The local structure of algebraic K-theory

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Preface

Algebraic K-theory draws its importance from its effective codification of a mathematical phenomenon which occurs in as separate parts of mathematics as number theory, geometric topology, operator algebra, homotopy theory and algebraic geometry. In reductionistic language the phenomenon can be phrased as

there is no canonical choice of coordinates.

As such, it is a meta-theme for mathematics, but the successful codification of this phenomenon in homotopy-theoretic terms is what has made algebraic K-theory into a valuable part of mathematics. For a further discussion of algebraic K-theory we refer the reader to chapter I below.

Calculations of algebraic K-theory are very rare, and hard to get by. So any device that allows you to get new results is exciting. These notes describe one way to get such results.

Assume for the moment that we know what algebraic K-theory is, how does it vary with its input?

The idea is that algebraic K-theory is like an analytic function, and we have this other analytic function called *topological cyclic homology* (TC) invented by Bökstedt, Hsiang and Madsen [6], and

the difference between K and TC is locally constant.

This statement will be proven below, and in its integral form it has not appeared elsewhere before.

The good thing about this is that TC is occasionally possible to calculate. So whenever you have a calculation of K-theory you have the possibility of calculating all the K-values of input “close” to your original calculation.

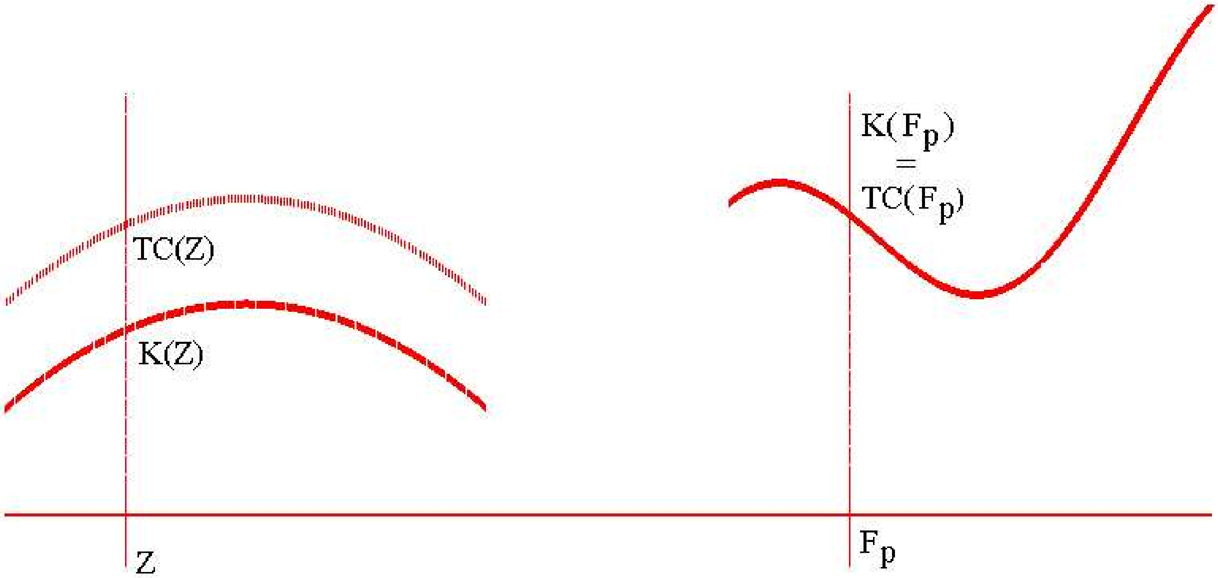


Figure 1: The difference between K and TC is locally constant. To the left of the figure you see that the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is quite substantial, but once you know this difference you know that it does not change in a "neighborhood" of $\mathbf{Z}$. In this neighborhood lies for instance all applications of algebraic K-theory of simply connected spaces, so here TC-calculations ultimately should lead to results in geometric topology as demonstrated by Rognes.

On the right hand of the figure you see that close to the finite field with p elements, K-theory and TC agrees (this is a connective and p-adic statement: away from the characteristic there are other methods that are more convenient). In this neighborhood you find many interesting rings, ultimately resulting in Hesselholt and Madsen's calculations of the K-theory of local fields.

So, for instance, if somebody (please) can calculate K-theory of the integers, many "nearby" applications in geometric topology (simply connected spaces) are available through TC-calculations (see e.g., [103],[102]). This means that calculations in motivic cohomology (giving K-groups of e.g., the integers) actually have bearings for our understanding of diffeomorphisms of manifolds!

On a different end of the scale, Quillen's calculation of the K-theory of finite fields give us access to "nearby" rings, ultimately leading to calculations of the K-theory of local fields [52]. One should notice that the illustration is not totally misleading: the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is substantial (though locally constant), whereas around the field $\mathbf{F}_p$ with p elements it is negligible.

Taking K-theory for granted (we'll spend quite some time developing it later), we should say some words about TC. Since K-theory and TC differ only by some locally constant term, they must have the same differential: $D_1K = D_1TC$. For ordinary rings A this differential is quite easy to describe: it is the *homology* of the category $\mathcal{P}_A$ of finitely generated projective modules.

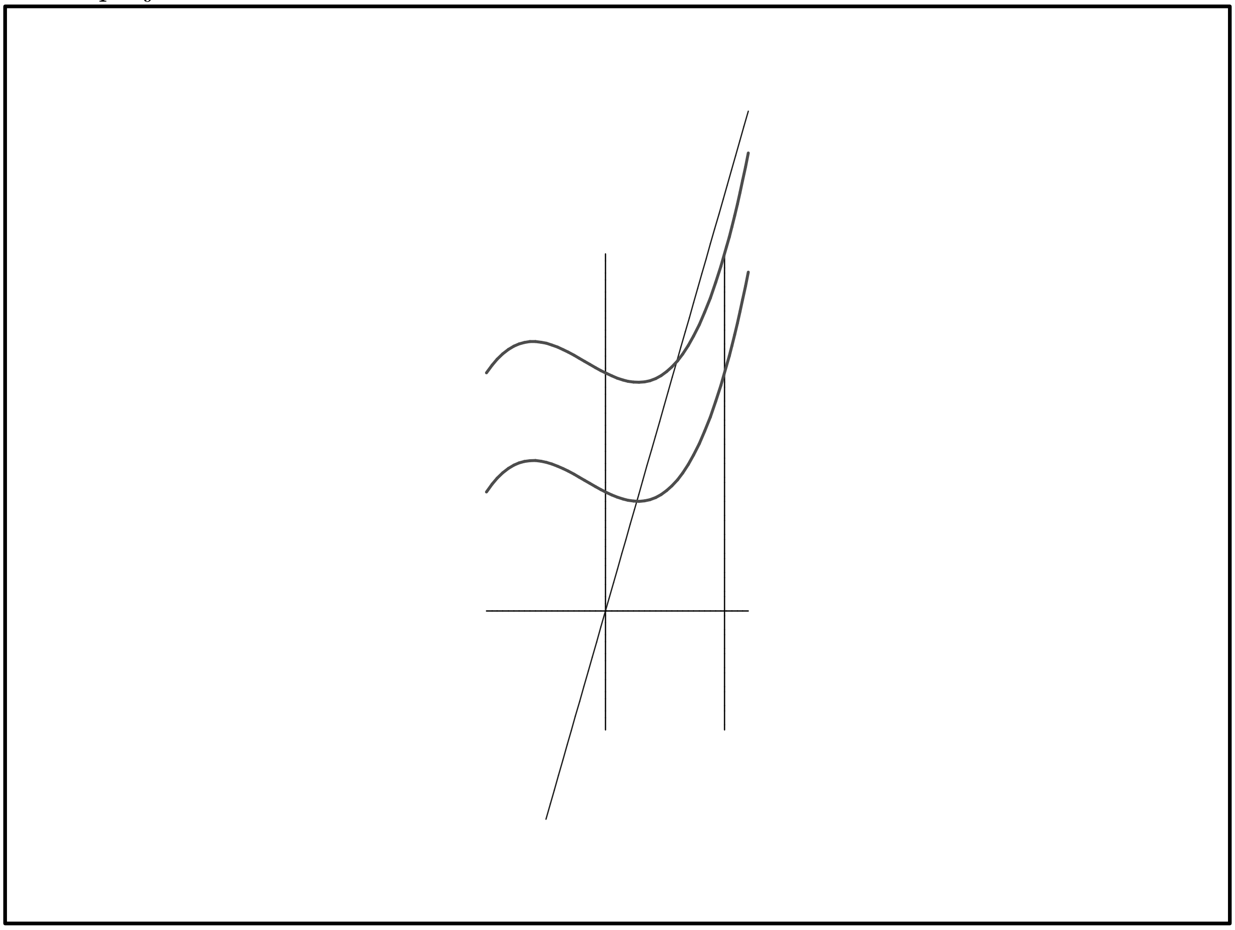


Figure 2: The differential of K and TC is equal at any point. For rings it is the homology of the category of finitely generated projective modules

The homology of a category is like Hochschild homology, and as Connes observed, certain models for these carry a circle action which is useful when comparing with K-theory. Only, in the case of the homology of categories it turns out that the ground ring over which to take Hochschild homology is not an ordinary ring, but the so-called sphere spectrum. Taking this idea seriously, we end up with Bökstedt's *topological Hochschild homology* THH.

One way to motivate the construction of TC from THH is as follows. There is a transformation $K \to THH$ which we will call the *Dennis trace map* and there is a model for THH for which the Dennis trace map is just the *inclusion of the fixed points under the circle action*. That is, the Dennis trace can be viewed as a composite

$$K \cong THH^{\mathbb{T}} \subseteq THH$$

where $\mathbb{T}$ is the circle group. The unfortunate thing about this statement is that it is *model dependent* in that fixed points do not preserve weak equivalences: if $X \to Y$ is a map of $\mathbb{T}$-spaces which is a weak equivalence of underlying spaces, normally the induced map $X^{\mathbb{T}} \to Y^{\mathbb{T}}$ won't be a weak equivalence. So, TC is an attempt to construct the $\mathbb{T}$-fixed points through techniques that **do** preserve weak equivalences.

It turns out that there is more to the story than this: THH possesses something called an *epicyclic structure* (which is not the case for all $\mathbb{T}$-spaces), and this allows us to approximate the $\mathbb{T}$-fixed points even better.

So in the end, the *cyclotomic trace* is a factorization

$$K \to TC$$

of the Dennis trace map.

This natural transformation is the theme for this book. There is another paper devoted to this transformation, namely Madsen's eminent survey [80]. We strongly encourage all readers to get a copy, and keep it close by while reading what follows.

It was originally an intention that readers who were only interested in discrete rings would have a path leading leading far into the material with minimal contact with ring spectra. This idea has to a great extent been abandoned since ring spectra and the techniques around them has become much more mainstream while these notes has matured. Some traces can still be seen in that chapter I does not depend at all on ring spectra, leading to the proof that stable K-theory of rings correspond to homology of the category of finitely generated projective modules. Topological Hochschild homology is however interpreted as a functor of ring spectra, so the statement that stable K-theory is THH requires some background on ring spectra.

The general plan of the book is as follows.

In section I.1 we give some general background on algebraic K-theory. The length of this introductory section is defended by the fact that this book is primarily concerned with algebraic K-theory; the theories that fill the last chapters are just there in order to shed light on K-theory, we are not really interested in them for any other reason. In I.2 we give Waldhausen's interpretation of algebraic K-theory and study in particular the case of radical extensions of rings. Finally I.3 compare stable K-theory and homology.

Chapter II aims at giving a crash course on ring spectra. In order to keep the presentation short we have limited us to present only the simplest version: Segal's Γ-spaces. This only gives us connective spectra, but that suffice for our purposes, and also fits well with Segal's version of algebraic K-theory which we are using heavily later in the book.

Chapter III can (and perhaps should) be skipped on a first reading. It only asserts that various reductions are possible. In particular K-theory of simplicial rings can be calculated degreewise "locally" (i.e., in terms of the K-theory of the rings appearing in each degree), simplicial rings are "dense" in (connective) ring spectra, and all definitions of algebraic K-theory we encounter give the same result.

In chapter IV topological Hochschild homology is at long last introduced. First for ring spectra, and then in a generality suitable for studying the correspondence with algebraic K-theory. The equivalence between the topological Hochschild homology of a ring and the homology of the category of finitely generated projective modules is established in IV.2, which together with the results in I.3 settle the equivalence between stable K-theory and topological Hochschild homology of rings.

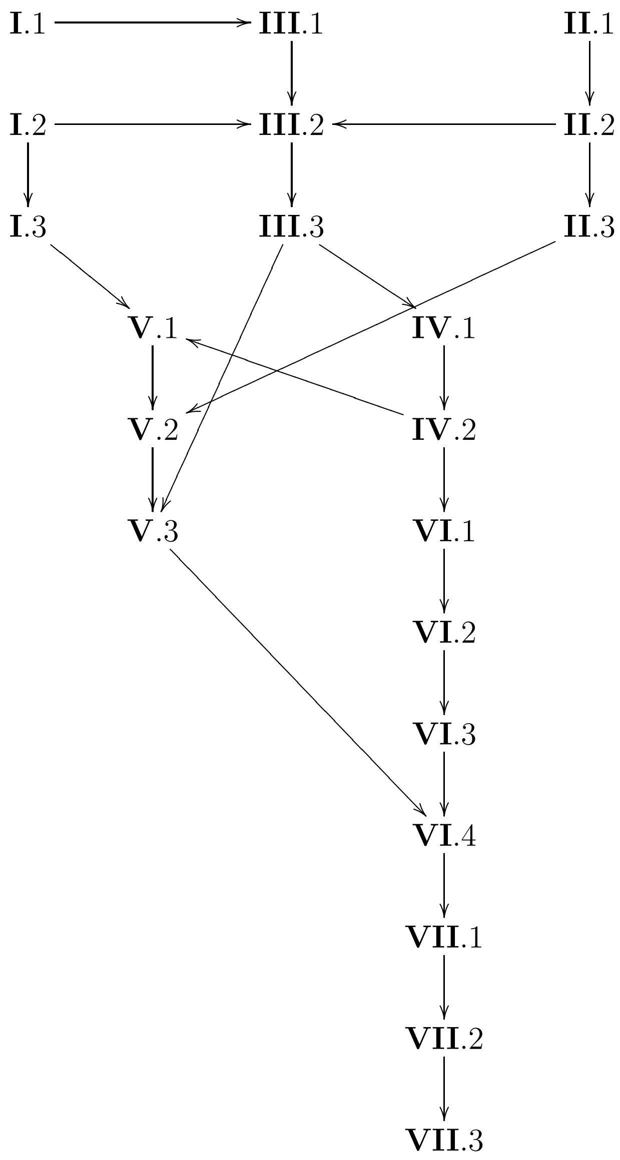
In order to push the theory further we need an effective comparison between K-theory and THH, and this is provided by the trace map $K \to THH$ in the following chapter.

