbrace{R \wedge \cdots \wedge R}^{n-i} \xrightarrow{1 \wedge e \wedge 1} \overbrace{R \wedge \cdots \wedge R}^{n+2},$$

where e is the unit map $S^0 \to R$; thus the i-th degeneracy inserts a unit in the $(i+1)$-st position. (Our assumption that the multiplication in R is strictly associative is necessary in order that the maps ∂_i and s_i defined in this way satisfy the simplicial identities).

We shall write

$$\tilde{\imath} : R \to THH(R)$$

for the inclusion of the 0-th simplicial filtration in $THH(R)$. If the multiplication in R is sufficiently commutative then $THH(R)$ inherits a ring-spectrum structure and $\tilde{\imath}$ is a ring map (see [3, Section 2]). If R is an E_∞ ring spectrum then $THH(R)$ inherits an E_∞ ring structure and $\tilde{\iota}$ is an E_∞ ring map (see [10]). Our definition of the spectrum ℓ, which is given in Section **??**, automatically implies that ℓ is an E_∞ ring spectrum, so we conclude that $THH(\ell)$ is also.

Now suppose that we are given a homology theory h_* with a multiplication and that we want to know $h_*(THH(R))$. In [4], Bökstedt introduced the following spectral sequence for this sort of calculation.

Proposition 3.1 *If h_* satisfies the strict Künneth formula*

$$h_*(X \wedge Y) \cong h_*X \otimes_{h_*S^0} h_*Y$$

then there is a spectral sequence

$$\mathbf{HH}_*(h_*(R)) \Rightarrow h_*(THH(R)), \tag{5}$$

where $\mathbf{HH}_*$ is defined with respect to the ground ring h_*S^0. For each $x \in h_*(R)$ the element

$$\iota_*(x) \in \mathbf{HH}_0(h_*(R))$$

survives to

$$\tilde{\iota}_*(x) \in h_*(THH(R)).$$

We warn the reader that there is no Hopf algebra structure in this spectral sequence.

It is likely, although we shall not attempt to prove it, that Proposition 3.1 holds without the assumption that h_* satisfies the strict Künneth formula (cf. [19, Theorem 13.1]).

Proof of Proposition 3.1. For any simplicial spectrum $X_\bullet$, we may apply the theory h_* to the simplicial filtration of $|X_\bullet|$ in the usual way to obtain a spectral sequence converging to $h_*(|X_\bullet|)$ (cf. [16, Theorem 11.14]). If $X_\bullet$ is "proper" then the E_2 term of this spectral sequence is the homology of the complex

$$\cdots \to h_*(X_n) \to \cdots \to h_*(X_1) \to h_*(X_0), \tag{6}$$

with differential

$$d = \sum (-1)^i (\partial_i)_*.$$

Now when $X_\bullet$ is $THH_\bullet(R)$ and h_* satisfies the strict Künneth formula this complex is just the Hochschild complex for $h_*(R)$, and we conclude that

$$E_2 \cong \mathbf{HH}_*(h_*(R))$$

as required. ♣

At the end of the next section we shall need to have somewhat tighter control of the spectral sequence (5). The information we need is provided by our next result.

Proposition 3.2 *There is a natural transformation*

$$\tilde{\sigma} : \Sigma R \to THH(R)$$

such that the element

$$\sigma_*(x) \in \mathbf{HH}_1(h_*(R))$$

survives to

$$\tilde{\sigma}_*(\Sigma x) \in h_*(THH(R)).$$

for each $x \in h_*(R)$.

Proof. Before we can define the natural transformation $\tilde{\sigma}$ we need some preliminary constructions. Let $S_\bullet(R)$ be the simplicial spectrum obtained by "replacing all $\wedge$'s in $THH_\bullet(R)$ by $\vee$'s." More precisely, the n-th simplicial degree of $S_n(R)$ is

$$\overbrace{R \vee \ldots \vee R}^{n+1}.$$

The i-th face operator

$$\partial_i : \overbrace{R \vee \ldots \vee R}^{n+1} \to \overbrace{R \vee \ldots \vee R}^{n}$$

is defined by the equation

$$\partial_i \circ I_j = \begin{cases} I_{j-1} & \text{if } i < j \\ \\ I_j & \text{if } i \geq j \text{ and } j < n \\ \\ I_0 & \text{if } i = j = n; \end{cases}$$

here

$$I_j : R \to R \vee \ldots \vee R$$

is the inclusion of the j-th wedge summand. The i-th degeneracy map s_i is defined by the equation

$$s_i \circ I_j = \begin{cases} I_{j+1} & \text{if } i < j \\ \\ I_j & \text{if } i \geq j. \end{cases}$$

We pause to determine the homotopy type of $|S_\bullet(R)|$.

Lemma 3.3 $|S_\bullet(R)| \simeq R \vee \Sigma R$

Proof of Lemma 3.3. Clearly we have

$$|S_\bullet(R)| \cong |S_\bullet(S^0)| \wedge R.$$

Now $S_\bullet(S^0)$ can be obtained by adding a disjoint basepoint to the standard simplicial decomposition of S^1 (see [3, page 20]), and so we have

$$|S_\bullet(R)| \cong (S^1)^+ \wedge R.$$

But for any space X, the space X^+ obtained by adding a disjoint basepoint splits stably as $S^0 \vee X$, so finally we have

$$|S_\bullet(R)| \simeq (S^0 \vee S^1) \wedge R \simeq R \vee \Sigma R$$

as required. ♣

For each n we can define a map

$$\omega_n : \overbrace{R \vee \ldots \vee R}^{n+1} \to \overbrace{R \wedge \ldots \wedge R}^{n+1}$$

by letting the restriction of ω_n to the j-th wedge summand be the composite

$$R \stackrel{\cong}{\to} \overbrace{S^0 \wedge S^0}^{j} \wedge R \wedge \overbrace{S^0 \wedge S^0}^{n-j} \xrightarrow{e\wedge 1 \wedge e} \overbrace{R \wedge \ldots \wedge R}^{n+1}.$$

Taken together, the ω_n give a map

$$\omega_\bullet : S_\bullet(R) \to THH_\bullet(R).$$

By passing to geometric realizations and using Lemma 3.3 we obtain a map[1]

$$\omega : R \vee \Sigma R \to THH(R)$$

The restriction of ω to the R summand is the map

$$\tilde{\iota} : R \to THH(R)$$

defined earlier. We can now define

$$\tilde{\sigma} : \Sigma R \to THH(R)$$

to be the restriction of ω to the ΣR summand.

[1]In [3, Section 3] and [4, Section 2] this map is denoted by λ.

To complete the proof of Proposition 3.2 it only remains to show that the transformation $\tilde{\sigma}$ has the desired relation to the spectral sequence (5).

Let $C_*(X_\bullet)$ denote the chain complex (6). A straightforward calculation shows that the homology of $C_*(S_\bullet(R))$ vanishes in all dimensions except 0 and 1, and in particular the spectral sequence associated to $S_\bullet(R)$ collapses. For each $x \in h_*(R)$ the element

$$I_{1*}x \in h_*(R \vee R)$$

is a 1-dimensional cycle in $C_*(S_\bullet(R))$ which represents a class $\overline{x}$ in $E_2(S_\bullet(R))$. If we write J for the inclusion of ΣR as a wedge summand in $|S_\bullet(R)|$, then $\overline{x}$ survives to

$$J_*(\Sigma x) \in h_*(|S_\bullet(R)|).$$

It follows that the image of $\overline{x}$ in $E_2(THH_\bullet(R))$ survives to $\omega_* J_*(\Sigma x)$, which by definition is $\tilde{\sigma}(\Sigma x)$, in $h_*(THH_\bullet(R))$. But the image of $I_{1*}x$ in the Hochschild complex $C_*(THH_\bullet(R))$ is $1 \otimes x$, and so the image of $\overline{x}$ in $E_2(THH_\bullet(R))$ is σx. We have now shown that σx survives to $\tilde{\sigma}(\Sigma x)$, as required to finish the proof of Proposition 3.2. ♣

We conclude this section with some remarks which will not be used in the rest of the paper.

Remark 3.4 Alan Robinson has defined a "topological" analog of $\mathrm{Tor}^S(M, N)$, which he denotes by $E \wedge_R F$ (see [19]). In analogy with the equation

$$\mathbf{HH}_* S = \mathrm{Tor}^{S \otimes S^{\mathrm{op}}}(S, S)$$

one presumably has

$$THH(R) \simeq R \wedge_{R \wedge R^{\mathrm{op}}} R.$$

Remark 3.5 There is another way to relate Robinson's work to THH. Given a sufficiently good map of ring spectra $T \to R$, it should be possible to define a spectrum

$$THH_T(R)$$

(i.e., "topological Hochschild homology over the ground ring T") by replacing all the smash products in the definition of $THH(R)$ by $\wedge_T$ products. In particular, if $R = HS$ for a discrete ring spectrum S one should have a formula

$$\pi_* THH_{H\mathbf{Z}}\, HS = \mathbf{HH}_* S$$

relating ordinary Hochschild homology to topological Hochschild homology; this would give one way to construct the map ϕ mentioned in Subsection 1.3 of the introduction. It should also be the case that

$$\pi_*(THH_{H\mathbf{Z}}\, HS)^{hS^1} = \mathbf{HC}^-_* S;$$

cf. Subsection 1.4.

Remark 3.6 Here is one more way to motivate the THH construction. If G is a topological group, its classifying space BG is the geometric realization of the following simplicial space.

$$\begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ G \times G \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ G \\ \downarrow\uparrow\downarrow \\ * \end{array}$$

The face and degeneracy maps are given by the equations

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

and

$$s_i(g_1, \ldots, g_n) = (\ldots, g_i, 1, g_{i+1}, \ldots).$$

The same construction can be applied when G is merely an associative monoid.

