The Differential Calculus of Homotopy Functors

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0. Introduction

My purpose here is to explain a method in homotopy theory. The following result is perhaps the best example to date of a statement that can be proved by this method:

Theorem 1. *For any 2-connected map of topological spaces $Y \to X$ the fiber of $A(Y) \to A(X)$ and the fiber of $TC(Y) \to TC(X)$ are weakly homotopy equivalent.*

Here A is Waldhausen's algebraic K-theory functor from spaces to spectra, and TC is another functor which I will discuss below. "Fiber" means homotopy fiber.

If we write $\tilde{A}$ for the reduced functor $\tilde{A}(Y) = \text{fiber}(A(Y) \to A(*))$ and similarly for TC, then in the case when X is a point we have the statement:

Corollary 2. *For any 1-connected topological space Y the spectra $\tilde{A}(Y)$ and $\tilde{T}C(Y)$ are weakly homotopy equivalent.*

I will not say much now about the other functor TC, except that $TC(X)$ is closely related to the free loop space ΛX (the space of all unbased maps from the circle to X) and is easier to study than $A(X)$ from the point of view of algebraic topology.

The theorem stated above represents the work of several people. In particular, the definition of the functor TC, and of a map $A \to TC$ which is crucial to the proof, uses work of Bökstedt-Hsiang-Madsen. A p-completed version of Theorem 1 (proved by the method outlined below) was the main result of [BCCGHM]. The theorem stated above is only a marginal advance over this, since a rational version ([G1], Corollary on p. 349) has been known for some time. (The final steps in the proof of Theorem 1 will appear in [G6].)

1. Summary of the Method

The proof of Theorem 1 uses a kind of deformation theory. The goal is to describe the change in $A(X)$ produced by a given (small) change in X. It turns out that to achieve this it is enough to describe the *infinitesimal* change in $A(X)$ produced by an *infinitesimal* change in X. By this I mean: to give an approximate description of the change in $A(X)$ produced by a very small change in X. (A small change in X is a highly connected map $Y \to X$.)

In a little more detail, the method is this:

There is a natural map of spectra from $A(X)$ to $TC(X)$, called the cyclotomic trace map. Denote its homotopy fiber by $F(X)$. There is a constant c such that for any k-connected map of spaces $Y \to X$ the map of homotopy fibers

$$\mathrm{fiber}(A(Y) \to A(X)) \to \mathrm{fiber}(TC(Y) \to TC(X))$$

is $(c + 2k)$-connected. In other words, the map $F(Y) \to F(X)$ is about twice as highly connected as the map $Y \to X$ upon which it depends. By a certain general principle (Proposition 5 below), it follows that the map $F(Y) \to F(X)$ is in fact ∞-connected when the map $Y \to X$ is at least 2-connected. (In other words, up to weak equivalence $F(X)$ depends only on $\pi_1(X)$ if X is connected.) This yields the conclusion of Theorem 1.

The general principle used above is analogous to the following fact from differential calculus: If a function f (in a suitable domain, and satisfying suitable differentiability hypotheses) is such that $|f(x) - f(y)| < C|x - y|^2$, then f is locally constant. A more familiar statement of this fact is that if the derivative of f is identically zero then f is locally constant.

Section 2 explains the idea "derivative of a homotopy functor". Section 3 states the general principle mentioned above. Section 4 discusses what one needs to know about Waldhausen's functor A in order to apply the principle here. Section 5 describes the other functor TC. Section 6 discusses that part of the proof which involves the map from A to TC. Details may be found in [BCCGHM, G2, G3, G4, and G6].

2. Differentiation of a Functor

For a more detailed account of the ideas below, see [G2].

2.1 The Definition

The idea can be made quite general, but for concreteness let us suppose that F is a functor from spaces to spectra. We always assume that it is a *homotopy functor*, meaning that it takes equivalences to equivalences. (Throughout, an *equivalence* of spaces or spectra means a weak homotopy equivalence.)

In calculus the concept of derivative, or differential, of a function f at a point x is a way of systematically describing the quantity $f(y) - f(x)$ with an accuracy like $|y - x|^2$. In a similar way the next two definitions serve to describe the $2k$-homotopy type of the fiber of $F(Y) \to F(X)$ when the map $Y \to X$ is k-connected.