In chapter VI topological cyclic homology is introduced. This is the most involved of the chapters in the book, since there are so many different aspects of the theory that have to be set in order. However, when the machinery is set properly up, Goodwillie's ICM-conjecture is proven in a couple of pages at the beginning of chapter VII. The chapter ends with a quick and inadequate review of the various calculations of algebraic K-theory that have resulted from trace methods.

The appendices collect some material that is used freely throughout the notes. Most of the material is available elsewhere in the literature, but has been collected for the convenience of the reader, and some material is of a sort that would distract the discussion in the book proper, and hence has been pushed back to an appendix.

Acknowledgments: This book owes a lot to many people. The first author especially wants to thank Marcel Bökstedt, Bjørn Jahren, Ib Madsen and Friedhelm Waldhausen for their early and decisive influence on his view on mathematics. These notes have existed for quite a while on the net, and we are grateful for the helpful comments we have received from a number of people, in particular from Morten Brun, Harald Kittang, John Rognes, Stefan Schwede and Paul Arne Østvær. A significant portion of the notes were written while visiting Stanford University, and the first author is grateful to Gunnar Carlsson and Ralph Cohen for inviting me and asking me to give a course based on these notes, which gave the impetus to try to finish the project.

For the convenience of the reader we provide the following leitfaden. It should not be taken too seriously, some minor dependencies are not shown, and many sections that are noted to depend on previous chapters can be read first looking up references when they appear. In particular, chapter III should be postponed upon a first reading.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V.1
IV.1
V.2
IV.2
V.3
VI.1
VI.2
VI.3
VI.4
VII.1
VII.2
VII.3

Contents

Chapter I

Algebraic K-theory

{I}

In this chapter we define and discuss the algebraic K-theory functor. This chapter will mainly be concerned with the algebraic K-theory of rings, but we will extend this notion at the end of the chapter. There are various possible extensions, but we will mostly focus on a class that are close to rings.

In the first section we give a quick nontechnical overview of K-theory. Many of the examples are touched lightly on, and are not needed later on, but are included to give an idea of the scope of the theory.

In the second section we introduce Waldhausen's S-construction of algebraic K-theory and prove some of the basic facts.

The third section concerns itself with comparisons between K-theory and various homology theories.

1 Introduction

The first appearance of what we now would call truly K-theoretic questions are the investigations of J. H. C. Whitehead and Higman on the diffeomorphism classes of h-cobordisms. The name "K-theory" is much younger, and first appears in Grothendieck's work on the Riemann-Roch theorem. But, even though it was not called K-theory, we can get some motivation by studying the early examples.

1.1 Motivating example from geometry: Whitehead torsion

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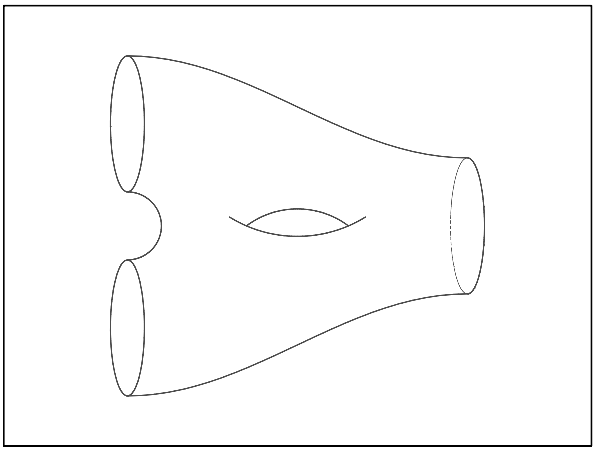


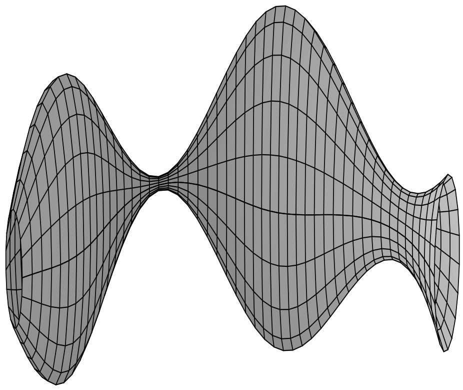
A cobordism W between a disjoint union M of two circles and a single circle N.

Let M and N be two smooth n-dimensional manifolds. A *cobordism* between M and N is an $n+1$-dimensional smooth compact manifold W with boundary the disjoint union of M and N (in the oriented case we assume that M and N are oriented, and W is an oriented cobordism from M to N if it is oriented so that the orientation agrees with that on M and is the opposite of that of N).

We are here interested in a situation where M and N are deformation retracts of W. Obvious examples are cylinders $M \times I$.

(For a more thorough treatment of the following example, see Milnor's very readable article [85])

More presicely: Let M be a compact, connected, smooth manifold of dimension $n > 5$. Suppose we are given an h-cobordism $(W; M, N)$, that is a compact smooth $n+1$ dimensional manifold W, with boundary the disjoint union of M and N, such that both the inclusions $M \subset W$ and $N \subset W$ are homotopy equivalences.



An h-cobordism $(W; M, N)$. This one is a cylinder.

subsec:QWh}

Question 1.1.1 *Is W diffeomorphic to $M \times I$?*

It requires some fantasy to realize that the answer to this question can be "no". In particular, in the low dimensions of the illustrations all h-cobordisms **are** cylinders.

However, this is not true in high dimensions, and the h-cobordism theorem 1.1.3 below gives a precise answer to the question.

To fix ideas, let $M = L$ be a *lens space* of dimension, say, $n = 7$. That is, the cyclic group of order l, $\pi = \mu_l = \{1, e^{2\pi i/l}, \ldots, e^{2\pi i(l-1)/l}\}$, acts on the seven sphere $S^7 =$

$\{\mathbf{x} \in \mathbf{C}^4 \text{ s.t. } |\mathbf{x}| = 1\}$ by complex multiplication

$$\pi \times S^7 \to S^7 \qquad (t, \mathbf{x}) \mapsto (t \cdot \mathbf{x})$$

and we let L be the quotient space $S^7/\pi = S^7/(\mathbf{x} \sim t \cdot \mathbf{x})$. Then L is a smooth manifold with fundamental group π.

Let

$$\ldots \xrightarrow{\partial} C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} \ldots \longrightarrow C_0$$

be the complex calculating the homology $H_* = H_*(W, L; \mathbf{Z}[\pi])$ of the inclusion $L = M \subseteq W$ (see section 7 and 9 in [85] for details). Each C_i is a finitely generated free $\mathbf{Z}[\pi]$ module, and has a preferred basis over $\mathbf{Z}[\pi]$ coming from the i simplices added to get from L to W in some triangulation. Define

$$B_i = im\{C_{i+1} \xrightarrow{\partial} C_i\}$$

and

$$Z_i = \ker\{C_i \xrightarrow{\partial} C_{i-1}\}.$$

We have short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0$$
$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0.$$

But since $L \subset W$ is a deformation retract, $H_* = 0$, and so $B_* = Z_*$.

Since each C_i is a finitely generated free $\mathbf{Z}[\pi]$ module, and we may assume each B_i free as well (generally we get by induction only that each B_i is "stably free", but in our lens space case this implies that B_i is free). Now, this means that we may **choose** arbitrary bases for B_i, but there can be nothing canonical about this choice. The strange fact is that this phenomenon is exactly what governs the geometry.

Let M_i be the matrix (in the chosen bases) representing the isomorphism {Mi}

$$B_i \oplus B_{i-1} \cong C_i$$

coming from a choice of section in

$$0 \longrightarrow B_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

1.1.2 K_1 and the Whitehead group {subsec:W

For any ring A we may consider the matrix rings $M_k(A)$ as a monoid under multiplication. The general linear group is the subgroup of invertible elements $GL_k(A)$. Take the limit $GL(A) = \lim_{k\to\infty} GL_k(A)$ with respect to the stabilization

$$GL_k(A) \xrightarrow{g \mapsto g \oplus 1} GL_{k+1}(A)$$

(thus every element $g \in GL(A)$ can be thought of as an infinite matrix

$$\begin{bmatrix} g' & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with $g' \in GL_k(A)$ for some $k < \infty$). Let $E(A)$ be the subgroup of elementary matrices (i.e. $E_k(A) \subset GL_k(A)$ is the subgroup generated by the elements e^a_{ij} with ones on the diagonal and a single off diagonal entry $a \in A$ in the ij position). The "Whitehead lemma" (see 1.2.2 below) implies that

$$K_1(A) = GL(A)/E(A)$$

is an abelian group. In the particular case where A is an integral group ring $\mathbf{Z}[\pi]$ we define the Whitehead group as the quotient

$$Wh(\pi) = K_1(\mathbf{Z}[\pi])/\{\pm\pi\}$$

via $\{\pm\pi\} \subseteq GL_1(\mathbf{Z}[\pi]) \to K_1(\mathbf{Z}[\pi])$.

Let $(W; M, N)$ be an h-cobordism, and let $M_i \in GL(\mathbf{Z}[\pi_1(M)])$ be the matrices described in 1.1 for the lens spaces above and similarly in general. Let $[M_i] \in Wh(\pi_1(M))$ be the corresponding equivalence classes and set

$$\tau(W, M) = \sum (-1)^i [M_i] \in Wh(\pi_1(M)).$$

The class $\tau(W, M)$ is called the *Whitehead torsion*.

{theo:hcob}

Theorem 1.1.3 *(Mazur (63), Barden (63), Stallings (65)) Let M be a compact, connected, smooth manifold of dimension > 5 with fundamental group $\pi_1(M) = \pi$, and let $(W; M, N)$ be an h-cobordism. The Whitehead torsion $\tau(W, M)$ is well defined, and τ induces a bijection*

$$\left\{ \begin{matrix} \text{diffeomorphism classes (rel. } M) \\ \text{of } h\text{-cobordisms } (W; M, N) \end{matrix} \right\} \longleftrightarrow Wh(\pi)$$

In particular, $(W; M, N) \cong (M \times I; M, M)$ if and only if $\tau(W, M) = 0$.

Example 1.1.4 One has managed to calculate $Wh(\pi)$ only for a very limited set of groups. We list a few of them; for a detailed study of Wh of finite groups, see [90]. The first three refer to the lens spaces discussed above (see page 375 in [85] for references).

1. $l = 1$, $M = S^7$. Exercise: show that $K_1\mathbf{Z} = \{\pm 1\}$, and so $Wh(0) = 0$. I.e: any h cobordism of S^7 is diffeomorphic to $S^7 \times I$.

2. $l = 2$. $M = P^7$, the real projective 7-space. Exercise: show that $K_1\mathbf{Z}[C_2] = \{\pm C_2\}$, and so $Wh(\mu_2) = 0$. I.e: any h cobordism of P^7 is diffeomorphic to $P^7 \times I$.

3. $l = 5$. $Wh(\mu_5) \cong \mathbf{Z}$ (generated by the invertible element $t + t^{-1} - 1 \in \mathbf{Z}[\mu_5]$ – the inverse is $t^2 + t^{-2} - 1$). I.e: there exists countably infinitely many non diffeomorphic h-cobordisms $(W; L, M)$.

4. Waldhausen [127]: If π is a free group, free abelian group, or the fundamental group of a submanifold of the three-sphere, then $Wh(\pi) = 0$.

5. Farrell and Jones [30]: If M is a closed Riemannian manifold with nonpositive sectional curvature, then $Wh(\pi_1 M) = 0$.

1.2 K_1 of other rings

{sec:$K_1

1. Commutative rings: The map from the units in A

$$A^* = GL_1(A) \to GL(A)/E(A) = K_1(A)$$

is split by the determinant map, and so the units of A is a split summand in $K_1(A)$. In certain cases (e.g. if A is local, or the integers in a number field, see next example) this is all of $K_1(A)$. We may say that $K_1(A)$ measures to what extent we can do Gauss elimination, in that $\ker\{det\colon K_1(A) \to A^*\}$ is the group of equivalence classes of matrices up to elementary row operations (i.e. multiplication by elementary matrices and multiplication of a row by an invertible element).

2. Let F be a number field (i.e. a finite extension of the rational numbers), and let $A \subseteq F$ be the ring of integers in F (i.e. the integral closure of $\mathbf{Z}$ in F). Then $K_1(A) \cong A^*$, and a result of Dirichlet asserts A^* is finitely generated of rank r_1+r_2-1 where r_1 (resp. $2r_2$) is the number of distinct real (resp. complex) embeddings of F.

3. Let $B \to A$ be an epimorphism of rings with kernel $I \subseteq rad(B)$ – the Jacobson radical of B (that is, if $x \in I$, then $1 + x$ is invertible in B). Then

$$(1+I)^\times \longrightarrow K_1(B) \longrightarrow K_1(A) \longrightarrow 0$$

is exact, where $(1+I)^\times \subset GL_1(B)$ is the group $\{1 + x | x \in I\}$ under multiplication (see e.g. page 449 in [4]). Moreover, if B is commutative and $B \to A$ is split, then

$$0 \longrightarrow (1+I)^\times \longrightarrow K_1(B) \longrightarrow K_1(A) \longrightarrow 0$$

is exact.

For later reference, we record the Whitehead lemma mentioned above. For this we need some definitions.