It would be natural to try to apply this construction to an associative ring spectrum R, replacing the G's by R's, the $\times$'s by $\wedge$'s, and $*$ by S^0. If one attempts to carry this out, however, it becomes apparent that there is no sensible way to define the first and last degeneracy maps ∂_0 and ∂_n. Further reflection shows that this is because there is no sensible way to define an augmentation map $R \to S^0$. This brings us back to the observation made at the beginning of Section 2: in the analogous algebraic situation, one compensates for the lack of an augmentation by using the Hochschild complex instead of the bar construction. Thus one can think of the THH construction as being the closest one can come to imitating the classifying space construction for a ring spectrum R. (The analogy is not precise, however, and in particular if the analog of the Hochschild construction is applied to a topological group G the result is not BG but instead is the free loop space $\mathrm{Map}(S^1, BG)$; see [11]).

4 Calculation of the E_2-term of the Adams spectral sequence

We remind the reader that p denotes an odd prime.

Let $\mathbf{M}$ denote the Moore spectrum $S^0 \cup_p e^1$. By definition we have

$$\pi_*(X; \mathbf{Z}/p) = \pi_*(X \wedge \mathbf{M}))$$

for any spectrum X, and we accordingly write

$$E_r(X; \mathbf{Z}/p)$$

for the classical Adams spectral sequence converging to $\pi_*(X \wedge \mathbf{M})$. We will index this spectral sequence as usual, so that

$$E_2^{t-s,s}(X; \mathbf{Z}/p) = \mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/p, H_*(X \wedge \mathbf{M}; \mathbf{Z}/p)).$$

The differentials d_r have bidegree $(-1, r)$, and $E_\infty^{t-s,*}$ is the associated graded of a filtration on $\pi_{t-s}(X; \mathbf{Z}/p)$. The Moore spectrum $\mathbf{M}$ is a ring spectrum (since p is odd), and it follows that the spectral sequence has a multiplicative structure (see [18, Theorem 2.3.3]).

The case $X = \ell$ is of particular importance for our work. In this case it is well known (cf. [18, page 75], and also see the proof of Theorem 4.1 below) that

$$E_2(\ell; \mathbf{Z}/p) \cong \mathbf{Z}/p\,[a_1] \cong E_\infty(\ell; \mathbf{Z}/p),$$

where a_1 is an element in bidegree $(2p-2, 1)$ representing $v_1 \in \pi_{2p-2}(\ell; \mathbf{Z}/p)$. We shall also write a_1 for the image of this element under the map

$$E_2(\ell; \mathbf{Z}/p) \to E_2(THH(\ell); \mathbf{Z}/p)$$

induced by the inclusion $\tilde{\imath} : \ell \to THH(\ell)$.

The purpose of this section is to prove the following theorem.

Theorem 4.1 *$E_2(THH(\ell); \mathbf{Z}/p)$ has the form*

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu],$$

where λ_1 is in bidegree $(2p-1, 0)$, λ_2 is in bidegree $(2p^2-1, 0)$, and μ is in bidegree $(2p^2, 0)$.

This, in turn, is a consequence of our next result. As usual, we write A_* for the dual of the Steenrod algebra.

Proposition 4.2 *As an algebra, $H_*(THH(\ell); \mathbf{Z}/p)$ has the form*

$$H_*(\ell; \mathbf{Z}/p) \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu],$$

where λ_1 is in degree $2p-1$, λ_2 is in degree $2p^2-1$, μ is in degree $2p^2$, and the inclusion of the first factor is the natural map

$$\tilde{\iota}_* : H_*(\ell; \mathbf{Z}/p) \to H_*(THH(\ell); \mathbf{Z}/p).$$

The A_*-coaction

$$\nu : H_*(THH(\ell); \mathbf{Z}/p) \to A_* \otimes H_*(THH(\ell); \mathbf{Z}/p)$$

is determined by the equations

$$\nu(\lambda_i) = 1 \otimes \lambda_i$$

and

$$\nu(\mu) = 1 \otimes \mu + \tau_0 \otimes \lambda_2.$$

Proof of Theorem 4.1. We need to calculate

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/p, H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p)).$$

We shall do this by using a standard change-of-rings theorem:

$$\mathrm{Ext}_\Gamma(\mathbf{Z}/p, \Gamma \Box_\Sigma N) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, N) \tag{7}$$

(see [18, Theorem A1.3.12]). Here Γ denotes a Hopf algebra, Σ a quotient Hopf algebra, and N a Σ-comodule; the $\Box$-product is defined on page 311 of [18]. (See pages 337–339 of [1] for a dual version of the following argument which avoids the $\Box$-product).

First we observe that if N is actually a Γ-comodule (more precisely if the Σ-comodule structure on N is induced by a Γ-comodule structure) then the map

$$\Gamma \otimes N \to \Gamma \otimes N$$

which takes $g \otimes n$ to

$$\sum_i g g_i \otimes n_i$$

(where, as usual, we have written

$$n \mapsto \sum_i g_i \otimes n_i$$

for the Γ-coaction on N) induces an isomorphism

$$(\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N \to \Gamma \Box_\Sigma N$$

(where the domain has the diagonal Γ-coaction). We can therefore rewrite (7) in this situation as follows:

$$\mathrm{Ext}_\Gamma(\mathbf{Z}/p, (\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, N). \tag{8}$$

Now let Γ be A_*, let Σ be the Hopf-algebra quotient of the inclusion

$$H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p) \to H_*(\mathbf{Z}/p; \mathbf{Z}/p) = A_*,$$

and let N be

$$\Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu].$$

Proposition 4.2 implies

$$H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p) \cong H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p) \otimes N,$$

and we have

$$\Gamma \Box_\Sigma \mathbf{Z}/p \cong H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p)$$

by [18, Lemma A1.1.16], so finally we have

$$H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p) \cong (\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N.$$

We can therefore conclude from equation (8) that

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/p, H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p)) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, N).$$

But

$$\Sigma = \Lambda(\chi\tau_1),$$

and in particular Proposition 4.2 shows that the Σ-coaction on N is trivial. Thus we have

$$\mathrm{Ext}_\Sigma(\mathbf{Z}/p, N) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, \mathbf{Z}/p) \otimes N,$$

and by [18, Lemma 3.1.9] this is

$$\mathbf{Z}/p\,[a_1] \otimes N$$

as required. ♣

Proof of Proposition 4.2. Of course, the first step in the proof of Proposition 4.2 is to calculate $H_*(THH(\ell); \mathbf{Z}/p)$ as an algebra by using the spectral sequence of Proposition 3.1. We shall denote this spectral sequence by

$$\widehat{E}^r(R)$$

in order to distinguish it from the Adams spectral sequence. To carry out the spectral sequence calculation, all we have to do is modify the proof of [4, Theorem 1.1].

Since

$$\widehat{E}^2(\ell) \cong \mathbf{HH}_*(H_*(\ell; \mathbf{Z}/p)),$$

we must begin by remembering what $H_*(\ell; \mathbf{Z}/p)$ is. In order to describe it we recall from [1, Lemma 16.8] that the canonical map

$$\varepsilon : \ell \to H\mathbf{Z}/p$$

(i.e., the map which represents the generator of $H^0(\ell;\mathbf{Z}/p)$) induces a monomorphism

$$H_*(\ell;\mathbf{Z}/p) \to H_*(H\mathbf{Z}/p;\mathbf{Z}/p) = A_*$$

with image

$$\mathbf{Z}/p\,[\chi\xi_1, \chi\xi_2, \ldots] \otimes \Lambda[\chi\tau_2, \chi\tau_3, \ldots];$$

here χ is the canonical anti-automorphism of A_*[2].

We can now apply Proposition 2.1 of Section 2 to conclude that

$$\widehat{E}^2(\ell) \cong H_*(\ell;\mathbf{Z}/p) \otimes \Lambda[\sigma(\chi\xi_1), \sigma(\chi\xi_2), \ldots] \otimes \Gamma[\sigma(\chi\tau_2), \sigma(\chi\tau_3), \ldots],$$

where $H_*(\ell;\mathbf{Z}/p)$ is in filtration-degree 0, $\sigma(\chi\xi_i)$ has bidegree $(2p^i - 2, 1)$, and $\sigma(\chi\tau_i)$ has bidegree $(2p^i - 1, 1)$.

The next step is to determine the differentials in the spectral sequence $\widehat{E}^r(\ell)$. This is easily done by comparing it with the spectral sequence $\widehat{E}^r(H\mathbf{Z}/p)$, whose behavior has been completely determined in [4]. We shall show in Section **??** that the map ε is an E_∞ ring map (more precisely, we should say that there is an E_∞ ring map in its homotopy class). In particular, it is an A_∞ ring map, and thus it induces a map

$$\varepsilon_* : \widehat{E}^r(\ell) \to \widehat{E}^r(H\mathbf{Z}/p).$$

A calculation similar to that for $\widehat{E}^2(\ell)$ shows that

$$\widehat{E}^2(H\mathbf{Z}/p) \cong A_* \otimes \Lambda[\sigma(\chi\xi_1), \sigma(\chi\xi_2), \ldots] \otimes \Gamma[\sigma(\chi\tau_0), \sigma(\chi\tau_1), \ldots].$$

By [4, Lemma 1.3], the only nontrivial differential in $\widehat{E}^r(H\mathbf{Z}/p)$ is given by the formula

$$d^{p-1}\,\gamma_j(\sigma(\chi\tau_i)) = \sigma(\chi\xi_{i+1}) \cdot \gamma_{j-p}(\sigma(\chi\tau_i)) \quad \text{if } j > p;$$

here we have written $\gamma_j(\sigma(\chi\tau_i))$ for the j-th divided power of $\sigma(\chi\tau_i)$. The same formula therefore holds in $\widehat{E}^{p-1}(\ell)$ for $i \geq 2$, and we conclude that

$$\widehat{E}^p(\ell) \cong H_*(\ell;\mathbf{Z}/p) \otimes \Lambda[\sigma(\chi\xi_1), \sigma(\chi\xi_2)] \otimes TP_p[\sigma(\chi\tau_2), \sigma(\chi\tau_3), \ldots],$$

where TP_p denotes a truncated polynomial algebra of height p (cf. [4, page 6]). Since all indecomposables in $\widehat{E}^p(\ell)$ are in filtrations 0 and 1 we can further conclude that

$$\widehat{E}^p(\ell) = \widehat{E}^\infty(\ell).$$

Proposition 3.2 implies that the elements $\sigma(\chi\xi_i)$ and $\sigma(\chi\tau_i)$ in $\widehat{E}^\infty$ are represented in $H_*(THH(\ell))$ by $\tilde{\sigma}_*(\Sigma(\chi\xi_i))$ and $\tilde{\sigma}_*(\Sigma(\chi\tau_i))$ respectively.