Definition 3. The *derivative* $\partial_x F(X)$ of F at the based space (X, x) is the homotopy colimit (as $k \to \infty$) of the spectra $\Omega^k\, \mathrm{fiber}(F(X \vee S^k) \to F(X))$.

The maps in the limit system are (loosely speaking) induced by the diagrams

$$\begin{tikzcd} & F(X \vee S^{k-1}) \arrow[r] \arrow[d] & F(X \vee D^k_+) \sim F(X) \arrow[d] \\ F(X) \sim & F(X \vee D^k_-) \arrow[r] & F(X \vee S^k) \end{tikzcd}$$

(Note that $F(X \vee D^k)$ is equivalent to $F(X)$.) Up to equivalence the derivative is determined by X and by knowing which component of X contains the point x. The spectrum $\partial_x F(X)$ is a functor of the based space (X, x), and any based map $(X, x) \to (Y, y)$ which is an equivalence induces an equivalence $\partial_x F(X) \to \partial_y F(Y)$.

There is a more general construction. If $f\colon Y \to X$ is a map of spaces, think of Y as a space over X, think of the mapping cylinder of f as the *fiberwise cone* of Y over X (another space over X), and denote it by $C_X Y$. Let $\Sigma_X Y$, the *fiberwise suspension* of Y over X, be the union along Y of two copies of $C_X Y$.

Definition 4. The *differential* of $(D_X F)(Y)$, defined for any map $Y \to X$, is the homotopy colimit of the spectra Ω^k fiber$(F(\Sigma^k_X Y) \to F(X))$,

The maps in the limit system are defined using diagrams

$$\begin{tikzcd} & F(\Sigma^{k-1}_X Y) \arrow[r] \arrow[d] & F(C_X \Sigma^{k-1}_X Y) \sim F(X) \arrow[d] \\ F(X) \sim & F(C_X \Sigma^{k-1}_X Y) \arrow[r] & F(\Sigma^k_X Y) \end{tikzcd}$$

For fixed X the differential $D_X F$ is a functor from spaces over X to spectra. It is a homotopy functor in the sense that it preserves equivalences, where a map of spaces over X is called an equivalence if as a map of spaces it is a (weak homotopy) equivalence. We have $(D_X F)(X) \sim *$ and $(D_X F)(X \vee S^0) \sim \partial_x F(X)$.

Note that there is a natural map

$$\text{fiber}(F(Y) \to F(X)) \to (D_X F)(Y)$$

The functor $D_X F$ is intended to be an excisive functor that approximates $Y \mapsto$ fiber$(F(Y) \to F(X))$, much as in calculus the differential of a function f at a point x is a linear function that approximates $f(y) - f(x)$. To explain this I need some language.

2.2 Excision

A commutative diagram $\mathscr{X}$ of spaces (or spectra)

$$\begin{tikzcd} \mathscr{X}(\varnothing) \arrow[r] \arrow[d] & \mathscr{X}(1) \arrow[d] \\ \mathscr{X}(2) \arrow[r] & \mathscr{X}(1, 2) \end{tikzcd}$$

is a *cofiber square* if the canonical map to $\mathscr{X}(1, 2)$ from the homotopy pushout (union along $\mathscr{X}(\varnothing)$ of the mapping cylinders of $\mathscr{X}(\varnothing) \to \mathscr{X}(1)$ and $\mathscr{X}(\varnothing) \to \mathscr{X}(2)$) is an equivalence. It is a *fiber square* (resp. k-connected) if the canonical map from $\mathscr{X}(\varnothing)$ to the homotopy pullback (fiber product over $\mathscr{X}(1, 2)$ of the path fibrations of $\mathscr{X}(1) \to \mathscr{X}(1, 2)$ and $\mathscr{X}(2) \to \mathscr{X}(1, 2)$) is an equivalence (resp. k-connected). Equivalently, a square diagram $\mathscr{X}$ of spectra is k-connected if the *iterated fiber*, the homotopy fiber of the map

$$\mathrm{fiber}(\mathscr{X}(\varnothing) \to \mathscr{X}(1) \to \mathrm{fiber}(\mathscr{X}(2) \to \mathscr{X}(1, 2))$$

of homotopy fibers, is $(k-1)$-connected.