{Def:comm

Definition 1.2.1 The *commutator* $[G, G]$ of a group G is the (normal) subgroup generated by all commutators $[g, h] = ghg^{-1}h^{-1}$. A group G is called *perfect* if it is equal to its commutator, or in other words, if $H_1 G = G/[G, G]$ vanishes. Any group G has a *maximal perfect subgroup*, which we call PG, and which is automatically normal. We say that G is *quasi-perfect* if $PG = [G, G]$. {maximal

An example of a perfect group is the alternating group on $n \geq 5$ letters. Further examples are provided by the

:Whitehead}

Lemma 1.2.2 *(The Whitehead lemma) Let A be a unital ring. Then $GL(A)$ is quasi-perfect with maximal perfect subgroup $E(A)$. I.e.*

$$[GL(A), GL(A)] = [E(A), GL(A)] = [E(A), E(A)] = E(A)$$

Proof: See e.g. page 226 in [4]. ∎

1.3 The Grothendieck group K_0

roup{D$K:K0$}

Definition 1.3.1 Let $\mathfrak{C}$ be a small category and $\mathcal{E}$ a collection of diagrams $c' \to c \to c''$ in $\mathfrak{C}$ closed under isomorphisms. Then $K_0(\mathfrak{C}, \mathcal{E})$ is the abelian group, defined (up to isomorphism) by the following universal property. Any function f from the set of isomorphism classes of objects in $\mathfrak{C}$ to an abelian group A such that $f(c) = f(c') + f(c'')$ for all sequences $c' \to c \to c''$ in $\mathcal{E}$, factors uniquely through $K_0(\mathfrak{C})$.

That is, $K_0(\mathfrak{C}, \mathcal{E})$ is the free abelian group on the set of isomorphism classes, modulo the relations of the type "$[c] = [c'] + [c'']$". So, it is not really necessary that $\mathcal{C}$ is small, the only thing we need to know is that the class of isomorphism classes form a set.

Most often the pair $(\mathfrak{C}, \mathcal{E})$ will be an *exact category* in the sense that $\mathfrak{C}$ is an additive category such that there exists an full embedding of $\mathfrak{C}$ in an abelian category $\mathfrak{A}$, such that $\mathfrak{C}$ is closed under extensions in $\mathfrak{A}$ and $\mathcal{E}$ consists of the sequences in $\mathfrak{C}$ which are short exact in $\mathfrak{A}$.

Any additive category is an exact category if we choose the exact sequences to be the split exact sequences, but there may be other exact categories with the same underlying additive category. For instance, the category of abelian groups is an abelian category, and hence an exact category in the natural way, choosing $\mathcal{E}$ to consist of the short exact sequences. These are not necessary split, e.g., $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z}$ is a short exact sequence which does not split.

The definition of K_0 is a case of "additivity": K_0 is a (really *the*) functor to abelian groups insensitive to extension issues. We will dwell more on this issue later, when we introduce the higher K-theories. Higher K-theory plays exactly the same rôle as K_0, except that the receiving category has a much richer structure than Abelian groups.

The choice of $\mathcal{E}$ will always be clear from the context, and we drop it from the notation and write $K_0(\mathfrak{C})$.

{ex:K_0}

Example 1.3.2 1. Let A be a unital ring. If $\mathfrak{C} = \mathcal{P}_A$, the category of finitely generated projective (left) A modules, with the usual notion of exact sequences, we often write $K_0(A)$ for $K_0(\mathcal{P}_A)$. Note that $\mathcal{P}_A$ is split exact, that is, all short exact sequences in $\mathcal{P}_A$ split. Thus we see that we could have defined $K_0(A)$ as the quotient of the free abelian group on the isomorphism classes in $\mathcal{P}_A$ by the relation $[P \oplus Q] \sim [P] + [Q]$. It follows that all elements in $K_0(A)$ can be written on the form $[F] - [P]$ where F is free.

{KfA}

2. Inside $\mathcal{P}_A$ sits the category $\mathcal{F}_A$ of finitely generated free A modules, and we let $K_0^f(A) = K_0(\mathcal{F}_A)$. If A is a principal ideal domain, then every submodule of a free module is free, and so $\mathcal{F}_A = \mathcal{P}_A$. This is so, e.g. for the integers, and we see that $K_0(\mathbf{Z}) = K_0^f(\mathbf{Z}) \cong \mathbf{Z}$, generated by the module of rank one. Generally, $K_0^f(A) \to K_0(A)$ is an isomorphism if and only if every finitely generated projective module is *stably free* (P and P' are said to be *stably isomorphic* if there is a $Q \in ob\mathcal{F}_A$ such that $P \oplus Q \cong P' \oplus Q$, and P is stably free if it is stably isomorphic to a free module). Whereas $K_0(A \times B) \cong K_0(A) \times K_0(B)$, K_0^f does not preserve products: e.g. $\mathbf{Z} \cong K_0^f(\mathbf{Z} \times \mathbf{Z})$, while $K_0(\mathbf{Z} \times \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$ giving an easy example of a ring where not all projectives are free.

3. Note that K_0 does not distinguish between stably isomorphic modules. This is not important in some special cases. For instance, if A is a commutative Noetherian ring of Krull dimension d, then every stably free module of rank $> d$ is free ([4, p. 239]).

{IBN}

4. The initial map $\mathbf{Z} \to A$ defines a map $\mathbf{Z} \to K_0^f(A)$ which is always surjective, and in most practical circumstances an isomorphism. If A has the invariance of basis property, that is, if $A^m \cong A^n$ if and only if $m = n$, then $K_0^f(A) \cong \mathbf{Z}$. Otherwise, $A = 0$, or there is an $h > 0$ and a $k > 0$ such that $A^m \cong A^n$ if and only if either $m = n$ or $m, n > h$ and $m \equiv n \mod k$. There are examples of rings with such h and k for all $h, k > 0$ (see [69] or [18]): let $A_{h,k}$ be the quotient of the free ring on the set $\{x_{ij}, y_{ji} | 1 \leq i \leq h, 1 \leq j \leq h+k\}$ by the matrix relations

$$[x_{ij}] \cdot [y_{ji}] = I_h, \text{ and } [y_{ji}] \cdot [x_{ij}] = I_{h+k}$$

Commutative (non-trivial) rings always have the invariance of basis property.

5. Let X be a CW complex, and let $\mathfrak{C}$ be the category of complex vector bundles on X with exact sequences meaning the usual thing. Then $K_0(\mathfrak{C})$ is the $K^0(X)$ of Atiyah and Hirzebruch [2]. Note that the possibility of constructing normal complements, assure that this is a split exact category.

6. Let X be a scheme, and let $\mathfrak{C}$ be the category of vector bundles on X. Then $K_0(\mathfrak{C})$ is the $K(X)$ of Grothendieck. This is an example of K_0 of an exact category which is not split exact.

1.3.3 Geometric example: Wall's finiteness obstruction

{subsec:G

Let A be a space which is dominated by a finite CW complex X (dominated means that there are maps $A \xrightarrow{i} X \xrightarrow{r} A$ such that $ri \simeq id_A$).

Question: is A homotopy equivalent to a finite CW complex?

The answer is yes if and only if a certain finiteness obstruction lying in $\tilde{K}_0(\mathbf{Z}[\pi_1 A]) = ker\{K_0(\mathbf{Z}[\pi_1 A]) \to K_0(\mathbf{Z})\}$ vanishes. So, for instance, if we know that $\tilde{K}_0(\mathbf{Z}[\pi_1 A])$ vanishes for algebraic reasons, we can always conclude that A is homotopy equivalent to a finite CW complex. As for K_1, calculations of $K_0(\mathbf{Z}[\pi])$ are very hard, but we give a short list.

1.3.4 K_0 of group rings

roup rings}

1. If C_p is a cyclic group of order prime order p less than 23, then $\tilde{K}_0(\mathbf{Z}[\pi])$ vanishes. $\tilde{K}_0(\mathbf{Z}[C_{23}]) \cong \mathbf{Z}/3\mathbf{Z}$ (Kummer, see [86, p. 30]).

2. Waldhausen [127]: If π is a free group, free abelian group, or the fundamental group of a submanifold of the three-sphere, then $\tilde{K}_0(\mathbf{Z}[\pi]) = 0$.

3. Farrell and Jones [30]: If M is a closed Riemannian manifold with nonpositive sectional curvature, then $\tilde{K}_0(\mathbf{Z}[\pi_1 M]) = 0$.

1.3.5 Facts about K_0 of rings

$ of rings}

1. If A is a commutative ring, then $K_0(A)$ has a ring structure. The additive structure comes from the direct sum of modules, and the multiplication from the tensor product.

2. If A is local, then $K_0(A) = \mathbf{Z}$.

3. Let A be a commutative ring. Define $rk_0(A)$ to be the kernel of the (split) surjection $rank\colon K_0(A) \to \mathbf{Z}$ associating the rank to a module. The modules P for which there exists a Q such that $P \otimes_A Q \cong A$ form a category. The isomorphism classes form a group under tensor product. This group is called the Picard group, and is denoted $Pic_0(A)$. There is a "determinant" map $rk_0(A) \to Pic_0(A)$ which is always surjective. If A is a Dedekind domain (see [4, p. 458–468]) may be reinterpreted as an isomorphism to the ideal class group $Cl(A)$.

4. Let A be the integers in a number field. Then Dirichlet tells us that $rk_0(A) \cong Pic_0(A) \cong Cl(A)$ is finite. For instance, if $A = \mathbf{Z}[e^{2\pi i/p}] = \mathbf{Z}[t]/\sum_{i=0}^{p-1} t^i$, the integers in the cyclotomic field $\mathbf{Q}(e^{2\pi i/p})$, then $K_0(A) \cong K_0(\mathbf{Z}[C_p])$ (1.3.41.).

5. If $f\colon B \to A$ is a surjection of rings with kernel I contained in the Jacobson radical, $rad(B)$, then $K_0(B) \to K_0(A)$ is injective ([4, p. 449]). It is an isomorphism if either

   (a) B is complete in the I-adic topology ([4]),

   (b) (B, I) is a Hensel pair ([34]) or

   (c) f is split (as K_0 is a functor).

1.3.6 Example from algebraic geometry

c geometry}

(Grothendieck's proof of the Riemann–Roch theorem. see Borel and Serre [11]) Let X be a quasiprojective non-singular variety over an algebraically closed field. Let $A(X)$ be the Chow ring of cycles under linear equivalence with product defined by intersection. Tensor product gives a ring structure on $K_0(X)$, and Grothendieck defines a natural ring morphism $ch\colon K_0(X) \to A(X) \otimes \mathbf{Q}$. For proper maps $f\colon X \to Y$ there are transfer maps $f_!\colon K_0(X) \to K_0(Y)$ and the Riemann–Roch theorem is nothing but a quantitative measure of the fact that

$$\begin{array}{ccc} K_0(X) & \xrightarrow{ch} & A(X) \otimes \mathbf{Q} \\ {\scriptstyle f_!}\downarrow & & {\scriptstyle f_!}\downarrow \\ K_0(Y) & \xrightarrow{ch} & A(Y) \otimes \mathbf{Q} \end{array}$$

fails to commute: $ch(f_!(x)) \cdot T(Y) = f_!(ch(x) \cdot T(X))$ where $T(X)$ is the value of the Todd class on the tangent bundle of X.

1.3.7 Number-theoretic example

{subsec:Nu

Let F be a number field and A its ring of integers. Then there is an exact sequence connecting K_1 and K_0:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(F) & \xrightarrow{\delta} & \\ \bigoplus_{\mathfrak{m}\in Max(A)} K_0(A/\mathfrak{m}) & \longrightarrow & K_0(A) & \longrightarrow & K_0(F) & \longrightarrow & 0 \end{array}$$

The zeta function of F is defined for $s \in \mathbf{C}$ to be

$$\zeta_F(s) = \sum_{I \text{ non-zero ideal in } A} |A/I|^{-s}$$

This series converges for $Re(s) > 1$, and admits an analytic continuation to the whole plane. It has a zero of order $r = rank(K_1(A))$ in $s = 0$, and

$$\lim_{s\to 0} \frac{\zeta_F(s)}{s^r} = -\frac{R|K_0(A)_{tor}|}{|K_1(A)_{tor}|}$$

where $|-_{tor}|$ denotes the cardinality of the torsion subgroup, and the regulator R depends on the map δ above.

This is related to the Lichtenbaum-Quillen conjecture, which is now confirmed at the prime 2 due to work of among many others Voevodsky, Suslin, Rognes and Weibel (see section 0.9 for references and a deeper discussion).

1.4 The Mayer–Vietoris sequence

We have said that $K_0(A)$ got its name before $K_1(A)$, and the reader may wonder why one chooses to regard them as related. Example 1.3.7 provides one reason, but that is cheating.

Historically, this was an insight of Bass, who proved that K_1 could be obtained from K_0 in analogy with the definition of $K^1(X)$ as $K^0(S^1 \wedge X)$ (cf. example 1.3.2.5). This is entailed by exact sequences connecting the two theories. As an example: if

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

is a cartesian square of rings and g (or f) surjective, then we have a long exact "Mayer–Vietoris" sequence

$$\begin{array}{l} K_1(A) \longrightarrow K_1(B) \oplus K_1(C) \longrightarrow K_1(D) \longrightarrow \\ K_0(A) \longrightarrow K_0(B) \oplus K_0(C) \longrightarrow K_0(D) \end{array}$$

However, it is not true that this continues. For one thing there is no simple analogy to the Bott periodicity $K^0(S^2 \wedge X) \cong K^0(X)$. Milnor proposed in [86] a definition of K_2 (see below) which would extend the Mayer–Vietoris sequence if **both** f and g are surjective, i.e. we have a long exact sequence

$$\begin{array}{l} K_2(A) \longrightarrow K_2(B) \oplus K_2(C) \longrightarrow K_2(D) \longrightarrow \\ K_1(A) \longrightarrow K_1(B) \oplus K_1(C) \longrightarrow K_1(D) \longrightarrow K_0(A) \longrightarrow \dots \end{array}$$

However, this was the best one could hope for:

o excision}

Example 1.4.1 Swan [118] gave the following example showing that there exist no functor K_2 giving such a sequence if only g is surjective. Let A be commutative, and consider the pullback diagram

$$\begin{array}{ccc} A[t]/t^2 & \xrightarrow{t \mapsto 0} & A \\ {\scriptstyle a+bt \mapsto \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle \Delta} \\ T_2(A) & \xrightarrow{g} & A \times A \end{array}$$

where $T_2(A)$ is the ring of upper triangular 2×2 matrices, g is the projection onto the diagonal, while Δ is the diagonal inclusion. As g splits $K_2(T_2(A)) \oplus K_2(A) \to K_2(A \times A)$ must be surjective, but, as we shall see below, $K_1(A[t]/t^2) \to K_1(T_2(A)) \oplus K_1(A)$ is not injective.