[2] We need to use χ in this description where Adams does not because we are thinking of $H_*(\ell;\mathbf{Z}/p)$ as $\pi_*(H\mathbf{Z}/p \wedge \ell)$ instead of $\pi_*(\ell \wedge H\mathbf{Z}/p)$.

Next we need to determine the multiplicative extensions in $H_*(THH(\ell);\mathbf{Z}/p)$. For this we use Dyer-Lashof operations. As we have seen in the previous section, $THH(\ell)$ is an E_∞ ring spectrum, and so its homology supports Dyer-Lashof operations

$$Q^i : H_n(THH(\ell);\mathbf{Z}/p)) \to H_{n+2i(p-1)}(THH(\ell);\mathbf{Z}/p)),$$

(see [21]). If x is an element of dimension $2s$ then $Q^s x = x^p$, ([21, Theorem 1.1(4)]) so in particular we have

$$(\tilde{\sigma}_*(\Sigma(\chi\tau_i)))^p = Q^{p^i}\tilde{\sigma}_*(\Sigma(\chi\tau_i)).$$

But Bökstedt shows that the map

$$\tilde{\sigma}_*\Sigma : H_n(R;\mathbf{Z}/p) \to H_{n+1}(THH(R);\mathbf{Z}/p)$$

commutes with Dyer-Lashof operations (see [4, Lemma 2.9]), and Steinberger has calculated the action of the Q^i in $H_*(\ell;\mathbf{Z}/p)$:

$$Q^{p^i}\chi\tau_i = \chi\tau_{i+1}$$

(see [21, Theorem 2.3]). We conclude that

$$(\tilde{\sigma}_*(\Sigma\chi\tau_i))^p = \tilde{\sigma}_*(\Sigma\chi\tau_{i+1})$$

for all $i \geq 2$, and hence that

$$(\tilde{\sigma}_*(\Sigma\chi\tau_2))^{p^i} = \tilde{\sigma}_*(\Sigma\chi\tau_{i+2})$$

for all $i \geq 0$. If we denote $\tilde{\sigma}_*\Sigma(\chi\xi_1)$ by λ_1, $\tilde{\sigma}_*\Sigma(\chi\xi_2)$ by λ_2, and $\tilde{\sigma}_*\Sigma(\chi\tau_2)$ by μ, we have now shown that

$$H_*(THH(\ell);\mathbf{Z}/p) \cong H_*(\ell;\mathbf{Z}/p) \otimes \Lambda(\lambda_1,\lambda_2) \otimes \mathbf{Z}/p\,[\mu]$$

as an algebra.

To complete the proof of Proposition 4.2 we need to determine the A_*-coaction on λ_1, λ_2, and μ. We shall give the calculation of $\nu(\lambda_2)$; the others are similar.

Since the map $\tilde{\sigma}_*\Sigma$ commutes with ν, we have

$$\nu(\lambda_2) = (1 \otimes \tilde{\sigma}_*\Sigma)\nu(\chi\xi_2).$$

Now $\nu(\chi\xi_2)$ is determined by Milnor's calculations: it is

$$1 \otimes \xi_2 + \xi_1 \otimes \xi_1^p + \xi_2 \otimes 1$$

(see [18, Theorem 3.1.1]). We therefore conclude that

$$\nu(\lambda_2) = 1 \otimes \lambda_2 + \xi_1 \otimes \tilde{\sigma}_*\Sigma(\xi_1^p) + \xi_2 \otimes \tilde{\sigma}_*\Sigma(1)$$

and it remains to show that the second and third terms are zero. But $\tilde{\sigma}_*\Sigma(\xi_1^p)$ represents the element $\sigma(\xi_1^p)$ in the spectral sequence, and this element is zero because σ is a derivation (equation (3) of Section 2). It follows that $\tilde{\sigma}_*\Sigma(\xi_1^p)$ is an element in filtration 0 with dimension $2p^2-2p+1$, and an inspection of the spectral sequence shows that the only such element is 0. Similarly, $\tilde{\sigma}_*\Sigma(1)$ is an element in filtration 0 with dimension 1, and again the only such element is 0. This completes the proof of Proposition 4.2. ♣

Remark 4.3 (Andy Baker and Larry Smith) Let us *assume* that the Brown-Peterson spectrum BP has an E_∞ structure. Starting from the equation

$$H_*(BP;\mathbf{Z}/p) = \mathbf{Z}/p\,[\chi\xi_1, \chi\xi_2, \ldots],$$

it is easy to see that $\widehat{E}^2(BP)$ has the form

$$H_*(BP;\mathbf{Z}/p)\otimes\Lambda[\sigma(\chi\xi_1),\sigma(\chi\xi_2),\ldots].$$

For dimensional reasons there cannot be any differentials, and we conclude that

$$H_*(THH(BP);\mathbf{Z}/p) = H_*(BP;\mathbf{Z}/p)\otimes\Lambda(\lambda_1,\lambda_2,\ldots), \tag{9}$$

where

$$\lambda_i = \tilde{\sigma}_*\Sigma(\chi\xi_i).$$

A similar calculation in rational cohomology, starting from the equation

$$H_*(BP;\mathbf{Q}) \cong \mathbf{Q}[v_1, v_2, \ldots],$$

shows that

$$H_*(THH(BP);\mathbf{Q}) = H_*(BP;\mathbf{Q})\otimes\Lambda(\lambda_1', \lambda_2', \ldots), \tag{10}$$

where

$$\lambda_i' = \tilde{\sigma}_*\Sigma(v_i).$$

Comparing equations (9) and (10) dimensionwise shows that $H_*(THH(BP);\mathbf{Z}_{(p)})$ must be torsion free. Now equation (9) and Theorem 1.3 of [38] imply that $THH(BP)$ is a wedge of suspensions of BP and that

$$\pi_*THH(BP)\cong\pi_*BP\otimes\Lambda(\lambda_1,\lambda_2,\ldots).$$

5 Localized mod-p homotopy of $THH(\ell)$

The object of this section is to prove the following result, which we will use in later sections to determine the differentials in the Adams spectral sequence $E_r(THH(\ell);\mathbf{Z}/p)$ for the mod-p homotopy of $THH(\ell)$.

Theorem 5.1 *The inclusion*

$$\tilde{\iota}:\ \ell \longrightarrow THH(\ell)$$

induces an isomorphism

$$\tilde{\iota}_*:\ v_1^{-1}\pi_*(\ell;\mathbf{Z}/p) \xrightarrow{\cong} v_1^{-1}\pi_*(THH(\ell);\mathbf{Z}/p).$$

The v_1-inverted mod p homotopy of an ℓ-module X with structure map $\alpha : \ell \wedge X \longrightarrow X$ is defined as a direct limit:

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p) = \lim_{\substack{\rightarrow \\ v_1}} \pi_*(X;\mathbf{Z}/p),$$

where the maps v_1 in the direct system

$$\pi_n(X;\mathbf{Z}/p) \longrightarrow \pi_{2(p-1)+n}(X;\mathbf{Z}/p)$$

send the homotopy class $[f]$ of a stable map $f : S^n \longrightarrow X \wedge \mathbf{M}$ to the homotopy class of $[\alpha \circ (v_1 \wedge f)]$, using v_1 also to denote a representative map $v_1 : S^{2(p-1)} \longrightarrow \ell$ and $\mathbf{M}$ to denote the mod p Moore spectrum.

This result is in some sense anticipated by Bökstedt's results on $THH(\mathbf{Z}_{(p)})$ in [4]. (Actually he discusses $THH(\mathbf{Z})$ and not $THH(\mathbf{Z}_{(p)})$, but it is clear that the results in [4] have parallels for $THH(\mathbf{Z}_{(p)})$.) It is an easy consequence of his computations that the inclusion of spectra

$$\iota:\ \mathbf{Z}_{(p)} \longrightarrow THH(\mathbf{Z}_{(p)})$$

induces an isomorphism in homotopy tensored with $\mathbf{Z}[1/p]$. In other words, inverting p in homotopy kills the difference between the Eilenberg-MacLane spectrum $\mathbf{Z}_{(p)}$ and $THH(\mathbf{Z}_{(p)})$. Our theorem states that something like this persists for $THH(\ell)$ if we consider multiplication by v_1 in mod-p homotopy instead of multiplication by p in p-local homotopy.

This theorem is a consequence of the following proposition.