A functor F (say from spaces, or spaces over X, to spectra) is *excisive* if it takes cofiber squares to fiber squares. This is a very strong condition. Homotopy functors occurring in nature usually satisfy a much weaker, but useful, condition, called *stable excision*: there is a constant c_2 such that if the maps $\mathscr{X}(\varnothing) \to \mathscr{X}(1)$ and $\mathscr{X}(\varnothing) \to \mathscr{X}(2)$ in a cofiber square are respectively k_1- and k_2-connected, then the diagram $F(\mathscr{X})$:

$$\begin{tikzcd}
F(\mathscr{X}(\varnothing)) \arrow[r] \arrow[d] & F(\mathscr{X}(1)) \arrow[d] \\
F(\mathscr{X}(2)) \arrow[r] & F(\mathscr{X}(1, 2))
\end{tikzcd}$$

is $(k_1 + k_2 + c_2)$-connected.

If F satisfies stable excision then, for each X, $D_X F$ satisfies excision; we may think of $D_X F$ as a (reduced) homology theory on the category of spaces over X. Moreover, stable excision for F implies that the map from $\mathrm{fiber}(F(Y) \to F(X))$ to $(D_X F)(Y)$ is approximately $2k$-connected for any k-connected map $Y \to X$.

2.3 The Principle

Theorem 1 is proved by applying the following principle with $F = \mathrm{fiber}(A \to TC)$ and $\varrho = 1$. The term "ϱ-analytic" will be explained in Section 3.

Proposition 5. *If F is a ϱ-analytic functor from spaces to spectra such that $(D_X F)(Y)$ is trivial for all X and all $Y \to X$, then for every $(\varrho + 1)$-connected map $Y \to X$ of spaces the map $F(Y) \to F(X)$ is an equivalence.*

"Trivial" means equivalent to a point (all homotopy groups are trivial). If F satisfies a suitable limit axiom, so that up to equivalence it is determined by its behavior on finite CW complexes, then it is enough to assume that $\partial_x F(X)$ rather than $D_X F$ is trivial.

3. Analytic Functors

"Analyticity" of a homotopy functor F has to do with the behavior of F with respect to cubical diagrams of spaces. By an n-cubical diagram we mean a functor $\mathscr{X}$ from the partially ordered set of all subsets of $\{1, \ldots, n\}$ to the category of spaces. Analyticity of F involves one condition, stable $(n-1)$st order excision, for each n. Stable first-order excision is stable excision as defined in Sect. 2.2.

Stable $(n-1)$st order excision concerns certain n-cubical diagrams $\mathscr{X}$, namely the *strong cofiber cubes*. Call $\mathscr{X}$ a strong cofiber cube if, for each $1 \le i < j \le n$ and $S \subset \{1, \ldots, n\} - \{i, j\}$, the diagram

$$\begin{tikzcd} \mathscr{X}(S) \arrow[r] \arrow[d] & \mathscr{X}(S \cup \{i\}) \arrow[d] \\ \mathscr{X}(S \cup \{j\} \arrow[r] & \mathscr{X}(S \cup \{i, j\}) \end{tikzcd}$$

is a cofiber square. The condition is that there is a constant c_n such that, whenever $\mathscr{X}$ is a strong cofiber cube in which the map $\mathscr{X}(\varnothing) \to \mathscr{X}(i)$ is k_i-connected for all i, with $k_i > \varrho$, then $F(\mathscr{X})$ is $(c_n + \Sigma k_i)$-connected. (An n-cubical diagram of spectra is called k-connected if its iterated fiber – the spectrum obtained by taking homotopy fibers in each of the n directions in turn – is $(k-1)$-connected.) Note that c_n is allowed to be negative.

For $n = 1$ this simply says that there is a constant c_1 such that for any k-connected map $\mathscr{X}(\varnothing) \to \mathscr{X}(1)$ of spaces the map $F(\mathscr{X}(\varnothing)) \to F(\mathscr{X}(1))$ is $(k + c_1)$-connected, at least if $k > \varrho$.

Definition 6. The functor F is ϱ-analytic if it satisfies $(n-1)$st order as above for all $n \ge 1$, and if the numbers c_n are bounded below by $c - \varrho n$ for some constant c.