Recall that, since A is commutative, $GL_1(A[t]/t^2)$ is a direct summand of $K_1(A[t]/t^2)$. The element $1{+}t \in A[t]/t^2$ is invertible (and not the identity), but $[1{+}t] \neq [1] \in K_1(A[t]/t^2)$ is sent onto $[1]$ in $K_1(A)$, and onto

$$[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)] \sim [\begin{pmatrix} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) & 0 \\ 0 & 1 \end{pmatrix}] = [\left[e_{12}^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}, e_{21}^{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}\right]] \sim [1] \in K_1(T_2(A))$$

where the inner brackets stand for commutator (which is trivial in K_1, by definition).

1.5 Milnor's $K_2(A)$

`s $K_2(A)$}`

Milnor's definition of $K_2(A)$ is given in terms of the Steinberg group, and turns out to be isomorphic to the second homology group $H_2E(A)$ of the group of elementary matrices. Another, and more instructive way to say this is the following. The group $E(A)$ is generated by the matrices e^a_{ij}, $a \in A$ and $i \neq j$, and generally these generators are subject to lots of relations. There are, however, some relations which are more important than others, and furthermore are universal in the sense that they are valid for any ring: the so-
`relations}` called *Steinberg relations*. One defines the Steinberg group $St(A)$ to be exactly the group generated by symbols x^a_{ij} for every $a \in A$ and $i \neq j$ subject to these relations. Explicitly:

$$x^a_{ij} x^b_{ij} = x^{a+b}_{ij}$$

and

$$[x^a_{ij}, x^b_{kl}] = \begin{cases} 1 & \text{if } i \neq l \text{ and } j \neq k \\ x^{ab}_{il} & \text{if } i \neq l \text{ and } j = k \\ x^{-ba}_{kj} & \text{if } i = l \text{ and } j \neq k \end{cases}$$

One defines $K_2(A)$ as the kernel of the surjection

$$St(A) \xrightarrow{x^a_{ij} \mapsto e^a_{ij}} E(A).$$

In fact,

$$0 \longrightarrow K_2(A) \longrightarrow St(A) \longrightarrow E(A) \longrightarrow 0$$

is a central extension of $E(A)$ (hence $K_2(A)$ is abelian), and $H_2(St(A)) = 0$, which makes it the "universal central extension" (see e.g. [66]).

The best references for K_i $i \leq 2$ are still Bass' [4] and Milnor's [86] books. Swan's paper [118] is recommended for an exposition of what optimistic hopes one might have to extend these ideas, and why some of these could not be realized (for instance, there is **no** functor K_3 such that the Mayer–Vietoris sequence extends, even if all maps are split surjective).

1.6 Higher K-theory

`{sec:High`

In the beginning of the seventies, suddenly there appeared a plethora of competing theories pretending to extend these ideas into a sequence of theories, $K_i(A)$ for $i \geq 0$. Some theories were more interesting than others, and many were equal. The one we are going to discuss in this paper is the Quillen K-theory, later extended by Waldhausen to a larger class of rings and categories.

As Quillen defines it, the K-groups are really the homotopy groups of a space $K(\mathfrak{C})$. He gave three equivalent definitions, one by the "plus" construction discussed in 1.6.1 below (we also use it in section III.1.3 but for most technical details we refer the reader to appendix A.1.6), one via "group completion" and one by what he called the Q-construction. That the definitions agree appeared in [41]. For a ring A, the homology of the space $K(A)$ is nothing but the group homology of $GL(A)$. Using the plus construction and homotopy theoretic methods, Quillen calculated in [97] $K(\mathbf{F}_q)$, where $\mathbf{F}_q$ is the field with q elements.

The advantage of the Q-construction is that it is more accessible to structural considerations. In the foundational article [99] Quillen uses the Q-construction to extend most of the general statements that were known to be true for K_0 and K_1.

However, given these fundamental theorems, of Quillen's definitions it is the plus construction that, has proven most directly accessible to calculations (this said, very few groups were in the end calculated directly from the definitions, and by now indirect methods such as motivic cohomology and the trace methods that are the topic of this book have extended our knowledge far beyond the limitations of direct calculations).

1.6.1 Quillen's plus construction

nstruction}

We will now describe a variant of Quillen's definition of the (connected cover of the) algebraic K-theory of an associative ring with unit A via the plus construction. We will be working in the category of simplicial sets (as opposed to topological spaces). The readers who are uncomfortable with this can consult appendix A1.6, and generally think of simplicial sets (often referred to as simply "spaces") as topological spaces instead. If X is a simplicial set, $H_*(X) = H(X; \mathbf{Z})$ will denote the homology of X with trivial integral coefficients, and if X is pointed we let $\tilde{H}_*(X) = H_*(X)/H_*(*)$.

ef:acyclic}

Definition 1.6.2 Let $f\colon X \to Y$ be a map of connected simplicial sets with connected homotopy fiber F. We say that f is acyclic if $\tilde{H}_*(F) = 0$.

We see that the fiber of an acyclic map must have perfect fundamental group (i.e. $0 = \tilde{H}_1(F) \cong H_1(F) \cong \pi_1 F/[\pi_1 F, \pi_1 F]$). Recall from 1.2.1 that any group π has a maximal perfect subgroup, which we call $P\pi$, and which is automatically normal.

If X is a connected space, X^+ is a space defined up to homotopy by the property that there exist an acyclic map $X \to X^+$ inducing the projection $\pi_1(X) \to \pi_1(X)/P\pi_1(X) = \pi_1(X^+)$ on the fundamental group. Here $P\pi_1(X) \subset \pi_1(X)$ is the maximal perfect subgroup.

1.6.3 Remarks on the construction

There are various models for X^+, and the most usual is Quillen's original (originally used by Kervaire [65] on homology spheres). That is, regard X as a CW complex, and add 2-cells to X to kill $P\pi_1(X)$, and then kill the noise created in homology by adding 3-cells. See e.g. [46] for details on this and related issues.

In our simplicial setting, we will use a slightly different model, giving us strict functoriality (not just in the homotopy category), namely the partial integral completion of [14, p. 219]. Just as K_0 was defined by a universal property for functions into abelian groups, the integral completion constructs a universal element over simplicial abelian groups (the "partial" is there just to take care of pathologies such as spaces where the fundamental group is not quasi-perfect). For the present purposes we only have need for the following properties of the partial integral completion, and we defer the actual construction to an appendix.

{prop:plu
{prop:plu

Proposition 1.6.4 *1. $X \mapsto X^+$ is an endofunctor of pointed simplicial sets, and there is a natural cofibration $q_X \colon X \to X^+$,*

{prop:plu

2. if X is connected, then q_X is acyclic, and

{prop:plu

3. if X is connected then $\pi_1(q_X)$ is the projection killing the maximal perfect subgroup of $\pi_1 X$

Then Quillen provides the theorem we need (for proof and precise simplicial formulation, see appendix A.1.6.3.1:

{theo:plu

Theorem 1.6.5 *For X connected, 1.6.4.2 and 1.6.4.3 characterizes X^+ up to homotopy under X.*

The integral completion will reappear as an important technical tool in chapter III.

Recall that the group $GL(A)$ was defined as the union of the $GL_n(A)$. Form the classifying space of this group, $BGL(A)$. Whether you form the classifying space before or after the limit is without consequence. Now, Quillen defines the connected cover of algebraic K-theory to be the realization $|BGL(A)^+|$ or rather, the homotopy groups,

$$K_i(A) = \begin{cases} \pi_i(BGLA^+) & \text{if } i > 0 \\ K_0(A) & \text{if } i = 0 \end{cases},$$

to be the K-groups of the ring A. In these notes we will use the following notation

{Def:alge

Definition 1.6.6 If A is a ring, then the algebraic K-theory space is

$$K(A) = BGL(A)^+$$

Now, the Whitehead lemma 1.2.2 tells us that $GL(A)$ is quasi-perfect with commutator $E(A)$, so $\pi_1 K(A) = GL(A)/PGL(A) = GL(A)/E(A)$ as expected. Furthermore, using the definition of $K_2(A)$ via the universal central extension, it is not too difficult to prove that the K_2's of Milnor and Quillen agree [87].

One might regret that this $K(A)$ has no homotopy in dimension zero, and this will be amended later. The reason we choose this definition is that the alternatives available to us at present all have their disadvantages. We might take $K_0(A)$ copies of this space, and although this would be a nice functor with the right homotopy groups, it will not agree with a more natural definition to come. Alternatively we could choose to multiply by $K_0^f(A)$ of 1.3.2.2 or $\mathbf{Z}$ as is more usual, but this has the shortcoming of not respecting products.

1.6.7 Other examples of use of the plus construction

1. Let $\Sigma_n \subset GL_n(\mathbf{Z})$ be the symmetric group of all permutations on n letters, and let $\Sigma_\infty = \lim_{n\to\infty} \Sigma_n$. Then the theorem of Barratt–Priddy–Quillen [114] states that $B\Sigma_\infty^+ \simeq \lim_{k\to\infty} \Omega^k S^k$, so $\pi_*(B\Sigma^+)$ are the stable homotopy groups of spheres.

2. Let X be a connected space whith abelian fundamental group. Then Kan and Thurston [60] has proved that X is, up to homotopy, a BG^+ for some strange group G. With a slight modification, the theorem can be extended to arbitrary connected X.

1.6.8 Alternative definition of $K(A)$

In case the partial integral completion bothers you; for $BGL(A)$ it can be substituted by the following construction: choose an acyclic cofibration $BGL(\mathbf{Z}) \to BGL(\mathbf{Z})^+$ once and for all (by adding particular 2 and 3 cells), and define algebraic K-theory by means of the pushout square

$$\begin{array}{ccc} BGL(\mathbf{Z}) & \longrightarrow & BGL(A) \\ \downarrow & & \downarrow \\ BGL(\mathbf{Z})^+ & \longrightarrow & BGL(A)^+ \end{array}$$

This will of course be functorial in A, and it can be verified that it has the right homotopy properties. However, at one point (e.g. in chapter III.) we will need functoriality of the plus construction for more general spaces. All the spaces which we will need in these notes can be reached by choosing to do our first plus not on $BGL(\mathbf{Z})$, but on BA_5. See appendix A.1.6.4 for more details.

1.7 Some of the results prior to 1990

or to 1990}

1. Quillen [97]: If $\mathbf{F}_q$ is the field with q elements, then

$$K_i(\mathbf{F}_q) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Z}/(q^j - 1)\mathbf{Z} & \text{if } i = 2j-1 \\ 0 & \text{if } i = 2j > 0 \end{cases}.$$

If $\bar{\mathbf{F}}_p$ is the algebraic closure of $\mathbf{F}_p$, then

$$K_i(\bar{\mathbf{F}}_p) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Q}/\mathbf{Z}[1/p] & \text{if } i = 2j-1 \\ 0 & \text{if } i = 2j > 0 \end{cases}.$$

The Frobenius automorphism $\Phi(a) = a^p$ induces multiplication by p^i on $K_{2i-1}(\bar{\mathbf{F}}_p)$, and the subgroup fixed by Φ^k is $K_{2i-1}(\mathbf{F}_{p^k})$.

2. Suslin [115]: "The algebraic K-theory of algebraically closed fields only depends on the characteristic, and away from the characteristic it always agrees with topological K-theory". More presicely:

   Let F be an algebraically closed field. $K_i(F)$ is divisible for $i \geq 1$. The torsion subgroup of $K_i(F)$ is zero if i is even, and

   $$\begin{cases} \mathbf{Q}/\mathbf{Z}[1/p] & \text{if } char(F) = p > 0 \\ \mathbf{Q}/\mathbf{Z} & \text{if } char(F) = 0 \end{cases}$$

   if i is odd. (see [117] for references.)

   On the spectrum level Suslin's results are: If p is a prime different from the characteristic of F, then

   $$K(F)^{\widehat{}}_p \simeq ku^{\widehat{}}_p$$

   (ku is complex K-theory and $^{\widehat{}}_p$ is p-completion) and if p is the characteristic of F, then

   $$K(F)^{\widehat{}}_p \simeq H\mathbf{Z}^{\widehat{}}_p.$$

3.
   - $K_0(\mathbf{Z}) = \mathbf{Z}$,
   - $K_1(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$,
   - $K_2(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$,
   - $K_3(\mathbf{Z}) = \mathbf{Z}/48\mathbf{Z}$, (Lee-Szczarba [70]).

4. Borel [10]: Let A be the integers in a number field F and n_j the order of the vanishing of the zeta function

   $$\zeta_F(s) = \sum_{I \text{ ideal in } A} |A/I|^{-s}$$

   at $s = 1 - j$. Then

   $$\operatorname{rank} K_i(A) = \begin{cases} 0 & \text{if } i = 2j > 0 \\ n_j & \text{if } i = 2j - 1 \end{cases}$$

   ex: If $A = \mathbf{Z}$, then

   $$n_j = \begin{cases} 1 & \text{if } j = 2k - 1 > 1 \\ 0 & \text{otherwise} \end{cases}$$

   Furthermore, Quillen [98] proves that the groups $K_i(A)$ are finitely generated.

5. [91] Let A be a perfect ring of characteristic p (meaning that the Frobenius homomorphism $a \mapsto a^p$ is an isomorphism), then $K_i(A)$ is uniquely p-divisible for $i > 0$.

6. Gersten [35]/Waldhausen [127]: If A is a free ring, then $K(A) \simeq K(\mathbf{Z})$.

7. Barratt-Priddy-Quillen [114]: the K-theory of the category of finite sets is equivalent to the sphere spectrum.

8. Waldhausen [127]: If G is a free group, free abelian group, or the fundamental group of a submanifold of the three-sphere, then there is a spectral sequence

$$E^2_{p,q} = H_p(G; K_q(\mathbf{Z})) \Rightarrow K_{p+q}(\mathbf{Z}[G])$$

9. Waldhausen [126]: The K-theory (in his sense) of the category of retractive spaces over a given space X, is equivalent to the product of the suspension spectrum of X and the differentiable Whitehead spectrum of X.

10. Goodwillie [39]: If $A \to B$ is a surjective map of rings such that the kernel is nilpotent, then the relative K-theory and the relative cyclic homology agree rationally:

$$K_i(A \to B) \otimes \mathbf{Q} \cong HC_{i-1}(A \to B) \otimes \mathbf{Q}.$$

11. Suslin/Panin:

$$K(\hat{\mathbf{Z}_p})\hat{} \simeq \operatorname{holim}_{\overleftarrow{n}} K(\mathbf{Z}/p^n\mathbf{Z})\hat{}$$

where $\hat{}$ denotes profinite completion.

1.8 Conjectures and such

1.9 Recent results

Lichtenbaum-Quillen and all the calculations using TC.