Proposition 5.2 *The inclusion*

$$\tilde{\iota} : \ell \longrightarrow THH(\ell)$$

induces an isomorphism

$$\tilde{\iota}_* : K(1)_*(\ell) \xrightarrow{\cong} K(1)_*(THH(\ell)).$$

The homology theory $K(1)$ is the first Morava K-theory, with

$$\pi_*(K(1)) = \mathbf{Z}/p\,[v_1, v_1^{-1}],$$

where the dimension of v_1 is $2(p-1)$. For an odd prime p $K(1)$ is the Adams summand of mod p periodic K-theory, and we therefore have an isomorphism

$$K(1)_*(X) \cong v_1^{-1}\pi_*(\ell \wedge X; \mathbf{Z}/p)$$

where ℓ is the Adams summand of complex K-theory, $\pi_*(\ell) = \mathbf{Z}_{(p)}[v_1]$.

Proof of Theorem 5.1. We start by choosing some notation: Take ℓ with the usual ℓ-module structure $\mu : \ell \wedge \ell \longrightarrow \ell$ and $THH(\ell)$ with $\alpha : \ell \wedge THH(\ell) \longrightarrow THH(\ell)$ being just the restriction of the multiplication on $THH(\ell)$.

Observe that we can make a commutative diagram

$$\begin{array}{ccc} \pi_*(l;\mathbf{Z}/p) & \xrightarrow{\tilde{\iota}_*} & \pi_*(THH(l);\mathbf{Z}/p) \\ \downarrow h & & \downarrow h \\ \pi_*(l\wedge l;\mathbf{Z}/p) & \xrightarrow{\tilde{\iota}_*} & \pi_*(l\wedge THH(l);\mathbf{Z}/p) \\ \downarrow \mu_* & & \downarrow \alpha_* \\ \pi_*(l;\mathbf{Z}/p) & \xrightarrow{\tilde{\iota}_*} & \pi_*(THH(l);\mathbf{Z}/p) \end{array}$$

where the h's denote Hurewicz maps. What is important here are the facts that the compositions $\mu_* \circ h$ and $\alpha_* \circ h$ are identities, so that α_* is a surjection, among other things. We can also localize the lower square in this diagram (but not the upper square!) to obtain the following diagram.

$$\begin{array}{ccc} v_1^{-1}\pi_*(l\wedge l;\mathbf{Z}/p) & \xrightarrow{\tilde{\iota}_*} & v_1^{-1}\pi_*(l\wedge THH(l);\mathbf{Z}/p) \\ \downarrow \mu_* & & \downarrow \alpha_* \\ v_1^{-1}\pi_*(l;\mathbf{Z}/p) & \xrightarrow{\tilde{\iota}_*} & v_1^{-1}\pi_*(THH(l);\mathbf{Z}/p) \end{array}$$

As we have observed that

$$K(1)_*(X) = v_1^{-1}\pi_*(\ell \wedge X; \mathbf{Z}/p)$$

Proposition 5.2 states that the upper arrow here is an isomorphism. As the localization of an epimorphism is an epimorphism, the right hand arrow α_* in the new diagram is an epimorphism, so we conclude that the lower $\tilde{\iota}_*$,

$$\tilde{\iota}_* : v_1^{-1}\pi_*(\ell;\mathbf{Z}/p) \longrightarrow v_1^{-1}\pi_*(THH(\ell);\mathbf{Z}/p),$$

is also surjective.

Now we must prove the lower $\tilde{\iota}_*$ is injective. Suppose that $x \in ker(\tilde{\iota}_*)$. By definition of localization we can find an integer m such that $v_1^m x = x' \in \pi_*(\ell;\mathbf{Z}/p)$. By choosing m larger if necessary we can arrange that

$$\tilde{\iota}_* : \pi_*(\ell;\mathbf{Z}/p) \longrightarrow \pi_*(THH(\ell);\mathbf{Z}/p)$$

carries x' to zero. By commutativity of the upper square of the first diagram,

$$\tilde{\iota}_*(h(x')) = 0,$$

so that injectivity of the localized $\tilde{\iota}_*$ implies there is m' such that

$$v_1^{m'} h(x') = 0.$$

Then

$$\begin{aligned} 0 &= \mu_*(v_1^{m'} h(x')) \\ &= v_1^{m'} \mu_*(h(x')) \\ &= v_1^{m'} x' \end{aligned}$$

in $\pi_*(\ell; \mathbf{Z}/p)$ so that in $v_1^{-1}\pi_*(\ell; \mathbf{Z}/p)$

$$0 = v_1^{m+m'} x.$$

We conclude that $x = 0$, so that

$$\tilde{\iota}_* : v_1^{-1}\pi_*(\ell; \mathbf{Z}/p) \longrightarrow v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p)$$

is also injective. ♣

To prove Proposition 5.2 we need the following lemma.

Lemma 5.3 *The Hochschild homology of $K(1)_*(\ell)$ with respect to the ground ring $\pi_*(K(1))$ is*

$$\mathbf{HH}_i^{\pi_*(K(1))}(K(1)_*(\ell)) = \begin{cases} K(1)_*(\ell) & \text{if } i = 0. \\ 0 & \text{if } i > 0. \end{cases}$$

Proof of 5.2. The inclusion of simplicial spectra $l \longrightarrow THH(l)$ induces a morphism of the spectral sequences for $K(1)_*(\ell)$ and $K(1)_*(THH(\ell))$ arising from the simplicial filtrations. For ℓ the spectral sequence is trivial, and for $THH(\ell)$ we can evaluate the E^2-term according to Proposition 3.1, since we know the homology theory $K(1)$ has a good Künneth theorem [18, page 133]. We find that the E^2-term of the spectral sequence is identified with the Hochschild homology of $K(1)_*(\ell)$ over $\pi_*(K(1))$. Then we have

$$K(1)_*(\ell) \cong E^\infty_{0,*} = E^2_{0,*} \longrightarrow E^2_{i,*} \cong \mathbf{HH}_i^{\pi_*(K(1))}(K(1)_*(\ell)).$$

But, according to the lemma above,

$$\mathbf{HH}_i^{\pi_*(K(1))}(K(1)_*(\ell)) = \begin{cases} K(1)_*(\ell), & \text{if } i = 0 \\ \\ 0, & \text{if } i > 0. \end{cases}$$

Thus we have an isomorphism of spectral sequences at the E^2-level and this fact immediately implies the proposition. ♣

The structural information about the algebra $K(1)_*(\ell)$ which we need to prove Lemma 5.3 is supplied by the following result.

Proposition 5.4 *$K(1)_0(\ell)$ is a direct limit of semisimple $\mathbf{Z}/p$-algebras. In fact,*

$$K(1)_0(\ell) = \lim_{n \geq 0} \prod^{p^n} \mathbf{Z}/p\,,$$

where the $n+1$-st map in the direct system is the p^n-fold power of the diagonal embedding

$$\mathbf{Z}/p \longrightarrow \prod^{p} \mathbf{Z}/p.$$

With this kind of hold on the structure of the algebra $K(1)_*(\ell)$ it is easy to prove Lemma 5.3.

Proof of Lemma 5.3. Recall the fact that Hochschild homology commutes with direct limits, the fact ([8, Theorem 5.3, page 173]) that for k-algebras A and B

$$\mathbf{HH}_*^k(A \times B) \cong \mathbf{HH}_*^k(A) \times \mathbf{HH}_*^k(B),$$

and the elementary computation

$$\mathbf{HH}_i^k(k) = \begin{cases} k, & \text{if } i = 0. \\ 0, & \text{if } i > 0. \end{cases}$$

Since

$$K(1)_*(\ell) = \pi_*(K(1)) \otimes K(1)_0(\ell),$$

the structural results of Proposition 5.4 imply $K(1)_*(\ell)$ is a limit of products of $\pi_*(K(1))$ with itself. Putting these facts together yields the proof of the Lemma immediately. ♣

We will give two proofs of Proposition 5.4, the first being an elementary computation suggested by the methods of [1]. Generally speaking, the background for the first proof is contained in [1, Part III, Chapter 17], where results are proved for the prime two and stated for odd primes. In the years since [1] was published notations and conventions have crystallized in ways that are not always compatible with the original source. The methods we use are adapted from the calculation of $E(1)_*(\ell)$, where $E(1)$ is the homology theory $v_1^{-1}l$ with coefficients $\pi_*(E(1)) = \mathbf{Z}_{(p)}[v_1, v_1^{-1}]$.

First Proof of Proposition 5.4. Note first that

$$K(1)_*(X) \cong E(1)_*(X; \mathbf{Z}/p)$$

so that a normal universal coefficients theorem gives $K(1)_*(X)$ from $E(1)_*(X)$. In our case $E(1)_*(\ell)$ is torsion-free [1, Proposition 17.2, page 354] so that we obtain

$$K(1)_*(\ell) = E(1)_*(\ell)/p\,E(1)_*(\ell).$$

Now we can use Adams' explicit description of $E(1)_*(\ell)$ as a subalgebra of

$$\mathbf{Q}[v_1, v_1^{-1}, w] = E(1)_*(\ell; \mathbf{Q}),$$

where v_1 and w both have degree $2(p-1)$. (At odd primes Adams uses u^{p-1} and v^{p-1} where we will be using v_1 and w, respectively.) Translating [1, Proposition 17.6, page 358] and its extension to odd primes into our notation we find that if one defines polynomials

$$\begin{aligned}
f_0(v_1, w) &= 1,\\
f_1(v_1, w) &= \frac{w - v_1}{(p+1)-1},\\
&\vdots\\
f_r(v_1, w) &= \frac{(w-v_1)}{((rp+1)-1)} \cdot \frac{(w-(p+1)v_1)}{((rp+1)-(p+1))} \cdots \frac{(w-((r-1)p+1)v_1)}{(rp+1)-((r-1)p+1)}\\
&= (p^r r!)^{-1}(w-v_1)(\cdots)(w-((r-1)p+1)v_1)
\end{aligned}$$

then one obtains $E(1)_*(\ell)$ embedded as a subalgebra of $\mathbf{Q}[v_1, v_1^{-1}, w]$ as the free $\mathbf{Z}_{(p)}[v_1, v_1^{-1}]$ module on $\{f_0, f_1, \ldots, f_r, \ldots\}$. By using techniques of the proof of this result one can figure out the structure of the ring $K(1)_*(\ell)$.