Most homotopy functors occurring in nature are ϱ-analytic for some ϱ, and in many cases it is a routine matter to verify this. The identity functor from spaces to spaces is 1-analytic, as is Waldhausen's functor A.

The proof of Proposition 5 uses an unusual inductive argument. It is not difficult, but I will not take time to explain it here; see [G3].

Proposition 5 expresses one of two main consequences of analyticity. The other, the existence of a "Taylor tower" for a functor in analogy with the Taylor series of a function, is not used in the proof of Theorem 1. It is explained in [G5].

4. The Derivative of K-Theory

In order to use Proposition 5 for proving Theorem 1, it is necessary first of all to know (up to natural equivalence) what the derivative of the functor A is. The answer turns out to be this:

Theorem 7. *For a based space (X, x) the spectrum $\partial_x A(X)$ is related by a chain of natural equivalences to $\Sigma^\infty(\Omega(X, x)_+)$.*

This is the unreduced suspension spectrum of the based loopspace of X. (The subscript "+" adds a disjoint basepoint.)

Theorem 7 (3.3 of [G2]) is proved indirectly; it is reduced to a corresponding statement (Theorem 7′ below) about smooth manifolds, using a major theorem of Waldhausen:

Theorem 8 (Waldhausen [W1]). *There is a natural weak equivalence of spectra between $A(X)$ and the product $\Sigma^\infty(X_+) \times Wh^{\mathrm{Diff}}(X)$, where $Wh^{\mathrm{Diff}}(X)$ is a natural double delooping of the differentiable pseudoisotopy spectrum $\mathscr{P}^{\mathrm{Diff}}(X)$.*

In view of Theorem 8, Theorem 7 may be rewritten:

Theorem 7′. *For a based space (X, x) the spectrum $\partial_x \mathscr{P}^{\mathrm{Diff}}(X)$ is related by a chain of natural equivalences to $\Omega^2\Sigma^\infty(\Omega(X, x)/X)$.*

It is notable that, while the relationship between K-theory and pseudoisotopy theory expressed in Theorem 8 is usually viewed as a way of reducing geometry to algebra, in this instance the flow of information is in the other direction. In this connection see also Sect. 5.3.

Recall that the underlying space of the spectrum $\mathscr{P}^{\mathrm{Diff}}(X)$ is essentially defined as a limit of spaces $P^{\mathrm{Diff}}(M)$ for manifolds M (compact, with boundary, of arbitrarily large dimension) of the homotopy type of X. The space $P^{\mathrm{Diff}}(M)$ is the simplicial group of all diffeomorphisms of $M \times I$ which are the identity along $(M \times 0) \cup (\partial M \times I)$.

Therefore, to "compute" $\partial_x \mathscr{P}^{\mathrm{Diff}}(X)$ is essentially to solve the following problem: For a smooth manifold M with an attached handle h of index $k \geq 3$, determine the $2k$-homotopy type of the fiber of $P^{\mathrm{Diff}}(M) \to P^{\mathrm{Diff}}(M \cup h)$. This is done in [G2] using Morlet's "disjunction lemma" and an old-fashioned differentiable general-position argument.

5. The Functor TC

I will now say something about the functor TC which occurs in the statement of Theorem 1. There are really two questions to address: How is it defined, and what does it turn out to be?

5.1 Definition of TC

I will not be very specific about this. TC is related to Bökstedt's "topological Hochschild homology" (THH). For details see [BCCGHM], [BHM], or [G4].

Recall that, according to one way of thinking about the K-theory of (based, connected) spaces, $A(BG)$ is the K-theory spectrum of the "ring up to homotopy" $\Omega^\infty\Sigma^\infty(|G|_+)$. The latter is to be thought of as the "group ring" $k[G]$ of the simplicial group G over the ground "ring" $k = QS^0$. Heuristically,

$$\begin{aligned} \text{connective spectrum} &= \text{infinite loop space} \\ &= \text{abelian group up to homotopy} \\ &= k\text{-module} \end{aligned}$$

and the group structure of G gives $k[G]$ a multiplication compatible with its additive structure. These ideas can be made precise by using a suitable notion of "ring up to homotopy", for example Bökstedt's notion of FSP (functor with smash product).