1.10 Where to read

Two very readable surveys on the K-theory of fields and related issues are [43] and [117]. The survey article [89] is also highly recommended. For the K-theory of spaces see [131]. Some introductory books about higher K-theory exist: [5], [112], [104] and [56], and a new one (which looks very promising to me) is currently being written by Weibel [133]. The "Reviews in K-theory 1940–84" [81], is also helpful (although with both Mathematical Reviews and Zentralblatt on the web it has lost some of its glory).

2 The algebraic K-theory spectrum.

Ideally, the so called "higher K-theory" is nothing but a reformulation of the idea behind K_0: the difference is that whereas K_0 had values in Abelian groups, K-theory has values in spectra. For convenience, we will follow Waldhausen and work with categories with cofibrations (see 2.1 below). When interested in the K-theory of rings we should, of course, apply our K-functor to the category $\mathcal{P}_A$ of finitely generated projective modules. The projective modules form a special example of what Quillen calls an exact category (see 1.3), which again is an example of a category with cofibrations.

There are many definitions of K-theory, each with their own advantages and disadvantages. Quillen started off the subject with no less than three: the plus construction, the group completion approach and the "Q"-construction. Soon more versions appeared, but luckily most turned out to be equivalent to Quillen's whenever given the same input. We will eventually meet three: Waldhausen's "S"-construction which we will discuss in just a moment, Segal's Γ-space approach (see chapter II.3), and Quillen's plus construction (see 1.6.1 and A.1.6).

2.1 Categories with cofibrations

{subsec:ca

The source for these facts is Waldhausen's [131] from which we steal indiscriminantly. That a category is *pointed* means that it has a chosen zero object 0 which is both initial and final.

{Def:cate

Definition 2.1.1 A *category with cofibrations* is a pointed category $\mathcal{C}$ together with a subcategory $co\mathcal{C}$ satisfying

{Def:cate

1. all isomorphisms are in $co\mathcal{C}$ {Def:cate
2. all maps from the zero object are in $co\mathcal{C}$ {Def:cate
3. if $A \to B \in co\mathcal{C}$ and $A \to C \in \mathcal{C}$, then the pushout

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C \coprod_A B \end{array}$$

   exists, and the lower horizontal map is in $co\mathcal{C}$.

We will call the maps in $co\mathcal{C}$ simply *cofibrations*. Cofibration may occasionally be written $\rightarrowtail$. A functor between categories with cofibrations is *exact* if it is pointed, takes cofibrations to cofibrations, and preserves the pushout diagrams in 3.

Example 2.1.2 (The category of finitely generated projective modules.) Let A be a ring (unital and associative as always) and let $\mathcal{M}_A$ be the category of all A modules. We will eventually let K-theory of the ring A be the K-theory of the category $\mathcal{P}_A$ of finitely generated projective right A-modules. The interesting structure of $\mathcal{P}_A$ as a category with cofibrations is to let the cofibrations be the injections $P' \rightarrowtail P$ in $\mathcal{P}_A$ such that the quotient P/P' is also in $\mathcal{P}_A$. That is, if $P' \rightarrowtail P \in \mathcal{P}_A$ is a cofibration if it is the first part of a short exact sequence

$$0 \to P' \rightarrowtail P \twoheadrightarrow P'' \to 0$$

of projective modules. In this case the cofibrations are split, i.e., to any cofibration $f\colon P' \to P$ there exist $g\colon P \to P'$ in $\mathcal{P}_A$ such that $gf = id_{P'}$.

A ring homomorphism $f\colon B \to A$ induces a pair of adjoint functors

$$\mathcal{M}_B \underset{f^*}{\overset{-\otimes_B A}{\rightleftarrows}} \mathcal{M}_A$$

where f^* is restriction of scalars. The adjunction isomorphism

$$\mathcal{M}_A(Q \otimes_B A, Q') \cong \mathcal{M}_B(Q, f^*Q')$$

is given by sending $L\colon Q \otimes_B A \to Q'$ to $q \mapsto L(q \otimes 1)$. When restricted to finitely generated projective modules $- \otimes_B A$ induces a map $K_0(B) \to K_0(A)$ making K_0 into a functor.

Usually authors are not too specific about their choice of $\mathcal{P}_A$, but unfortunately this may not always be good enough. For one thing the assignment $A \mapsto \mathcal{P}_A$ should be functorial, and the problem is the annoying fact that if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are maps of rings, then $(M \otimes_A B) \otimes_B C$ and $M \otimes_A C$ are generally only naturally isomorphic (not equal).

So whenever pressed, $\mathcal{P}_A$ is the following category.

{Def:fgp}

Definition 2.1.3 Let A be a ring. The category of *finitely generated projective A-modules* $\mathcal{P}_A$ is the following category with cofibrations. Its objects are the pairs (m, p), where m is a nonnegative integer and $p = p^2 \in M_m(A)$. A morphism $(m, p) \to (n, q)$ is an A-module homomorphism $im(p) \to im(q)$. A cofibration is a split monomorphism.

Since $p^2 = p$ we get that $im(p) \subseteq A^m \xrightarrow{p} im(p)$ is the identity, and $im(p)$ is a finitely generated projective module, and any finitely generated projective module in $\mathcal{M}_A$ is isomorphic to some such image, and so the full and faithful functor $\mathcal{P}_A \to \mathcal{M}_A$ sending (m, p) to $im(p)$ displays $\mathcal{P}_A$ as a category equivalent to the category of finitely generated projective objects in $\mathcal{M}_A$. Note that for any morphism $a\colon (m, p) \to (n, q)$ we may define

$$x_a\colon A^m \twoheadrightarrow im(p) \xrightarrow{a} im(q) \subseteq A^n,$$

and we get that $x_a = x_a p = q x_a$. In fact, when $(m, p) = (n, q)$, you get an isomorphism of rings

$$\mathcal{P}_A((n, p), (n, p)) \cong \{y \in M_n(A) | y = yp = py\}$$

via $a \mapsto x_a$, with inverse

$$y \mapsto \{im(p) \subseteq A^n \xrightarrow{y} A^n \xrightarrow{p} im(p)\}.$$

Note that the unit in the right side ring is the matrix p.

If $f\colon A \to B$ is a ring homomorphism, then $f_*\colon \mathcal{P}_A \to \mathcal{P}_B$ is given on objects by $f_*(m, p) = (m, f(p))$ ($f(p) \in M_m(B)$ is the matrix you get by using f on each entry in

p), and on morphisms $a\colon (m,p) \to (n,q)$ by $f_*(a) = f(x_a)|_{im(f(p))}$, which is well defined as $f(x_a) = f(q)f(x_a) = f(x_a)f(p)$. There is a natural isomorphism between

$$\mathcal{P}_A \longrightarrow \mathcal{M}_A \xrightarrow{M \mapsto M \otimes_A B} \mathcal{M}_B$$

and

$$\mathcal{P}_A \xrightarrow{f_*} \mathcal{P}_B \longrightarrow \mathcal{M}_B.$$

The assignment $A \mapsto \mathcal{P}_A$ is a functor from rings to exact categories.

Example 2.1.4 (The category of finitely generated free module) Let A be a ring. To conform with the strict definition of $\mathcal{P}_A$ in 2.1.3, we define the category $\mathcal{F}_A$ of finitely generated free A-modules as the full subcategory of $\mathcal{P}_A$ with objects of the form $(n,1)$, where 1 is the identity $A^n = A^n$. The inclusion $\mathcal{F}_A \subseteq \mathcal{P}_A$ is "cofinal" in the sense that given any object $(n,p) \in ob\mathcal{P}_A$ there exists another object $(m,q) \in ob\mathcal{P}_A$ such that $(m,q) \oplus (n,p) = (m+n, q \oplus p)$ is isomorphic to a free module. This will have the consequence that the K-theories of $\mathcal{F}_A$ and $\mathcal{P}_A$ only differ at K_0. {Def:fgf}

2.1.5 K_0 of categories with cofibrations

If $\mathcal{C}$ is a category with cofibrations, we let the "short exact sequences" be the cofiber sequences $c' \rightarrowtail c \twoheadrightarrow c''$, meaning that $c' \rightarrowtail c$ is a cofibration and the sequence fits in a pushout square

$$\begin{array}{ccc} c' & \longrightarrow & c \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c'' \end{array}$$

This set is the set of objects of a category which we will call $S_2\mathcal{C}$. The maps are commutative diagrams

$$\begin{array}{ccccc} c' & \longrightarrow & c & \longrightarrow & c'' \\ \downarrow & & \downarrow & & \downarrow \\ d' & \longrightarrow & d & \longrightarrow & d'' \end{array}$$

Note that we can define cofibrations in $S_2\mathcal{C}$ too: a map like the one above is a cofibration if the vertical maps are cofibrations and the map from $c \coprod_{c'} d'$ to d is a cofibration.

{lem:$S_2

Lemma 2.1.6 *With these definitions $S_2\mathcal{C}$ is a category with cofibrations.*

Proof: Firstly, we have to prove that a composite of two cofibrations

$$\begin{array}{ccccc} c' & \longrightarrow & c & \longrightarrow & c'' \\ \downarrow & & \downarrow & & \downarrow \\ d' & \longrightarrow & d & \longrightarrow & d'' \\ \downarrow & & \downarrow & & \downarrow \\ e' & \longrightarrow & e & \longrightarrow & e'' \end{array}$$

again is a cofibration. The only thing to be checked is that the map from $c\coprod_{c'} e'$ to e is a cofibration, but this follows by 2.1.1.1. and 3. since

$$c\coprod_{c'} e' \cong c\coprod_{c'} d' \coprod_{d'} e' \rightarrowtail d\coprod_{d'} e' \rightarrowtail e$$

The axioms 2.1.1.1 and 2.1.1.2 are clear, and for 2.1.1.3 we reason as follows. Consider the diagram

$$\begin{array}{ccccc} d' & \longrightarrow & d & \longrightarrow & d'' \\ \uparrow & & \uparrow & & \uparrow \\ c' & \longrightarrow & c & \longrightarrow & c' \\ \downarrow & & \downarrow & & \downarrow \\ e' & \longrightarrow & e & \longrightarrow & e'' \end{array}$$

where the rows are objects of $S_2\mathcal{C}$ and the upwards pointing maps constitute a cofibration in $S_2\mathcal{C}$. Taking the pushout (which you get by taking the pushout of each column) the only nontrivial part of 2.1.1.3. is that we have to check that $(e'\coprod_{c'} d')\coprod_{d'} d \to e\coprod_c d$ is a cofibration. But this is so since it is the composite

$$(e'\coprod_{c'} d')\coprod_{d'} d \cong e'\coprod_{c'} d \coprod (e'\coprod_{c'} c)\coprod_{c} d \to e\coprod_{c} d$$

and the last map is a cofibration since $e'\coprod_{c'} c \to e$ is. ∎

There are three important functors

$$d_0, d_1, d_2 \colon S_2\mathcal{C} \to \mathcal{C}$$

sending a sequence $\mathbf{c} = \{c' \rightarrowtail c \twoheadrightarrow c''\}$ to $d_0(\mathbf{c}) = c''$, $d_1(\mathbf{c}) = c$ and $d_2(\mathbf{c}) = c'$.

all exact.}

Lemma 2.1.7 *The functors $d_i \colon S_2\mathcal{C} \to \mathcal{C}$ are all exact.*

Proof: See [131, p. 323]. ∎

We now give a reformulation of the definition of K_0. We let $\pi_0 i\mathcal{C}$ be the set of isomorphism classes of $\mathcal{C}$. That a functor F from categories with cofibrations to abelian groups is "under $\pi_0 i$" then means that it comes equipped with a natural map $\pi_0 i\mathcal{C} \to F(\mathcal{C})$, and a map between such functors must respect this structure.

Lemma 2.1.8 *K_0 is the universal functor F under $\pi_0 i$ to abelian groups satisfying additivity, i.e., such that the natural map*

$$F(S_2\mathcal{C}) \xrightarrow{(d_0,d_2)} F(\mathcal{C}) \times F(\mathcal{C})$$

is an isomorphism.

Proof: First one shows that K_0 satisfies additivity. Consider the splitting $K_0(\mathcal{C}) \times K_0(\mathcal{C}) \to K_0(S_2\mathcal{C})$ which sends $([a],[b])$ to $[a \rightarrowtail a \vee b \to b]$. We have to show that the composite

$$K_0(S_2\mathcal{C}) \xrightarrow{(d_0,d_2)} K_0(\mathcal{C}) \times K_0(\mathcal{C}) \longrightarrow K_0(S_2\mathcal{C})$$

sending $[a' \rightarrowtail a \to a'']$ to $[a' \rightarrowtail a' \vee a'' \to a''] = [a' = a' \to 0] + [0 \to a'' = a'']$ is the identity. But this is clear from the diagram

$$\begin{array}{ccccc} a' & = & a' & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \\ a' & \longrightarrow & a & \longrightarrow & a'' \\ \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & a'' & = & a'' \end{array}$$

in $S_2S_2\mathcal{C}$. Let F be any other functor under $\pi_0 i$ satisfying additivity. By additivity the function $\pi_0(i\mathcal{C}) \to F(\mathcal{C})$ satisfies the additivity condition used in the definition of K_0 in 1.3.1; so there is a unique factorization $\pi_0(i\mathcal{C}) \to K_0(\mathcal{C}) \to F(\mathcal{C})$ which for the same reason must be functorial. ∎

The question is: can we obtain deeper information about the category $\mathcal{C}$ if we allow ourselves a more fascinating target category than abelian groups? The answer is yes. If we use a category of spectra instead we get a theory – K-theory – whose homotopy groups we have already seen some of.

2.2 Waldhausen's S construction

We now give Waldhausen's definition of the K-theory of a category with (isomorphisms and) cofibrations. (According to Waldhausen, the "S" is for "Segal" as in Graeme B. Segal. According to Segal his construction was close to the "block-triangular" version given for additive categories in 2.2.4 below. Apparently, Segal and Quillen were aware of this construction even before Quillen discovered his Q-construction, but it was not before Waldhausen reinvented it that it became apparent that the S-construction was truly useful. In fact, in a letter to Segal [96], Quillen comments: "... But it was only this spring that I succeeded in freeing myself from the shackles of the simplicial way of thinking and found the category Q($\underline{\underline{B}}$)".)

For any category $\mathcal{C}$, the *arrow category* $Ar\mathcal{C}$ (not to be confused with the *twisted* arrow category $T\mathcal{C}$ used sometimes), is the category whose objects are the morphisms in $\mathcal{C}$, and where a morphism from $f: a \to b$ to $g: c \to d$ is a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} a & \longrightarrow & c \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ b & \longrightarrow & d \end{array}$$

Consider the ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category, and consider the arrow category $Ar[n]$.