First we will prove the identity

$$f_r \cdot f_s = \sum_{i=0}^{r} \binom{s}{r-i}\binom{s+i}{s} v_1^{r-i} f_{s+i}$$

in $E(1)_*(\ell)$ for $r \leq s$. Once we know the special case

$$f_1 \cdot f_r = \binom{r}{1} v_1 f_r + \binom{r+1}{r} f_{r+1}$$

the general case will be proved by induction on r for fixed s.

To prove the special case, we note, following [1], that $f_1 \cdot f_r$ is homogeneous of total degree $2(r+1)(p-1)$, so it has an expansion

$$f_1 \cdot f_r = c_0 v_1^{r+1} + c_1 v_1^r f_1 + \cdots + c_r v_1 f_r + c_{r+1} f_{r+1},$$

where $c_i \in \mathbf{Z}_{(p)}$. ¿From the definition of f_r it is clear that

$$f_r(1, sp+1) = \binom{s}{r},$$

where the binomial coefficient is interpreted as 0 if $s < r$, so that successively substituting $v_1 = 1, w = sp + 1$ for $0 \le s \le r + 1$, we can determine the coefficients c_i. The answers are

$$c_0 = \cdots = c_{r-1} = 0,$$

and

$$\begin{aligned} c_r &= \binom{r}{1}, \\ c_{r+1} &= \binom{r+1}{r}. \end{aligned}$$

To prove the general case, suppose that for $r < s$ we have

$$f_r \cdot f_s = \sum_{i=0}^{r} \binom{s}{r-i} \binom{s+i}{s} v_1^{r-i} f_{s+i}.$$

Then

$$\begin{aligned} \binom{r}{1} v_1 f_r f_s + \binom{r+1}{r} f_{r+1} f_s &= (f_1 f_r) f_s \\ &= f_1 (f_r f_s) \\ &= \sum_{i=0}^{r} \binom{s}{r-i} \binom{s+i}{s} v_1^{r-i} f_1 f_{s+i} \end{aligned}$$

Hence

$$\begin{aligned} \binom{r+1}{r} f_{r+1} f_s &= \sum_{i=0}^{r} \binom{s}{r-i} \binom{s+i}{s} v_1^{r-i} f_1 f_{s+i} - \binom{r}{1} v_1 f_r f_s \\ &= \sum_{i=0}^{r} \binom{s}{r-i} \binom{s+i}{s} v_1^{r-i} \left[\binom{s+i}{1} v_1 f_{s+i} + \binom{s+i+1}{s+i} f_{s+i+1} \right] \\ &\quad - \binom{r}{1} v_1 \left(\sum_{j=0}^{r} \binom{s}{r-j} \binom{s+j}{s} v_1^{r-j} f_{s+j} \right) \\ &= \binom{r+1}{r} \sum_{i=0}^{r+1} \binom{s}{r+1-i} \binom{s+i}{s} v_1^{r+1-i} f_{s+i} \end{aligned}$$

after patient fiddling with the binomial coefficients, and we can cancel the extra factor to get the identity we want.

Using the mod-p reduction of this identity we can prove that the reduction of

$$\{f_1, f_p, \ldots, f_{p^i}, \ldots\}$$

generates $K(1)_*(\ell)$. To prove f_s is a polynomial in the f_{p^i} for $p^i \leq s$ suppose that the p-adic expansion of s begins

$$s = s_j p^j + s_{j+1} p^{j+1}$$

where $1 \leq s_j \leq p-1$ and $s_{j+1} \geq 0$. If $s - p^j = 0$, there is nothing to do, so suppose that $s - p^j > 0$. By our identity

$$f_{p^j} \cdot f_{s-p^j} = \sum_{i=0}^{p^j} \binom{s-p^j}{p^j - i} \binom{s-p^j+i}{s-p^j} v_1^{p^j - i} f_{s-p^j+i}$$

so that

$$\binom{s}{p^j} f_s = f_{p^j} \cdot f_{s-p^j} - \sum_{i=0}^{p^j-1} \binom{s-p^j}{p^j - i} \binom{s-p^j+i}{s-p^j} v_1^{p^j - i} f_{s-p^j+i}.$$

But the coefficient of f_s has nonzero reduction mod-p, since it is a p-adic unit by [13, page 115], [9, page 270], [20], or by hand, so induction on s proves the assertion. Thus we have a surjection

$$m : \mathbf{Z}/p\,[v_1, v_1^{-1}] \otimes \mathbf{Z}/p\,[f_1, f_p, \ldots] \longrightarrow K(1)_*(\ell).$$

Now we determine the relations satisfied by the f_{p^j} in $K(1)_*(\ell)$ and get generators for the kernel of m following similar strategy. In $\mathbf{Q}[v_1, v_1^{-1}, w]$ we have

$$\begin{aligned} (f_{p^j})^p &= c_0 v_1^{p^{j+1}} + c_1 v_1^{p^{j+1}-1} f_1 + \cdots + \\ &\quad c_{p^j} v_1^{p^{j+1}-p^j} f_{p^j} + \cdots + c_{p^{j+1}} f_{p^{j+1}} \end{aligned}$$

Substituting $v_1 = 1$, and then successively $w = (sp+1)$ for $0 \leq s \leq p^j$ we get

$$0 = c_0 = \cdots = c_{p^j-1}, \; 1 = c_{p^j}.$$

Substituting $v_1 = 1$, and then successively $w = (sp+1)$ for $p^j + 1 \leq s \leq p^{j+1}$ we find

$$\binom{s}{p^j}^p = \binom{s}{p^j} + c_{p^j+1} \binom{s}{p^j+1} + \cdots + c_s$$

so that $p | c_s$ for s in this range, by induction on s. Thus, the relation satisfied by the residue class $f_{p^j} \in K(1)_*(\ell)$ is

$$(f_{p^j})^p = v_1^{p^{j+1}-p^j} f_{p^j}.$$

Introduce the degree zero Laurent polynomial

$$g_{p^j} = v_1^{-p^j} f_{p^j}$$

and the relation takes the form

$$(g_{p^j})^p = g_{p^j}.$$

We can finally state that m induces a surjection

$$m : \mathbf{Z}/p\,[v_1, v_1^{-1}][g_{p^j} :\ j \geq 0]/(g_{p^j}^p - g_{p^j} :\ j \geq 0) \longrightarrow K(1)_*(\ell)$$

and we claim this is an isomorphism. It suffices to consider the restriction

$$m_0 : \mathbf{Z}/p\,[g_{p^j} :\ j \geq 0]/(g_{p^j}^p - g_{p^j} :\ j \geq 0) \longrightarrow K(1)_0(\ell).$$

We suppose $x \in ker(m_0)$. Then there is an $n \geq 0$ such that

$$x \in A_n = \mathbf{Z}/p\,[g_{p^j} :\ n \geq j \geq 0]/(g_{p^j}^p - g_{p^j})$$

which is a $\mathbf{Z}/p$-vector space of dimension p^{n+1}. The image $m_0(A_n)$ is the subspace of $K(1)_0(\ell)$ spanned by $\{f_0 = 1, \ldots, v_1^{-p^{n+1}+1} f_{p^{n+1}-1}\}$. This space supports p^{n+1} independent functionals ϕ_s where

$$\phi_0(f_i) = \delta_{i0}$$

and

$$\phi_s(f) = f(1, sp+1) \bmod p$$

for $1 \leq s \leq p^{n+1} - 1$, so it is also of dimension p^{n+1}. It follows that $x = 0$. Hence $ker(m_0) = 0$. Since we retrieve m by tensoring with $\mathbf{Z}/p\,[v_1, v_1^{-1}]$, m is also an isomorphism.

Since it is well known that

$$\mathbf{Z}/p\,[x]/(x - x^p) \cong \prod_{i=1}^{p} \mathbf{Z}/p,$$

the isomorphisms m_0 and m can be unwound to give the rest of the proof. ♣

Our second proof of Proposition 5.4 follows a suggestion of M.J. Hopkins that a proof could be extracted from a result of Miller and Ravenel [17] in stable homotopy theory. We follow the exposition in [18] between pages 223 and 226. Obviously the prerequisites for this proof are much more demanding than those for the first proof, but we can also prove more.