For such a "ring" R, Bökstedt defines a K-theory spectrum $K(R)$. Both the Quillen K-theory of rings and the Waldhausen K-theory of spaces are included as special cases (the cases of a discrete ring R and a group ring $k[G]$ respectively). He also defines a spectrum $THH(R)$; heuristically it is the simplicial object

$$\begin{array}{c} R \\ \uparrow\downarrow\uparrow \\ R\otimes R \\ \uparrow\downarrow\uparrow\downarrow\uparrow \\ R\otimes R\otimes R \\ \cdots \end{array}$$

with face and degeneracy maps given by the product and unit of R, respectively, as in the definition of the standard chain complex for Hochschild homology. The "tensor products" are meant to be over k and are really smash products of spectra.

Bökstedt defines a map of spectra $K(R) \to THH(R)$; it is modeled on the "trace map" from K-theory to Hochschild homology defined by Dennis for an ordinary ring R.

Very roughly speaking, TC is related to THH as cyclic homology is related to Hochschild homology. For any FSP there is a spectrum $TC(R)$ with a map $TC(R) \to THH(R)$. The trace $K(R) \to THH(R)$ lifts to a map $K(R) \to TC(R)$, called the *cyclotomic trace.* (After p-completion this is the same as the map of that same name constructed in [BHM]).

Let the simplicial group G be a loop group for the space X, and let R be $k[G]$. In this case we sometimes write $TC(X)$ instead of $TC(R)$. Thus in this case the cyclotomic trace is a map $A(X) \to TC(X)$. It is this which is used in the proof of the theorem.

5.2 Description of TC

From a computational point of view the main thing to know about $TC(X)$ is that it is related in a certain way to the free loop space $\Lambda X = \mathrm{Map}(S^1, X)$. Again let G be a simplicial loop group for X.

First of all, it is fairly easy to see that $THH(k[G])$ is equivalent to $\Sigma^\infty(\Lambda X_+)$. This is essentially because ΛX is equivalent to the realization of the simplicial space

$$\begin{array}{c} G \\ \uparrow\downarrow\uparrow \\ G\times G \\ \uparrow\downarrow\uparrow\downarrow\uparrow \\ G\times G\times G \\ \cdots \end{array}$$

(the "cyclic bar construction" or "cyclic nerve" of G).

To describe $TC(X)$ we must consider some additional structure that the space ΛX has. Let the circle group S^1 act on ΛX in the usual way, and let $\Delta_p: \Lambda X \to \Lambda X$ be the pth power map (composition with the standard map $S^1 \to S^1$ of degree p).

It turns out that the functor $X \mapsto TC(X)$ is very closely related to the functor $X \mapsto B(X) = \Sigma^\infty \Sigma((ES^1 \times_{S^1} \Lambda X)_+)$, although to say exactly how they are related it is apparently necessary to consider separately the profinite homotopy type and the rational homotopy type.

Concerning the profinite type, the statement is that after p-completion (p a prime) the spectrum $TC(X)$ becomes part of a fiber square

$$\begin{tikzcd} TC(X) \arrow[r] \arrow[d] & B(X) \arrow[d, "\mathrm{Trf}"] \\ \Sigma^\infty \Lambda X_+ \arrow[r, "1-\Delta_p"'] & \Sigma^\infty \Lambda X_+ \end{tikzcd}$$

Here Trf is the S^1-transfer associated to the bundle

$$(\Lambda X \sim)\ ES^1 \times \Lambda X \to ES^1 \times_{S^1} \Lambda X$$

and $1 - \Delta_p$ is the difference between two stable maps, the identity and the map induced by Δ_p.

This, it turns out, has the consequence that for 1-connected spaces X there is a natural equivalence, after p-completion and passage to reduced functors, between $TC(X)$ and

$$\Sigma^\infty(X_+) \times \mathrm{fiber}(e \circ \mathrm{Trf}: B(X) \to \Sigma^\infty(X_+))$$

where the map is the composition of the transfer and the map induced by evaluation $\Lambda X \to X$ at a point in the circle.