{Def:IIWal

Definition 2.2.1 Let $\mathcal{C}$ be a category with cofibrations. Then $S\mathcal{C} = \{[n] \mapsto S_n\mathcal{C}\}$ is the simplicial category which in degree n is the category $S_n\mathcal{C}$ of functors $C\colon Ar[n] \to \mathcal{C}$ satisfying the following properties

1. For all $j \geq 0$ we have that $C(j = j) = 0$ (the preferred null object in $\mathcal{C}$)

2. if $i \leq j \leq k$, then $C(i \leq j) \rightarrowtail C(i \leq k)$ is a cofibration, and

$$\begin{array}{ccc} C(i \leq j) & \longrightarrow & C(i \leq k) \\ \downarrow & & \downarrow \\ C(j = j) & \longrightarrow & C(j \leq k) \end{array}$$

   is a pushout.

To get one's hand on each $S_n\mathcal{C}$, think of the objects as strings

$$C_{01} \rightarrowtail C_{02} \rightarrowtail \ldots \rightarrowtail C_{0n}$$

with compatible choices of cofibers $C_{ij} = C_{0j}/C_{0i}$, or equivalently as triangles

$$\begin{array}{ccccccccc} C_{01} & \rightarrowtail & C_{02} & \rightarrowtail & C_{03} & \rightarrowtail \cdots \rightarrowtail & C_{0,n-1} & \rightarrowtail & C_{0n} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & C_{12} & \rightarrowtail & C_{13} & \rightarrowtail \cdots \rightarrowtail & C_{1,n-1} & \rightarrowtail & C_{1n} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & C_{23} & \rightarrowtail \cdots \rightarrowtail & C_{2,n-1} & \rightarrowtail & C_{2,n} \\ & & & & & & \downarrow & & \downarrow \\ & & & & & \ddots & \vdots & & \vdots \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & C_{n-2,n-1} & \rightarrowtail & C_{n-2,n} \\ & & & & & & & & \downarrow \\ & & & & & & & & C_{n-1,n} \end{array}$$

with horizontal arrows cofibrations and every square a pushout (the null object is placed in the corners below the diagonal).

The first thing one should notice is that

Lemma 2.2.2 *There is a natural isomorphism $K_0(\mathcal{C}) \cong \pi_1(obS\mathcal{C})$.*

Proof: Since obviously $obS_0\mathcal{C}$ is trivial, $\pi_1(obS\mathcal{C})$ is the quotient of the free group on the pointed set $ob\mathcal{C} = obS_1\mathcal{C}$ by the relation that $[c'] = [c'']^{-1}[c]$ for every $c' \rightarrowtail c \twoheadrightarrow c'' \in obS_2\mathcal{C}$ (this uses the Kan loop group description of the fundamental group of a space with only one zero simplex, see the appendix A.1.1.9). An isomorphism $c' \xrightarrow{\cong} c$ can be considered as an element $c' \xrightarrow{\cong} c \to 0 \in obS_2\mathcal{C}$, and so $[c'] = [c]$. Since we then have that

$$[c'][c''] = [c'' \vee c'] = [c' \vee c''] = [c''][c']$$

we get an abelian group, and so $K_0(\mathcal{C})$ is the quotient of the free abelian group on the isomorphism classes of $\mathcal{C}$ by the relation $[c'] + [c''] = [c]$, which is just the formula arrived at in 1.3 ∎

Thus we have that $K_0(A) = K_0(\mathcal{P}_A)$ is the fundamental group of $obS\mathcal{P}_A$ if we choose the cofibrations to be the split monomorphisms, and it can be shown that $K_i(A)$ is $\pi_{i+1}(obS\mathcal{P}_A)$ for the other groups we discussed in the introduction (namely $i = 1$ and $i = 2$).

2.2.3 Additive categories

Recall that an *$\mathcal{A}b$-category* [79] is a category where the morphism sets are abelian groups and where composition is bilinear (also called *linear category*). An *additive category* is an $\mathcal{A}b$-category with all finite products.

Let $\mathfrak{C}$ be an additive category, regarded as a category with cofibrations by letting the cofibrations be the split inclusions. With this choice we call $\mathfrak{C}$ a *split exact category.*

In these cases it is easier to see how the S-construction works. Note that if

$$c = (c_{0,1}, \dots, c_{i-1,i}, \dots, c_{n-1,n})$$

is a sequence of objects, then the sum diagram $\psi_n c$ with

$$(\psi_n c)_{ij} = \bigoplus_{i \le k \le j} c_{k-1,k}$$

and maps the obvious inclusions and projections, is an element in $S_n\mathfrak{C}$. Since $\mathfrak{C}$ is split exact every element of $S_n\mathfrak{C}$ is isomorphic to such a diagram. Maps between two such sum diagrams can be thought of as upper triangular matrices:

`{Def:$T_n`

Definition 2.2.4 Let $\mathcal{C}$ be an $\mathcal{A}b$-category. For every $n > 0$, we define $T_n\mathcal{C}$ – the $n \times n$ *upper triangular matrices* on $\mathcal{C}$ – to be the category with objects $ob\mathcal{C}^n$, and morphisms

$$T_n\mathcal{C}((c_1, \dots, c_n), (d_1, \dots, d_n)) = \bigoplus_{1 \le j \le i \le n} \mathcal{C}(c_i, d_j)$$

with composition given by matrix multiplication

{lem:T=S}

Lemma 2.2.5 *Let $\mathfrak{C}$ be additive. Then the assignment ψ_q given in the discussion above defines a full and faithful functor*

$$\psi_q \colon T_q\mathfrak{C} \to S_q\mathfrak{C}$$

which is an equivalence of categories since $\mathfrak{C}$ is split

2.3 The equivalence $obS\mathcal{C} \to BiS\mathcal{C}$

An amazing – and very useful – property about the simplicial set of objects of the S-construction is, considered as a functor from categories with cofibrations to simplicial sets, it transforms natural isomorphisms to homotopies, and so is invariant up to homotopy to equivalences of categories.

This is reminiscent to the classifying space construction, but slightly weaker in that the classifying space takes all natural transformations to homotopies, whereas obS only does this to natural isomorphisms.

Assume all categories are small. If $\mathcal{C}$ is a category, then $i\mathcal{C} \subseteq \mathcal{C}$ is the subcategory with all objects, but only isomorphisms as morphisms.

For every $n \geq 0$, regard $[n] = \{0 < 1 < \cdots < n\}$ as a category (if $a \leq b$ there is a unique map $a \leftarrow b$), and maps in Δ as functors (hence we may regard Δ as a full subcategory of the category of small categories). The *classifying space* (or *nerve*) of a small category $\mathcal{C}$ is the space (simplicial set) $B\mathcal{C}$ defined by

$$[q] \mapsto B_q\mathcal{C} = \{c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q \in \mathcal{C}\} = \{\text{functors } [q] \to \mathcal{C}\}.$$

Note that $B[q] = \Delta[q]$. The standard fact that natural transformations induce homotopies come from the fact that a natural transformation is the same as a functor $\mathcal{C} \times [1] \to \mathcal{D}$, and $B(\mathcal{C} \times [1]) \cong B\mathcal{C} \times B[1] = B\mathcal{C} \times \Delta[1]$. (see appendix A.1.1.5 for related topics).

{obShtfu}

Lemma 2.3.1 *If*

$$f, g\colon \mathcal{C} \to \mathcal{D}$$

are isomorphic exact functors, then they induce homotopic maps

$$obS\mathcal{C} \to obS\mathcal{D}.$$

Hence $\mathcal{C} \mapsto obS\mathcal{C}$ sends equivalences of categories to homotopy equivalences of spaces.

Proof: (the same proof as in [131, 1.4.1]). We define a homotopy

$$H\colon obS\mathcal{C} \times B[1] \longrightarrow obS\mathcal{D}$$

as follows. Regard the isomorphism $f \cong g$ as a functor $F\colon \mathcal{C} \times [1] \to \mathcal{D}$. Let $c\colon Ar[n] \to \mathcal{C}$ be an object of $S_n\mathcal{C}$, and $\phi \in B_n[1] = \Delta([n],[1])$. Then $H(c,\phi)$ is the composite

$$Ar[n] \longrightarrow Ar[n] \times [n] \xrightarrow{(c,\phi)} \mathcal{C} \times [1] \xrightarrow{F} \mathcal{D}$$

where the fist map sends $i \leq j$ to $(i \leq j, j)$. This is an element in $S_n\mathcal{D}$ since $f \cong g$ is an isomorphism. ∎

obSinobNtS}

Corollary 2.3.2 *If $t\mathcal{C} \subset i\mathcal{C}$ is a subcategory of the isomorphisms containing all objects, then the inclusion of the zero skeleton is an equivalence*

$$obS\mathcal{C} \xrightarrow{\simeq} BtS\mathcal{C}$$

where $tS_q\mathcal{C} \subseteq S_q\mathcal{C}$ is the subcategory whose morphisms are transformations coming from $t\mathcal{C}$.

Proof: This follows by regarding the bisimplicial object

$$\{[p],[q]\mapsto B_p tS_q\mathcal{C}\}$$

as $obS_q\mathbf{N}_p(\mathcal{C},t\mathcal{C})$, where $\mathbf{N}_p(\mathcal{C},t\mathcal{C})$ is a full subcategory of the **category** $\mathbf{N}_p\mathcal{C}$ of functors $[p]\to\mathcal{C}$ and natural transformations between these. The objects of $\mathbf{N}_p(\mathcal{C},t\mathcal{C})$ are the chains of maps in $t\mathcal{C}$, i.e., $ob\mathbf{N}(\mathcal{C},t\mathcal{C})=B_pt\mathcal{C}$.

Consider the functor $\mathcal{C}\to\mathbf{N}_p(\mathcal{C},t\mathcal{C})$ given by sending c to the chain of identities on c (here we need that all identity maps are in $t\mathcal{C}$). It is an equivalence of categories. A splitting being given by e.g., sending $c_0\leftarrow\cdots\leftarrow c_p$ to c_0: the natural isomorphism to the identity on $\mathbf{N}_p(\mathcal{C},t\mathcal{C})$ being given by

$$\begin{array}{ccccccccc}
c_0 & \xleftarrow{\alpha_1} & c_1 & \xleftarrow{\alpha_2} & c_2 & \xleftarrow{\alpha_3} & \dots & \xleftarrow{\alpha_p} & c_p \\
\| & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_1\alpha_2}\downarrow & & & & {\scriptstyle\alpha_1\alpha_2\dots\alpha_p}\downarrow \\
c_0 & = & c_0 & = & c_0 & = & \dots & = & c_0
\end{array}.$$

Considering $obS\mathcal{C}\to BtS\mathcal{C}$ as a map of bisimplicial spaces, we see that by 2.3.1 it is a homotopy equivalence

$$obS\mathcal{C}=obS\mathbf{N}_0(\mathcal{C},t\mathcal{C})\to obS\mathbf{N}_p(\mathcal{C},t\mathcal{C})=B_ptS\mathcal{C}$$

in every degree, and so by A.1.5.0.2 a weak equivalence of diagonals. ∎

2.3.3 Additivity

The fundamental theorem of the S-construction is the additivity theorem. For proofs we refer the reader to [131] or [83]. This result is actually not used explicitly anywhere in these notes, but it is our guiding theorem for all of K-theory. In fact, it shows that the S-construction is a true generalization of K_0, giving the same sort of universality for K-theory considered as a functor into spectra (see below).

{additivi

Theorem 2.3.4 *Let $\mathcal{C}$ be a category with cofibrations. The natural map*

$$obS(S_2\mathcal{C})\to obS(\mathcal{C})\times obS(\mathcal{C})$$

is a weak equivalence. ∎

2.4 The spectrum

Continuing where lemma 2.1.6 and 2.1.7 left off, one checks that the definition of $S\mathcal{C}$ guarantees that it is in fact a simplicial category with cofibrations.

To be precise,

Definition 2.4.1 Let $\mathcal{C}$ be a category with cofibrations. A cofibration $c \rightarrowtail d \in S_q\mathcal{C}$ is a map such that for $0 < i \leq q$ the maps

$$c_{0i} \rightarrowtail d_{0i}$$

and

$$d_{0,i-1} \coprod_{c_{0,i-1}} c_{0i} \rightarrowtail d_{0i}$$

are all cofibrations.

Note that if $c \rightarrowtail d$ is a cofibration then it follows that all the maps $c_{ij} \rightarrowtail d_{ij}$ are cofibrations.

This means that we may take S of each $S_n\mathcal{C}$, and in this way obtain a bisimplicial object $SS\mathcal{C}$, and by iteration, a sequence of (multi)-simplicial objects $S^{(m+1)}\mathcal{C} = SS^{(m)}\mathcal{C}$.

Recall that a *spectrum* is a sequence of pointed spaces, $m \mapsto X_m$, $m \geq 0$, together with maps $S^1 \wedge X_m \to X_{m+1}$. See appendix A.1.2 for further development of the basic properties of spectra, but recall that given a spectrum X, we define its homotopy groups as

$$\pi_q X = \lim_{\overrightarrow{k}} \pi_{k+q} X_k$$

(where the colimit is taken along the adjoint of the structure maps. A map of spectra $f\colon X \to Y$ is a *pointwise equivalence* if $f_n\colon X_n \to Y_n$ is a weak equivalence for every n, and a *stable equivalence* if it induces an isomorphism $\pi_*(f)\colon \pi_*X \to \pi_*Y$.

We will study another model for spectra much closer in chapter II. Morally, spectra are beefed up versions of chain complexes, but in reality they give you much more.

Note that $S_0\mathcal{C} = *$, i.e., $S\mathcal{C}$ is *reduced*. If we consider the space $obS\mathcal{C}$ it will also be reduced, and the inclusion of the 1-skeleton $obS_1\mathcal{C} = ob\mathcal{C}$ gives a map

$$S^1 \wedge ob\mathcal{C} \to obS\mathcal{C}$$

This means that the multi-simplicial sets

$$m \mapsto obS^{(m)}\mathcal{C} = ob\underbrace{S \dots S}_{m \text{ times}}\mathcal{C}$$

form a spectrum after taking the diagonal.