Second Proof of Proposition 5.4. In the language of BP-theory, we want to calculate $K(1)_*(BP\langle 1\rangle)$, and it is not much more work to calculate all the rings

$$K(n)_*(BP\langle n\rangle) = v_n^{-1} k(n)_*(BP\langle n\rangle)$$

for all $n \geq 1$. The coefficient rings of the spectra here are

$$BP\langle n\rangle_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n]$$

where degree $v_k = 2(p^k - 1)$,

$$k(n)_* = \mathbf{Z}/p\,[v_n],$$

and

$$K(n)_* = \mathbf{Z}/p\,[v_n, v_n^{-1}].$$

$BP\langle n\rangle$ may be constructed from the spectrum BP,

$$BP_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n, \ldots],$$

by a Baas-Sullivan construction

$$BP \longrightarrow BP\langle n\rangle$$

which realizes the quotient map

$$\mathbf{Z}_{(p)}[v_1, \ldots, v_n, \ldots] \longrightarrow \mathbf{Z}_{(p)}[v_1, \ldots, v_n, \ldots]/(v_{n+1}, v_{n+2}, \ldots)$$

on coefficients. One has that

$$BP_*BP = BP_*[t_1, t_2, \ldots]$$

so that the dual form of one of the universal coefficients theorems for generalized homology theories [1, Proposition 13.5, page 285] gives

$$\begin{aligned} k(n)_*(BP) &\cong k(n)_* \otimes_{BP_*} BP_*BP \\ &= \mathbf{Z}/p\,[v_n][t_1, t_2, \ldots] \end{aligned}$$

and

$$\begin{aligned} k(n)_*(BP\langle n\rangle) &\cong k(n)_* \otimes_{BP_*} BP_*(BP\langle n\rangle) \\ &= \mathbf{Z}/p\,[v_n][t_1, t_2, \ldots]/(\eta_R(v_{n+1}), \eta_R(v_{n+2}), \ldots), \end{aligned}$$

by definition of $BP\langle n\rangle$ and by definition of the right unit η_R. Now we have also for $k \geq 1$ the congruence

$$\eta_R(v_{n+k}) \equiv v_n t_k^{p^n} - v_n^{p^k} t_k \text{ modulo } (\eta_R(v_{n+1}), \ldots, \eta_R(v_{n+k-1}))$$

in $k(n)_*(BP)$ [18, page 224]. It follows that

$$k(n)_*(BP\langle n\rangle) \cong \mathbf{Z}/p\,[v_n][t_1, t_2, \ldots]/(v_n t_k^{p^n} - v_n^{p^k} t_k : k \geq 1)$$

and

$$K(n)_*(BP\langle n\rangle) \cong \mathbf{Z}/p\,[v_n, v_n^{-1}][t_1, t_2, \ldots]/(v_n t_k^{p^n} - v_n^{p^k} t_k : k \geq 1).$$

When $n = 1$ we can introduce new variables of degree 0

$$u_1 = v_1^{-1} t_1, \ldots, u_k = v_1^{-p^{k-1} - \cdots - 1} t_k, \ldots,$$

whereupon the presentation changes to

$$K(1)_*(BP\langle 1\rangle) \cong \mathbf{Z}/p\,[v_1, v_1^{-1}][u_1, u_2, \ldots]/(u_k^p - u_k : k \geq 1),$$

which is what we had before. ♣

6 The localized Adams spectral sequence

This section contains the last of the preliminary computations we need before we compute $\pi_*(THH(\ell); \mathbf{Z}/p)$ with the classical Adams spectral sequence, denoted $E_r(THH(\ell); \mathbf{Z}/p)$ in section **??**. ¿From here on this is the only spectral sequence we will be dealing with, so for the rest of the paper we simplify the notation to just E_r. In theorem 4.1 we computed the E_2-term of the spectral sequence and the result is

$$\begin{aligned} E_2^{*,*} &= \operatorname{Ext}_{A_*}^{*,*}(\mathbf{Z}/p, H\mathbf{Z}/p_*(THH(l) \wedge \mathbf{M})), \\ &\cong \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu], \end{aligned}$$

where bidegree $a_1 = (2(p-1), 1)$, bidegree $\lambda_i = (2p^i - 1, 0)$, and bidegree $\mu = (2p^2, 0)$. As a_1 is a permanent cycle, this is a spectral sequence of $\mathbf{Z}/p\,[a_1]$-algebras, so we can invert a_1 to obtain a spectral sequence of $\mathbf{Z}/p\,[a_1, a_1^{-1}]$ algebras, which we will denote $a_1^{-1}E_r$. Though we make no assertion concerning the convergence of the localized spectral sequence, the isomorphism

$$v_1^{-1}\pi_*(\ell; \mathbf{Z}/p) \cong v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p)$$

established in Theorem 5.1 determines $a_1^{-1}E_\infty$. Since we have inverted only bihomogeneous elements $a_1^{-1}E_r$ is still bigraded, and this extra fact enables us to identify the pattern of differentials which gives the required E_∞-term.

Theorem 6.1 *In the localized spectral sequence $a_1^{-1}E_r$ one has*

$$a_1^{-1}E_2 \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu]$$

and

$$a_1^{-1}E_\infty \cong \mathbf{Z}/p\,[a_1, a_1^{-1}].$$

The pattern of differentials may be described recursively as follows: The n-th nonzero differential occurs in

$$a_1^{-1}E_{r(n)} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}]$$

and is given by the formulas

$$d_{r(n)}(\lambda_n) = 0,\ d_{r(n)}(\lambda_{n+1}) = 0,\ \text{and}\ d_{r(n)}(\mu^{p^{n-1}}) \doteq a_1^{r(n)}\lambda_n.$$

The $r(n)$ are given recursively by the definitions

$$r(1) = p,\ r(2k) = p\,r(2k-1),\ \text{and}\ r(2k+1) = p\,r(2k) + p,$$

and the λ_n are defined for $n \geq 3$ by $\lambda_n = \lambda_{n-2}\mu^{p^{n-3}(p-1)}$.

We use the symbol $\doteq$ for equality up to a multiple by a nonzero element of $\mathbf{Z}/p$.

For example, this says that the first few steps in the spectral sequence are

$$E_p \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu],$$

when $d_p(\mu) \doteq a_1^p \lambda_1$;

$$E_{p^2} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_2, \lambda_1 \mu^{p-1}) \otimes \mathbf{Z}/p\,[\mu^p],$$

when $d_{p^2}(\mu^p) \doteq a_1^{p^2} \lambda_2$;

$$E_{p^3+p} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1 \mu^{p-1}, \lambda_2 \mu^{p^2-p}) \otimes \mathbf{Z}/p\,[\mu^{p^2}],$$

when $d_{p^3+p}(\mu^{p^2}) \doteq a_1^{p^3+p} \lambda_1 \mu^{p-1}$; and

$$E_{p^4+p^2} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_2 \mu^{p^2-p}, \lambda_1 \mu^{p^3-p^2+p-1}) \otimes \mathbf{Z}/p\,[\mu^{p^3}],$$

when $d_{p^4+p^2}(\mu^{p^3}) \doteq a_1^{p^4+p^2} \lambda_2 \mu^{p^2-p}$.

Proof. As we have said in the opening remarks, we will show that the isomorphism

$$v_1^{-1}\pi_*(\ell; \mathbf{Z}/p) \cong v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p)$$

of Theorem 5.1 determines the E_∞-term of the spectral sequence and that there is exactly one way to obtain this E_∞-term from the given E_2-term.

According to Theorem 5.1,

$$\tilde{\iota}\, : \ell \longrightarrow THH(\ell)$$

induces an isomorphism

$$\tilde{\iota_*} : v_1^{-1}\pi_*(\ell; \mathbf{Z}/p) \xrightarrow{\cong} v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p).$$

Therefore we have a commutative diagram

$$\begin{array}{ccc}
\pi_*(l; \mathbf{Z}/p) & \longrightarrow & \pi_*(THH(l); \mathbf{Z}/p) \\
\downarrow & & \downarrow \\
v_1^{-1}\pi_*(l; \mathbf{Z}/p) & \xrightarrow{\cong} & v_1^{-1}\pi_*(THH(l); \mathbf{Z}/p),
\end{array}$$

where the vertical arrows are the localization maps. As the righthand localization map is injective, we conclude that

$$\tilde{\iota_*} : \pi_*(\ell; \mathbf{Z}/p) \longrightarrow \pi_*(THH(\ell); \mathbf{Z}/p)$$

is injective.

In section **??** we showed that

$$\tilde{\imath} \, : \ell \longrightarrow THH(\ell)$$

induces a map of Adams spectral sequences, which may be identified with the canonical algebra inclusion

$$\mathbf{Z}/p\,[a_1] \longrightarrow \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu]$$

on E_2-terms. The domain spectral sequence has no differentials, so we have

$$\mathbf{Z}/p\,[a_1] \longrightarrow E_\infty^{*,*},$$

and this map is injective, since we have just observed that the map on homotopy is. Now we use the surjectivity of the induced map on localized homotopy. This implies that if $x \in \pi_*(THH(\ell); \mathbf{Z}/p)$, then there is a natural number N such that $v_1^N x$ lies in the image of $\pi_*(\ell; \mathbf{Z}/p)$ in $\pi_*(THH(\ell); \mathbf{Z}/p)$. Multiplication by v_1 in homotopy is represented by multiplication by a_1 in the Adams spectral sequence [1, Lemma 17.11(ii), page 361], so we can draw conclusions about the spectral sequence E_r. Take an infinite cycle $\xi \in E_\infty^{n,s}$, not already in $\mathbf{Z}/p\,[a_1]$, (Thus $2(p-1) < n$.), representing $x \in \pi_n(THH(\ell); \mathbf{Z}/p)$, which is itself not in the image of $\pi_n(\ell; \mathbf{Z}/p)$. If $v_1^N x$ lies in the image of $\pi_*(l; \mathbf{Z}/p)$, then $a_1^N \xi = 0$ in the spectral sequence. This is not saying that the homotopy element represented by ξ becomes zero when multiplied by v_1^N, but rather that a jump in the Adams filtration degrees has occurred as we multiplied by higher and higher powers of v_1. All we can say is that $v_1^N x$ is now represented by an infinite cycle in $E_\infty^{n',s'}$, where now $2(p-1)s' = n'$. We believe that the reader will find it helpful to sketch the spectral sequence at this point. These remarks are then most conveniently expressed in terms of the localized spectral sequence $a_1^{-1} E_r$ as the assertion

$$a_1^{-1} E_\infty^{*,*} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}].$$