Concerning the rational type, the statement is that for 2-connected maps $Y \to X$ there is a natural equivalence, after rationalization, between the fiber of $TC(Y) \to TC(X)$ and the fiber of $B(Y) \to B(X)$. (This is not, however, induced by a natural map $TC \to B$ or $B \to TC$.)

5.3 Generalizations

Theorem 1 can be generalized so as to apply to more than the K-theory of spaces. There is considerable evidence for the following:

Conjecture 9. *For any 1-connected map $R \to S$ of FSP's the resulting map of spectra from* fiber$(K(R) \to K(S))$ *to* fiber $(TC(R) \to TC(S)$ *is an equivalence.*

This can be deduced from Theorem 1 in some cases, namely those in which $\pi_0(R)$ $(= \pi_0(S))$ is an integral group ring $\mathbb{Z}[\pi]$. In particular, it is true for the map $QS^0 = k = R \to S = \mathbb{Z}$. Unfortunately, this does not yet amount to a computation of the fiber of $A(*) \to K(\mathbb{Z})$ in any real sense, because $TC(\mathbb{Z})$ is still a fairly mysterious object.

6. The Derivative of TC

After producing a map from A to TC, it remains to show that it induces an equivalence $\partial_x A(X) \to \partial_x TC(X)$. This is done in two steps.

The first step is to show that $\partial_x A(X)$ and $\partial_x TC(X)$ are abstractly equivalent, in the sense that these two functors from based spaces (X, x) to spectra are related by a chain of natural equivalences. I have already said that $\partial_x A(X)$ is abstractly equivalent to $\Sigma^\infty \Omega(X, x)_+$. The same is true of $\partial_x TC(X)$. Of course I cannot begin to explain why, since I have not even defined TC here, but to get the idea I invite the reader to work out the equivalences (see Section 2 of [G2]):

$$\partial_x \Sigma^\infty A(X)_+ \sim \mathrm{Map}(S^1, \Sigma^\infty \Omega(X, x)_+)$$
$$\partial_x B(X)_+ \sim \Sigma^\infty \Omega(X, x)_+.$$

The second step is to prove:

Lemma 10. *The cyclotomic trace $A \to TC$ induces an equivalence $\partial_x A(X) \to \partial_x TC(X)$.*

The trick in proving this is to begin with the case when X is the suspension ΣY of a connected space Y.

To see that this special case is enough, one classifies all the natural maps $\Sigma^\infty \Omega(X, x)_+ \to \Sigma^\infty \Omega(X, x)_+$ in the homotopy category of homotopy functors from based spaces to spectra. It turns out that the only maps which are equivalences when X is a simply-connected suspension are those which are equivalences for all X.

The argument which proves the lemma in the case $X = \Sigma Y$ is essentially the main argument of [CCGH]. It relies on a tool which is only available in the suspension case: the cyclotomic trace can be composed with another natural map as follows:

$$\coprod_{n \geq 1} D_n(Y) \to \Omega \tilde{A}(\Sigma Y) \to \Omega \widetilde{TC}(\Sigma Y).$$

Here $D_n(Y)$ is the divided power $\Sigma^\infty(E(\mathbb{Z}/n)_+ \wedge_{\mathbb{Z}/n} Y^{[n]})$. (I am writing $Y^{[n]}$ for the smash product of n copies of Y.) The composed map above induces a map of derivatives

$$\partial_y \left(\coprod_{n \geq 1} D_n(Y) \right) \to \partial_y \Omega \widetilde{TC}(\Sigma Y) \sim \Omega \Sigma^\infty \Omega \Sigma Y_+$$

which, more or less by direct examination, is seen to be an equivalence. It follows that the map

$$\Omega \Sigma^\infty \Omega \Sigma Y_+ \sim \partial_y \Omega \tilde{A}(\Sigma Y) \to \partial_y \Omega \widetilde{TC}(\Sigma Y) \sim \Omega \Sigma^\infty \Omega \Sigma Y_+$$

induced by the cyclotomic trace is a split surjection, and from this one concludes without much trouble that it is an equivalence.

As a by-product this yields the main result of [CCGH], which can now be viewed as a special case of Corollary 2:

Theorem 11. *For connected spaces Y there is a natural equivalence of spectra*

$$\Omega A(\Sigma Y) \sim \Omega A(*) \times \coprod_{n \geq 1} D_n(Y).$$

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