A consequence of the additivity theorem 2.3.4 is that this spectrum is almost an "Ω-spectrum": more precisely the adjoint maps $obS^{(m)}\mathcal{C} \to \Omega obS^{(m+1)}\mathcal{C}$ are equivalences for all $m > 0$. We won't need this fact.

Let for any category $i\mathcal{D} \subseteq \mathcal{D}$ be the subcategory with the same objects, but with only the isomorphisms as morphisms. As before, we get a map $S^1 \wedge Bi\mathcal{C} \to BiS\mathcal{C}$, and hence another spectrum $m \mapsto BiS^{(m)}\mathcal{C}$.

For each n the degeneracies induce an inclusion

$$obS^{(n)}\mathcal{C} = B_0 iS^{(n)}\mathcal{C} \to BiS^{(n)}\mathcal{C}$$

giving a map of spectra. That the two spectra are pointwise equivalent (that is, the maps $obS^{(n)}\mathcal{C} = B_0 iS^{(n)}\mathcal{C} \to BiS^{(n)}\mathcal{C}$ are all weak equivalences of spaces) follows from corollary 2.3.2.

Definition 2.4.2 Let $\mathcal{C}$ be a category with cofibrations. Then

$$\mathbf{K}(\mathcal{C}) = \{m \mapsto obS^{(m)}\mathcal{C}\}$$

is the K-theory spectrum of $\mathcal{C}$ (with respect to the isomorphisms).

In these notes we will only use this definition for categories with cofibrations which are $\mathcal{A}b$-categories. Exact categories are particular examples of $\mathcal{A}b$-categories with cofibrations, and we will never need the further restrictions in the definition of exact categories, even though we will give all statements for exact categories only.

The additivity theorem 2.3.4 can be restated as a property of the K-theory spectrum: The natural map

$$\mathbf{K}(S_2\mathcal{C}) \longrightarrow \mathbf{K}(\mathcal{C}) \times \mathbf{K}(\mathcal{C})$$

is a pointwise equivalence (i.e.,

$$obS^{(n)}(S_2\mathcal{C}) \longrightarrow S^{(n)}(\mathcal{C}) \times obS^{(n)}(\mathcal{C})$$

is a weak equivalence for all n). One should note that the claim that the map is a *stable* equivalence follows almost automatically by the construction (see [131, 1.3.5]).

Definition 2.4.3 (K-theory of rings) Let A be a ring (unital and associative as always). Then we define the K-theory of A, $\mathbf{K}(A)$, to be $\mathbf{K}(\mathcal{P}_A)$, the K-theory of the category of finitely generated projective right A-modules.

K-theory behaves nicely with respect to "cofinal" inclusions, see e.g., [113], and we cite the only case we need: the inclusion $\mathcal{F}_A \subseteq \mathcal{P}_A$ induces a homotopy fiber sequence of spectra

$$\mathbf{K}(\mathcal{F}_A) \longrightarrow \mathbf{K}(\mathcal{P}_A) \longrightarrow H(K_0(A)/K_0^f(A))$$

where $H(M)$ is the Eilenberg-MacLane spectrum of an abelian group M (a spectrum whose only nonzero homotopy group is $K_0(A)/K_0^f(A)$ in dimension zero. See section II.1 for a construction). Hence the homotopy groups of $\mathbf{K}(\mathcal{F}_A)$ and $\mathbf{K}(A) = \mathbf{K}(\mathcal{P}_A)$ coincide in positive dimensions.

2.5 K-theory of split radical extensions

Recall that if B is a ring, the Jacobson radical $rad(M)$ of an B-module M is the intersection of all the kernels of maps from M to simple modules [4, p. 83]. Of particular importance to us is the case of a nilpotent ideal $I \subseteq B$. Then $I \subseteq rad(B)$ since $1 + I$ consists of units.

We now turn to the very special task of giving a suitable model for $\mathbf{K}(B)$ when $f\colon B \to A$ is a split surjection with kernel I contained in the Jacobson radical $rad(B) \subseteq B$. We have some low dimensional knowledge about this situation, namely 1.2.3. and 1.3.2.5. which tell us that $K_0(B) \cong K_0(A)$ and that the multiplicative group $(1+I)^\times$ maps surjectively onto the kernel of $K_1(B) \twoheadrightarrow K_0(A)$. Some knowledge of K_2 was also available already in the seventies (see e.g., [22] [125] and [77])

We use the strictly functorial model explained in 2.1.3 for the category of finitely generated projective modules $\mathcal{P}_A$ where an object is a pair (m, p) where m is a natural number and $p \in M_m A$ satisfies $p^2 = p$.

Lemma 2.5.1 *Let $f\colon B \to A$ be a split surjective k-algebra map with kernel I, and let $j\colon A \to B$ be a splitting. Let $c = (m, p) \in \mathcal{P}_A$ and $P = im(p)$, and consider $\mathcal{P}_B(j_*c, j_*c)$ as a monoid under composition. The kernel of the monoid map*

$$f_*\colon \mathcal{P}_B(j_*c, j_*c) \to \mathcal{P}_A(f_*j_*c, f_*j_*c) = \mathcal{P}_A(c, c)$$

is isomorphic to the monoid of matrices $x = 1 + y \in M_m(B)$ such that $y \in M_n I$ and $y = yj(p) = j(p)y$. This is also naturally isomorphic to the set $\mathcal{M}_A(P, P \otimes_A j^*I)$. The monoid structure induced on $\mathcal{M}_A(P, P \otimes_A j^*I)$ is given by

$$\alpha \cdot \beta = (1+\alpha) \circ (1+\beta) - 1 = \alpha + \beta + \alpha \circ \beta$$

for $\alpha, \beta \in \mathcal{M}_A(P, P \otimes_A I)$ where $\alpha \circ \beta$ is the composite

$$P \xrightarrow{\beta} P \otimes_A I \xrightarrow{\alpha \otimes 1} P \otimes_A I \otimes_A I \xrightarrow{\text{multiplication in } I} P \otimes_A I$$

Proof: Identify $\mathcal{P}_B(j_*c, j_*c)$ as the matrices $x \in M_n(B)$ such that $x = xj(p) = j(p)x$ and likewise for $\mathcal{P}_A(c, c)$. The kernel consists of the matrices x for which $f(x) = p$ (the identity!), that is the matrices of the form $j(p) + y$ with $y \in M_n(I)$ such that $y = yj(p) = j(p)y$. As sets this is isomorphic to the claimed monoid, and the map $j(p) + y \mapsto 1 + y$ is a monoid isomorphism since $(j(p)+y)(j(p)+z) = j(p)^2 + yj(p) + j(p)z + yz = j(p) + y + z + yz \mapsto 1 + y + z + yz = (1+y)(1+z)$. The identification with $\mathcal{M}_A(P, P \otimes_A j^*I)$ is through the composite

$$\begin{aligned} Hom_A(P, P \otimes_A j^*I) &\cong Hom_A(P, j^*(P \otimes_A I)) \cong Hom_B(P \otimes_A B, P \otimes_A I) \\ &\xrightarrow{\phi \mapsto 1+\phi} Hom_B(P \otimes_A B, P \otimes_A B) \\ &\cong Hom_B(im(j(p)), im(j(p))) = \mathcal{P}_B(j_*c, j_*c) \end{aligned}$$

where the last isomorphism is the natural isomorphism between

$$\mathcal{P}_A \xrightarrow{j_*} \mathcal{P}_B \longrightarrow \mathcal{M}_B$$

and

$$\mathcal{P}_A \longrightarrow \mathcal{M}_A \xrightarrow{-\otimes_A B} \mathcal{M}_B.$$

∎

Lemma 2.5.2 *In the same situation as the preceding lemma, if $I \subset Rad(B)$, then the kernel of*

$$f_*\colon \mathcal{P}_B(j_*c, j_*c) \longrightarrow \mathcal{P}_A(c,c)$$

is a group.

Proof: Suslin ICM86 To see this, assume first that $P \cong A^n$. Then

$$\mathcal{M}_A(P, P\otimes_A I) \cong M_n I \subseteq M_n(rad(B)) = rad(M_n(B))$$

(we have that $M_n(rad(B)) = Rad(M_n(B))$ since $\mathcal{M}_B(B^n,-)$ is an equivalence from B-modules to $M_n(B)$-modules, [4, p. 86]), and so $(1+M_n(I))^\times$ is a group. If P is a direct summand of A^n, say $A^n = P\oplus Q$, and $\alpha\in\mathcal{M}_A(P, P\otimes_A I)$, then we have a diagram

$$\begin{array}{ccc} P\otimes_A B & \xrightarrow{1+\alpha} & P\otimes_A B \\ \downarrow & & \downarrow \\ A^n\otimes_A B & \xrightarrow{1+(\alpha,0)} & A^n\otimes_A B \end{array}$$

where the vertical maps are split injections. By the discussion above $1+(\alpha,0)$ must be an isomorphism, forcing $1+\alpha$ to be one too. ∎

All the above holds true if instead of considering module categories, we consider the S construction of Waldhausen applied n times to the projective modules. More precisely, let now c be some object in $S^{(n)}_p\mathcal{P}_A$. Then the set of morphisms $S^{(n)}_p\mathcal{M}_A(c, c\otimes_A I)$ is still isomorphic to the monoid of elements sent to the identity under

$$S^{(n)}_p\mathcal{P}_B(j_*c, j_*c) \xrightarrow{f_*} S^{(n)}_p\mathcal{P}_A(c,c)$$

and, if I is radical, this is a group. We will usually suppress the simplicial indices and speak of elements in some unspecified dimension.

{def:Dandl

Definition 2.5.3 We need a few technical definitions. Let

$$0 \longrightarrow I \longrightarrow B \xrightarrow{f} A \longrightarrow 0$$

be a split extension of k-algebras with $I\subset Rad(B)$, and choose a splitting $j\colon A\to B$ of f.

Let $t\mathcal{P}_B \subseteq \mathcal{P}_B$ be the subcategory with the same objects, but with morphisms only the endomorphisms taken to the identity by f_*. Note that, since $I\subseteq rad(B)$, all morphisms in $t\mathcal{P}_B$ are automorphisms.

Let

$$tS^{(n)}_q\mathcal{P}_B \subseteq iS^{(n)}_q\mathcal{P}_B$$

be the subcategory with the same objects, but with morphisms transformations of diagrams in $S^{(n)}_q\mathcal{P}_B$ consisting of morphisms in $t\mathcal{P}_B$.

Consider the sequence of (multi) simplicial exact categories $n\mapsto \mathcal{D}^n_A B$ given by

$$ob\mathcal{D}^n_A B = obS^{(n)}\mathcal{P}_A \text{ and } \mathcal{D}^n_A B(c,d) = S^{(n)}\mathcal{P}_B(j_*c, j_*d)$$

Let $t\mathcal{D}^n_A B \subset \mathcal{D}^n_A B$ be the subcategory containing all objects, but whose only morphisms are the automorphisms $S^{(n)}\mathcal{M}_A(c, c\otimes_A I)$ considered as the subset $\{b \in S^{(n)}\mathcal{P}_B(j_*c, j_*c) | f_*b = 1\} \subseteq \mathcal{D}^n_A B(c,c)$.

We set

$$\mathbf{K}_A B = \{n \mapsto Bt\mathcal{D}^n_A B = \coprod_{m\in S^{(n)}\mathcal{P}_A} B\left(S^{(n)}\mathcal{M}_A(m, m\otimes_A I)\right)\} \tag{2.5.3}$$ {def:KAB}

where the bar construction is taken with respect to the group structure.

Recall that in the eyes of K-theory there really is no difference between the special type of automorphisms coming from t and all isomorphisms since by corollary 2.3.2 the inclusions

$$obS^{(n)}\mathcal{P}_B \subseteq BtS^{(n)}\mathcal{P}_B \subseteq BiS^{(n)}\mathcal{P}_B$$

are both weak equivalences.

Note that $\mathcal{D}^n_A B$ depends not only on I as an A bimodule, but also on the multiplicative structure it inherits as an ideal in B. We have a factorization

$$S^{(n)}\mathcal{P}_A \xrightarrow{j_!} \mathcal{D}^n_A B \xrightarrow{j_\#} S^{(n)}\mathcal{P}_B$$

where $j_!$ is the identity on object, and j_* on morphisms, and $j_\#$ is the fully faithful functor sending $c \in obt\mathcal{D}^n_A B = obS^{(n)}\mathcal{P}_A$ to $j_*c \in obS^{(n)}\mathcal{P}_B$ (and the identity on morphisms). We see that $\mathbf{K}_A B$ is a subspectrum of $\{n \mapsto BiS^{(n)}\mathcal{P}_B\}$ via

$$t\mathcal{D}^n_A B \longrightarrow tS^{(n)}\mathcal{P}_B \subseteq iS^{(n)}\mathcal{P}_B$$

{theo:KAB}

Theorem 2.5.4 *Let $f\colon B \to A$ be a split map of k-algebras with splitting j and kernel $I \subset Rad(B)$. Then*

$$\mathcal{D}^n_A B \xrightarrow{j_\#} S^{(n)}\mathcal{P}_B, \textit{ and its restriction } t\mathcal{D}^n_A B \xrightarrow{j_\#} tS^{(n)}\mathcal{P}_B$$

are (degreewise) equivalences of simplicial exact categories, and so the chain

$$\mathbf{K}_A B(n) = Bt\mathcal{D}^n_A B \subseteq BtS^{(n)}\mathcal{P}_B \supseteq obS^{(n)}\mathcal{P}_B = \mathbf{K}(B)(n)$$

consists of weak equivalences.

Proof: To show that

$$\mathcal{D}^n_A B \xrightarrow{j_\#} S^{(n)}\mathcal{P}_B$$

is an equivalence, all we have to show is that every object in $S^{(n)}\mathcal{P}_B$ is isomorphic to something in the image of $j_\#$. We will show that $c \in S^{(n)}\mathcal{P}_B$ is isomorphic to $j_*f_*c = j_\#(j_!f_*c)$.

Let $c = (m, p) \in ob\mathcal{P}_B$, $P = im(p)$. Consider the diagram with short exact columns

$$\begin{array}{ccc} im(p)\cdot I & \dashrightarrow & im(jf(p))\cdot I \\ \downarrow & & \downarrow \\ im(p) & \overset{\eta_p}{\dashrightarrow} & im(jf(p)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ f^*im(f(p)) & = & f^*im(fjf(p)) \end{array}$$

Since $im(p)$ is projective there exist a (not necessarily natural) lifting η_p. Let C be the cokernel of η_p. A quick diagram chase shows that $C \cdot I = C$. Since $im(f(p))$, and hence C, is finitely generated Nakayama's lemma tells us that C is trivial. This implies that η_p is surjective, but $im(f(p))$ is also projective, so η_p must be split surjective. Call the splitting ϵ. Since $\pi\epsilon = \pi'\eta_p\epsilon = \pi'$ the argument above applied to ϵ shows that ϵ is also surjective. Hence η_p is an isomorphism. Thus, every object $c \in ob\mathcal{P}_B$ is isomorphic to $j_*(f_*c)$.