Now we claim that the pattern of differentials which delivers this E_∞-term can only be that pattern described in the statement of the theorem. Before we argue this point, recall that we have already mentioned the fact that the localized spectral sequence retains its bigrading, since the multiplicative set

$$S = \{a_1^i : i \geq 0\}$$

of elements we have inverted consists of bihomogeneous elements. One checks easily that defining the bidegree of the fraction m/a_1^i to be $(n - 2i(p-1), s - i)$, if the bidegree of m is (n, s), leads to a well defined notion of bidegree in the localized spectral sequence. Most importantly, we still have

$$d_r : a_1^{-1} E_r^{n,s} \longrightarrow a_1^{-1} E_r^{n-1,s+r}.$$

Start with the localized E_2-term

$$a_1^{-1}E_2 \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu],$$

where the bidegrees of a_1, λ_i, and μ are $(2(p-1), 1)$, $(2p^i - 1, 0)$ and $(2p^2, 0)$, respectively. Since the bidegree of d_r is $(-1, r)$, λ_1 is an infinite cycle. Considering the bigrading again, we see that $d_r(\lambda_2) = 0$, for all r, except possibly when $r = p+1$. But if $d_{p+1}(\lambda_2)$ is not zero, we have $d_{p+1}(\lambda_2) \doteq a_1^{p+1}$, where $\doteq$ means equal up to a nonzero scalar factor, which is not consistent with the expression for the E_∞-term. Therefore λ_2 is also an infinite cycle. Since we have a spectral sequence of algebras, we can conclude that μ is not an infinite cycle, and that the first nonzero differential in the spectral sequence is determined by what it does on μ. By bigrading considerations the only possible nonzero differential on μ is

$$d_p(\mu) \doteq a_1^p \lambda_1.$$

Then one easily calculates

$$a_1^{-1}E_{p+1} = \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_2, \lambda_3) \otimes \mathbf{Z}/p\,[\mu^p],$$

where $\lambda_3 = \lambda_1 \mu^{p-1}$, as in the statement of the theorem. Notice that the bidegrees of the new $\mathbf{Z}/p\,[a_1, a_1^{-1}]$-algebra generators are $(2p^2-1, 0)$, $(2p^3 - 2p^2 + 2p - 1, 0)$, and $(2p^3, 0)$.

We have already argued that λ_2 is an infinite cycle, and a calculation of bidegrees shows there is one possible differential which could be nonzero on λ_3, namely, $d_{p^2+1}(\lambda_3) \doteq a_1^{p^2+1}$. But this is also ruled out, since it would contradict what we know about the E_∞-term. Thus λ_3 is an infinite cycle, and, therefore, μ^p cannot be an infinite cycle, and we need to determine which differential takes μ^p out of the spectral sequence.

Since $r \geq p+1$, $d_r(\mu^p) = a_1^r \lambda_3$ is impossible, and the only possibility is that for some r

$$d_r(\mu^p) \doteq a_1^r \lambda_2.$$

Calculating bidegrees, we find that this can only happen, and, in fact, must happen, when $r = p^2$.

Proceeding inductively, assume that the first $n-1$ differentials in the localized spectral sequence occur at the times specified in the theorem and are given by the formulas. Then we find that

$$a_1^{-1}E_{r(n-1)+1} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}],$$

The argument that the next differential is $d_{r(n)}$ follows the same pattern as the argument given for the second differential. One will have to note that the bidegree of λ_m is $(2p^m - 2p^{m-1} + \cdots + 2p - 1, 0)$ if m is odd, and $(2p^m - 2p^{m-1} + \cdots + 2p^2 - 1, 0)$, if m is even. Also one will need to know that

$$r(n) = p^n + p^{n-2} + \cdots + p,$$

if n is odd, and

$$r(n) = p^n + p^{n-2} + \cdots + p^2,$$

if n is even. By inductive hypothesis one knows that λ_n is an infinite cycle. Then one argues that the only way λ_{n+1} could fail to be an infinite cycle is if $d_{r(n)}(\lambda_{n+1}) \doteq a_1^{r(n)+1}$, which is impossible since $E_\infty = \mathbf{Z}/p\,[a_1, a_1^{-1}]$. These two facts imply that for some $r > r(n-1)$, $d_r(\mu^{p^{n-1}}) \neq 0$, and calculating bidegrees one more time one sees the only possibility is

$$d_{r(n)}(\mu^{p^{n-1}}) \doteq a_1^{r(n)}\lambda_n.$$

This completes the proof of the theorem. ♣

7 The mod p homotopy of $THH(\ell)$

In the following theorem we show how the differentials in the localized Adams spectral sequence determine the differentials in the Adams spectral sequence itself. One of the interesting things about this example is that the spectral sequence has infinitely many non-zero differentials, and that we can determine them all. Notice that if one removes the explicit arithmetic from the proof below one has a general result on the Adams spectral sequence for the mod-p homotopy of an ℓ-module spectrum X. This result is stated at the end of the section.

Theorem 7.1 *Let E_r be the Adams spectral sequence convergent to the mod p homotopy of $THH(\ell)$. Then the nonzero differentials in E_r are exactly the $d_{r(n)}$, where the function $r(n)$ is as in Theorem 6.1.*

Moreover, the E_∞-term is a direct sum of cyclic $\mathbf{Z}/p\,[a_1]$-modules with the following generators. $1 \in E_\infty^{0,0}$ generates a free $\mathbf{Z}/p\,[a_1]$ module summand of E_∞, and all the generators of $\mathbf{Z}/p\,[a_1]$-torsion summands of E_∞ arise in the following way.

If the p-adic expansion of $m > 0$ starts $m = m_{n-1}p^{n-1} + m_n p^n$, where $p-1 > m_{n-1} > 0$ and $m_n \geq 0$, then

$$\lambda_n \mu^{p^{n-1}(m_{n-1}-1)} \mu^{p^n m_n} \text{ and } \lambda_{n+1}\lambda_n \mu^{p^{n-1}(m_{n-1}-1)} \mu^{p^n m_n}$$

are generators of the summands of E_∞ isomorphic to $\mathbf{Z}/p\,[a_1]/(a_1^{r(n)})$. There is no torsion present other than $a_1^{r(n)}$-torsion for every n.

Proof. Put $r(0) = 1$, put $S_0 = \{\ \}$, and take the other notation from the preceding theorem. $P(n)$ is the following statement:

There is a set

$$S_n \subset E_{r(n)+1}^{*,0}$$

such that the $\mathbf{Z}/p\,[a_1]$-module $E^{*,*}_{r(n)+1}$ may be decomposed as an internal direct sum

$$E^{*,*}_{r(n)+1} = \langle S_n \rangle \oplus \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_{n+1}, \lambda_{n+2}) \otimes \mathbf{Z}/p\,[\mu^{p^n}],$$

where $\langle S_n \rangle$ is the $\mathbf{Z}/p\,[a_1]$-module generated by S_n. For each $x \in S_n$ we have

$$a_1^{r(n)} x = 0$$

so that

$$\langle S_n \rangle \subset \oplus_{0 \le s < r(n)} E^{*,s}_{r(n)+1}.$$

Concerning the differentials in the spectral sequence, $E_{r(n)+1} = E_{r(n+1)}$, and $d_{r(n+1)}$ is determined by the formulas

$$\begin{aligned} d_{r(n+1)}(x) &= 0 \text{ for } x \in S_n \cup \{\lambda_{n+1}, \lambda_{n+2}\} \\ d_{r(n+1)}(\mu^{p^n}) &= a_1^{r(n+1)} \lambda_{n+1}. \end{aligned}$$

As one proves the statements $P(n)$, one reexamines each inductive step in the proof and unwinds what has happened to get the decomposition of E_∞ into cyclic $\mathbf{Z}/p\,[a_1]$-modules.

Remark: It will be clear from the proof of $P(n)$ that $\langle S_n \rangle$ is also an ideal of $E^{*,*}_{r(n)+1}$, and that the elements of S_n are all of square zero. A second glance shows that there are many more multiplicative relations satisfied by the elements of S_n so that the $E_{r(n)+1}$-term is complicated as an algebra. The algebra structure on E_∞ is therefore very complicated, and we have not tried to explicitly determine it.

The part of the statement $P(0)$ concerning the algebraic structure of the E_2-term of the spectral sequence is the content of theorem 4.1, so we have to discuss the differential. As E_2 is a_1-torsion free, the natural map

$$E_2 \longrightarrow a_1^{-1} E_2$$

is injective and preserves the bigrading, so the first nonzero differential in the domain is determined by the first nonzero differential in the target. Referring to the preceding theorem we get that the first nonzero differential in the Adams spectral sequence is $d_{r(1)}$ and that the following formulas are valid:

$$d_{r(1)}(\lambda_1) = 0, \; d_{r(1)}(\lambda_2) = 0, \text{ and } d_{r(1)}(\mu) = a_1^{r(1)} \lambda_1.$$

This proves $P(0)$.

To get the feel for the induction step, let us work out the proof of $P(1)$ with some care. Recall that $r(1) = p$, so that by $P(0)$ we may calculate the differential on a typical element

$$\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^{pm_1+m_0} \in E_p^{*,0},$$

where $\epsilon_i = 0$ or 1 and $0 \le m_0 \le p-1$, obtaining

$$d_p(\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^{pm_1+m_0}) \doteq \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^{pm_1}(m_0\mu^{m_0-1}a_1^p\lambda_1),$$

Then it is clear that $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^{pm_1+m_0}$ is a cycle if $\epsilon_1 = 1$, or if $m_0 = 0$. We also see that the $\mathbf{Z}/p\,[a_1]$-generators of the a_1-torsion submodule of $E_{p+1}^{*,*}$ may be taken to be

$$S_1 = \{\lambda_2^{\epsilon_2}\lambda_1\mu^{pm_1}, \lambda_2^{\epsilon_2}(\lambda_1\mu)\mu^{pm_1}, \ldots, \lambda_2^{\epsilon_2}(\lambda_1\mu^{p-2})\mu^{pm_1}, \text{where } m_1 \ge 0\} \subset E_{p+1}^{*,0}$$

In fact, in E_{p+1} $a_1^p x = 0$ for each $x \in S_1$, so that

$$\langle S_1\rangle \subset \oplus_{0\le s<p} E_{p+1}^{*,s},$$

since a_1 has bidegree $(2(p-1), 1)$. Cycles which generate the $\mathbf{Z}/p\,[a_1]$-free part of E_{p+1} are of the form

$$z = \lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}(\mu^p)^{m_1},$$

where $\lambda_3 = \lambda_1\mu^{p-1}$, $\epsilon_i = 0$ or 1, and $m_1 \ge 0$. This gives us our $\mathbf{Z}/p\,[a_1]$-module decomposition

$$E_{p+1}^{*,*} = \langle S_1\rangle \oplus \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_2, \lambda_3) \otimes \mathbf{Z}/p\,[\mu^p]$$

Now we can consider the next differential in the spectral sequence. The submodule $\langle S_1\rangle$ complementary to the subalgebra

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_2, \lambda_3) \otimes \mathbf{Z}/p\,[\mu^p]$$

is a torsion module, and the subalgebra is torsion-free. Subsequent differentials are the d_r for $r > p$, have bidegree $(-1, r)$, and are derivations of $\mathbf{Z}/p\,[a_1]$-algebras. Since

$$\langle S_1\rangle \subset \oplus_{0\le s<p} E_{p+1}^{*,s},$$

so that there are no nonzero elements of $\langle S_1\rangle$ of filtration degree greater than p, and since algebra demands that $d_r(\langle S_1\rangle) \subset \langle S_1\rangle$, we find that the elements of $\langle S_1\rangle$ are cycles for d_r for every $r > p$. Algebraic reasons do not rule out the possibility that the next nonvanishing differential on an element of the complementary subalgebra

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_2, \lambda_3) \otimes \mathbf{Z}/p\,[\mu^p]$$

has a component in $\langle S_1\rangle$, but the filtration argument does in fact rule this out. Thus $\langle S_1\rangle$ is a $\mathbf{Z}/p\,[a_1]$-submodule of E_∞ annihilated by a_1^p. We will show that the higher differentials introduce higher a_1-torsion so that S_1 is a set of $\mathbf{Z}/p\,[a_1]$-module generators for the summands of E_∞ isomorphic to $\mathbf{Z}/p\,[a_1]/(a_1^p)$ as claimed in the theorem. Also, we have concluded that the next nonvanishing differential on

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_2, \lambda_3) \otimes \mathbf{Z}/p\,[\mu^p]$$

has only a torsion-free component, so it is detected in the localized spectral sequence. Putting this discussion together with the preceding theorem we obtain the rest of $P(1)$, namely, that $E_{p+1} = E_{r(2)}$, and that the formulas

$$d_{r(2)}(x) = 0 \text{ for } x \in S_1 \cup \{\lambda_2, \lambda_3\}$$

and

$$d_{r(2)}(\mu^p) = a_1^{r(2)}\lambda_2$$

determine the next differential.

The argument for deducing $P(n)$ from $P(n-1)$ is in outline exactly the same as the argument we have just finished, so we will just quickly repeat the steps. According to $P(n-1)$ we have the decomposition

$$E^{*,*}_{r(n)} = \langle S_{n-1}\rangle \oplus \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}],$$

and we can calculate $E_{r(n)+1}$ from $P(n-1)$, first deriving the formula

$$d_{r(n)}(\lambda_n^{\epsilon_n}\lambda_{n+1}^{\epsilon_{n+1}}\mu^{p^n m_n + p^{n-1}m_{n-1}}) \doteq \lambda_n^{\epsilon_n}\lambda_{n+1}^{\epsilon_{n+1}}\mu^{p^n m_n} m_{n-1}(\mu^{p^{n-1}})^{m_{n-1}-1}(a_1^{r(n)}\lambda_n),$$

where $\epsilon_i = 0$ or 1, $0 \le m_{n-1} \le p-1$, and $m_n \ge 0$. One finds that the a_1-torsion submodule of $E^{*,*}_{r(n)+1}$ is spanned by

$$\begin{aligned} S_n = {} & S_{n-1} \cup \\ & \{\lambda_{n+1}^{\epsilon_{n+1}}\lambda_n\mu^{p^n m_n}, \lambda_{n+1}^{\epsilon_{n+1}}(\lambda_n\mu^{p^{n-1}})\mu^{p^n m_n}, \ldots, \lambda_{n+1}^{\epsilon_{n+1}}(\lambda_n\mu^{p^{n-1}(p-2)})\mu^{p^n m_n}, \text{where } m_n \ge 0\} \\ & \subset E^{*,0}_{r(n)+1} \end{aligned}$$

and that each $x \in S_n - S_{n-1}$ generates a submodule isomorphic to $\mathbf{Z}/p\,[a_1]/(a_1^{r(n)})$. This gives us

$$\langle S_n\rangle \subset \oplus_{0\le s<r(n)} E^{*,s}_{r(n)+1}$$

The complementary a_1-torsion-free submodule is seen to be the algebra

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_{n+1}, \lambda_{n+2}) \otimes \mathbf{Z}/p\,[\mu^{p^n}],$$

where $\lambda_{n+2} = \mu^{p^{n-1}(p-1)}\lambda_n$. Then one argues with the filtration of $\langle S_n\rangle$ and the bidegree of d_r that the next differential is determined in the localized spectral sequence. By reference to theorem 6.1 we obtain $E_{r(n)+1} = E_{r(n+1)}$ and the desired formulas for $d_{r(n+1)}$, completing the proof of $P(n)$.

Thus, by the stage E_∞ the torsion-free subalgebra has been reduced to $\mathbf{Z}/p\,[a_1]$ and one has seen that the inductive steps from $P(n-1)$ to $P(n)$ present E_∞ as a sum of cyclic modules over $\mathbf{Z}/p\,[a_1]$ of the required a_1-order. ♣

References

[1] Adams, J.F. *Stable homotopy and generalized homology.* University of Chicago Press, 1974.

[2] Araki, S. and Toda, H. *Multiplicative structures in mod q cohomology theories, I.* Osaka J. Math. **2**(1965), 71-115.

[3] Bökstedt, M. *Topological Hochschild homology.* Preprint, 1987.

[4] Bökstedt, M. *The topological Hochschild homology of* $\mathbf{Z}$ *and* $\mathbf{Z}/p$. Preprint, 1987.

[5] Bökstedt, M. *The natural transformation from* $K(\mathbf{Z})$ *to* $THH(\mathbf{Z})$. Preprint, 1987.

[6] Bökstedt, M. and Waldausen, F. *The map* $BSG \longrightarrow A(*) \longrightarrow QS^0$.

[7] Bruner, R.R. *The homotopy theory of* H_∞ *ring spectra.* Lecture Notes in Mathematics, Vol 1176, 88-128. Springer, 1986.

[8] Cartan, H. and Eilenberg, S. *Homological algebra.* Princeton University Press, 1956.

[9] Dickson, L.E. *History of the theory of numbers, Vol. I.* Carnegie Insitute of Washington, 1919.

[10] Elmendorf, A.D. *Foundations of topological Hochschild homology.* In preparation.

[11] Goodwillie, T. *Cyclic homology, derivations, and the free loopspace.* Topology **24**(1985), 187-215.

[12] Kane, Richard. Memoir of the AMS.

[13] Kummer, D. *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen.* Journal für die Reine und Angewandte Mathematik **44**(1852), 93-146.

[14] Lang, S. *Algebra.* Addison-Wesley, 1965.

[15] May, J.P. *Simplicial objects in algebraic topology.* Van Nostrand, 1967.

[16] May, J.P. *Geometry of iterated loop spaces.* Lecture Notes in Mathematics, Vol 271. Springer, 1972.

[17] Miller, H.R. and Ravenel, D.C. *Morava stabilizer algebras and the localization of Novikov's* E_2*-term.* Duke Math.J.**44** (1977), 433–447.

[18] Ravenel, D.C. *Complex cobordism and stable homotopy groups of spheres.* Academic Press, 1986.

[19] Robinson, A. *Derived tensor products in stable homotopy theory.* Topology **22**(1983), 1-18.

[20] Singmaster, D. *Divisibility of binomial and multinomial coefficients by primes and prime powers.* A collection of manuscripts related to the Fibonacci sequence, 18th anniversary volume (V. Hoggatt, Jr. and M. Bicknell-Johnson, eds), The Fibonacci Association, Santa Clara, California, 1980, 98–113.

[21] Steinberger, M. *Homology operations for H_∞ and H_n ring spectra.* Lecture Notes in Mathematics, Vol 1176, 56-87. Springer, 1986.

[22] Waldhausen, F. *Algebraic K-theory of spaces, a manifold approach.*

[23] Waldhausen, F. *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy.*

[24] Waldhausen, F. *Algebraic K-theory of spaces, II.*

[25] unexplained. *AA*

[26] unexplained. *II*

[27] unexplained. *GG*

[28] unexplained. *WW*

[29] unexplained. *MM*

[30] unexplained. *SV*

[31] unexplained. *Jones*

[32] unexplained. *Wlecture*

[33] unexplained. *GoCal*

[34] unexplained. *GoLet*

[35] unexplained. *BHM*

[36] unexplained. *RavAmJ*

[37] unexplained. *Einfty*

[38] unexplained. *BrownPet*

[39] unexplained. E_2 *term*