Let $c \in obS^{(n)}\mathcal{P}_B$. Then c and j_*f_*c are splittable diagrams with isomorphic vertices. Choosing isomorphisms on the "diagonal" we can extend these to the entire diagram, and so c and j_*f_*c are indeed isomorphic as claimed, proving the first assertion.

To show that

$$t\mathcal{D}^n_A B \xrightarrow{j_\#} tS^{(n)}\mathcal{P}_B$$

is an equivalence, note first that this functor is also fully faithful. We know that any $c \in obtS^{(n)}\mathcal{P}_B = obS^{(n)}\mathcal{P}_B$ is isomorphic in $S^{(n)}\mathcal{P}_B$ to j_*f_*c, and the only thing we need to show is that we can choose this isomorphism in t. Let $\iota\colon c \to j_*f_*c \in iS^{(n)}\mathcal{P}_B$ be any isomorphism. Consider

$$c \xrightarrow{\iota} j_*f_*c = j_*f_*j_*f_*c \xrightarrow{j_*f_*(\iota^{-1})} j_*f_*c$$

Since $f_*(j_*f_*(\iota^{-1}) \circ \iota) = f_*(\iota^{-1}) \circ f_*(\iota) = 1_{f_*c}$ the composite $j_*f_*(\iota^{-1}) \circ \iota$ is an isomorphism in $tS^n_q\mathcal{P}$ from c to $j_\#(j_!f_*c)$. ∎

We set

{def:tild

Definition 2.5.5

$$\tilde{\mathbf{K}}_A B = \mathbf{K}_A B/\mathbf{K}(A) = \{n \mapsto \bigvee_{m \in S^{(n)}\mathcal{P}_A} B\left(S^{(n)}\mathcal{M}_A(m, m \otimes_A I)\right)\}$$

and theorem 2.5.4 says that

$$\tilde{\mathbf{K}}_A B \xrightarrow{\sim} \mathbf{K}(B)/\mathbf{K}(A)$$

is a (pointwise) equivalence of spectra. The latter spectrum is stably equivalent to the fiber of $\mathbf{K}(B) \to \mathbf{K}(A)$. To see this, consider the square

$$\begin{array}{ccc} \mathbf{K}(B) & \longrightarrow & \mathbf{K}(A) \\ \downarrow & & \downarrow \\ \mathbf{K}(B)/\mathbf{K}(A) & \longrightarrow & * \end{array}$$

It is a (homotopy) cocartesian square of spectra, and hence (homotopy) cartesian. (In spectrum dimension n this is a cocartesian square, and the spaces involved are at least $n-1$ connected, and so all maps are $n-1$ connected. Then Blakers-Massey A.1.10.0.8 tells us that it is $(n-1)+(n-1)-1 = 2n-3$ (homotopy) cartesian.) This means that the (homotopy) fiber of the upper horizontal map maps by a weak equivalence to the (homotopy) fiber of the lower horizontal map.

2.5.6 "Analyticity properties" of $\mathbf{K}_A(B)$

Although we are not using the notion of calculus of functors in these notes, we will in many cases come quite close. The next lemma, which show how $\mathbf{K}_A(B)$ behaves under certain inverse limits, can be viewed as an example of this. A twist, which will reappear later is that we do not ask whether the functor turns "cocartesianness" into "cartesianness", but rather to what extent the functor preserves inverse limits. The reason for this is that in many cases the coproduct structure of the source category can be rather messy, whereas some forgetful functor tells us exactly what the limits should be.

Let $\mathcal{A}$ be the category of split radical extensions over a given ring A. The category $s\mathcal{A}$ of simplicial objects in $\mathcal{A}$ then inherits the notion of k-cartesian cubes via the forgetful functor down to simplicial sets. By "final maps" in an n cube we mean the maps induced from the n inclusions of the subsets of cardinality $n-1$ in $\{1,\dots,n\}$. If $A\ltimes\mathcal{P}\in s\mathcal{A}$ it makes sense to talk about $\mathbf{K}(A\ltimes\mathcal{P})$ by applying the functor in every degree, and diagonalizing.

For the basics on cubes see appendixA.1.10.

lem:IKanal}

Lemma 2.5.7 *Let $A\ltimes\mathcal{P}$ be a strongly cartesian n-cube in $s\mathcal{A}$ such all the final maps are k connected. Then $\mathbf{K}(A\ltimes\mathcal{P})$ is $(1+k)n$ cartesian.*

Proof: Fix q, $p=(p_1,\dots,p_q)$ and $c\in obS^{(q)}_p\mathcal{P}_A$. The cube $S^{(q)}_p\mathcal{M}_A(c, c\otimes_A\mathcal{P})$ is also strongly cartesian (it is so as a simplicial set, and so as a simplicial group), and the final maps are still k connected. Taking the bar of this gives us a strongly cartesian cube $\mathcal{Y}(c)=BS^{(q)}_p\mathcal{M}_A(c,c\otimes_A\mathcal{P})$, but whose final maps will be $k+1$ connected. By ref(?) this means that $\mathcal{Y}(c)$ will be $(k+2)n-1$ cocartesian. The same will be true for

$$\coprod_{c\in obS^{(q)}_p\mathcal{P}_A}\mathcal{Y}(c)$$

Varying p and remembering that each multi-simplicial space is q reduced in the p direction, we see that the resulting cube is $q+(k+2)n-1$ cocartesian. Varying also q this means that the cube of spectra $\mathbf{K}(A\ltimes\mathcal{P})$ is $(k+2)n-1$ cocartesian, or equivalently $(k+2)n-1-(n-1)=(k+1)n$ cartesian. ∎

The importance of this lemma will become apparent as we will approximate elements in $\mathcal{A}$ by means of cubical diagrams in $s\mathcal{A}$ where all but the initial node will be "reduced" in the sense that the zero skeletons will be exactly the trivial extension $A=A$.

2.6 Categories with cofibrations and weak equivalences

The definition above does not cover more general situations where we are interested in incorporating some structure of weak equivalences, e.g., simplicial rings. Waldhausen [131] covers this case also, and demands only that the category of weak equivalences $w\mathcal{C} \subseteq \mathcal{C}$ contains all isomorphisms and satisfy the gluing lemma, that is, if the left horizontal maps in the commutative diagram

$$\begin{array}{ccccc} d & \leftarrowtail & c & \rightarrow & e \\ \downarrow & & \downarrow & & \downarrow \\ d' & \leftarrowtail & c' & \rightarrow & e' \end{array}$$

are cofibrations and the vertical maps are weak equivalences, then the induced map

$$d \coprod_c d \to d' \coprod_{c'} e'$$

is also a weak equivalence. In this case $S\mathcal{C}$ inherits a subcategory of weak equivalences, $wS\mathcal{C}$ satisfying the same conditions by declaring that a map is a weak equivalence if it is on all nodes. We iterate this construction and define

$$\mathbf{K}(\mathcal{C}, w) = \{m \mapsto BwS^{(m)}\mathcal{C}\}. \tag{2.6.0}$$

Corollary 2.3.2 then says that

$$\mathbf{K}(\mathcal{C}) \xrightarrow{\simeq} \mathbf{K}(\mathcal{C}, i)$$

is an equivalence of spectra.

One should note that there really is no need for the new definition, since the old covers all situations by the following situations. If we let $\mathbf{N}_q\mathcal{C}$ be the category of functors $[q] \to \mathcal{C}$ and natural transformations between these, we can let $\mathbf{N}_q(\mathcal{C}, w)$ be the full subcategory of $\mathbf{N}_q\mathcal{C}$ with $ob\mathbf{N}_q(\mathcal{C}, w) = B_q w\mathcal{C}$. Letting q vary this is a simplicial category with cofibrations, and we have an isomorphism

$$\mathbf{K}(\mathcal{C}, w)(m) = BwS^{(m)}\mathcal{C} \cong obS^{(m)}\mathbf{N}(\mathcal{C}, w) = \mathbf{K}(\mathbf{N}(\mathcal{C}, w)).$$

2.7 Other important facts about the K-theory spectrum

The following theorems are important for the general framework of algebraic K-theory and we include them for the reader's convenience. We will neither need for the development of the theory nor prove them, but we still want to use them in some examples and draw the reader's attention to them.

Theorem 2.7.1 (Additivity theorem: section 1.4 in [131] and [83]) *Let $\mathcal{C}$ be a category with cofibrations and weak equivalences $w\mathcal{C}$. Then*

$$BwSS_2\mathcal{C} \to BwS\mathcal{C} \times NwS\mathcal{C}$$

is an equivalence, and the structure map $BwS^{(m)}\mathcal{C} \to \Omega BwS^{(m+1)}\mathcal{C}$ is an equivalence for $m > 0$.

Theorem 2.7.2 (Approximation theorem [131])

Theorem 2.7.3 (Localization theorem [131], [41] and [99])

Theorem 2.7.4 (Devissage theorem [99])

Theorem 2.7.5 (Resolution theorem [99])

3 Stable K-theory is homology

{I3}

In this section we will try to connect K-theory to homology. This is done by considering "small perturbations" in input in K-theory, giving a linear theory: the "directional derivative" of K-theory. This is then compared with the classical concept of homology, and the two are shown to be equal.

3.1 Split surjections with square-zero kernels

If A is a unital ring, and P is any A bimodule (with no multiplicative structure as part of the data), we define the ring $A \ltimes P$ simply to be $A \oplus P$ as an A bimodule, and with multiplication $(a', p')(a, p) = (a'a, a'p + p'a)$, that is $P^2 = 0$ when P is considered as the kernel of the projection $A \ltimes P \twoheadrightarrow A$.

Algebraically, this is considered to be a small deformation of A. And the difference between $K(A \ltimes P)$ and $K(A)$ reflects the local structure of K-theory. The goal is of to measure this difference.

Considered as a functor from A bimodules, $P \mapsto \mathbf{K}(A \ltimes P)$ is not *additive*, even if we
otadditive} remove the part coming from $\mathbf{K}(A)$. That is, if we let

$$\tilde{\mathbf{K}}(A \ltimes P) = fiber\{\mathbf{K}(A \ltimes P) \longrightarrow \mathbf{K}(A)\}$$

then the natural map $\tilde{\mathbf{K}}(A \ltimes (P \oplus Q)) \to \tilde{\mathbf{K}}(A \ltimes P) \times \tilde{\mathbf{K}}(A \ltimes Q)$ is not an equivalence. For instance do we have by [62] that $\pi_2\tilde{\mathbf{K}}(\mathbf{Z} \ltimes P) \cong \bigwedge^2 P \oplus P/2P$ for all abelian groups P. Hence

$$\begin{aligned}
\pi_2\tilde{\mathbf{K}}(\mathbf{Z} \ltimes (P \oplus Q)) &\cong \bigwedge^2 (P \oplus Q) \oplus (P \oplus Q)/2(P \oplus Q) \\
&\cong \left(\bigwedge^2 P \oplus P/2P\right) \oplus \left(\bigwedge^2 Q \oplus Q/2Q\right) \oplus P \otimes Q \\
&\cong \pi_2\tilde{\mathbf{K}}(\mathbf{Z} \ltimes P) \oplus \pi_2\tilde{\mathbf{K}}(\mathbf{Z} \ltimes Q) \oplus (P \otimes Q)
\end{aligned}$$

where the tensor product expresses the nonlinearity.

There are means of forcing linearity upon a functor, which will eventually give stable K-theory, and the aim of this section is to prove that this linear theory is equivalent to the homology of the category of finitely generated projective A-modules.

3.2 The homology of a category

Let $\mathcal{C}$ be an $\mathcal{A}b$-category (that is: a category enriched in $\mathcal{A}b$, the category of Abelian groups, see appendix B. $\mathcal{A}b$-categories are also known as "linear categories" and unfortunately, some call them "additive categories", a term we reserve for pointed $\mathcal{A}b$-categories with sum). The important thing to remember is that the homomorphism sets are really abelian groups, and composition is bilinear.

We say that $\mathcal{C}$ is *flat* if the morphism sets are flat as abelian groups. A $\mathcal{C}$-bimodule is an $\mathcal{A}b$-functor (linear functor) $\mathcal{C}^o \otimes \mathcal{C} \to \mathcal{A}b$. The category $\mathcal{A}b^{\mathcal{C}^o\otimes\mathcal{C}}$ of bimodules forms an abelian category with enough projectives, so we are free to do homological algebra. If $\mathcal{C}$ is flat, the *Hochschild homology* of $\mathcal{C}$ with coefficients in $M \in \mathcal{A}b^{\mathcal{C}^o\otimes\mathcal{C}}$ is customarily defined {Def:HHof
as

$$\mathrm{Tor}_*^{\mathcal{A}b^{\mathcal{C}^o\otimes\mathcal{C}}}(M, \mathcal{C})$$

(see [88]). There is a standard simplicial abelian group (complex) whose homotopy groups calculate the Hochschild homology groups, namely

$$HH(\mathcal{C}, M)_q = \bigoplus_{c_0,\dots,c_q \in ob\mathcal{C}} M(c_0, c_q) \otimes \bigotimes_{1\leq i\leq q} \mathcal{C}(c_i, c_{i-1})$$

with face and degeneracies as in Hochschild homology (see [88], and also below).

Let $\mathcal{C}$ be any category. It is not uncommon to to call functors $\mathcal{C}^o \times \mathcal{C} \to \mathcal{A}b$ "bifunctors". We note immediately that, by adjointness of the free and forgetful functors

$$\mathcal{E}ns \overset{\mathbf{Z}}{\rightleftarrows} \mathcal{A}b$$

connecting abelian groups to sets, a "bifunctor" is nothing but a $\mathbf{Z}\mathcal{C}$-bimodule in the $\mathcal{A}b$-enriched world (see B); that is an $\mathcal{A}b$-functor $\mathbf{Z}\mathcal{C}^o \otimes \mathbf{Z}\mathcal{C} \to \mathcal{A}b$. So, for any "bifunctor" (i.e. $\mathbf{Z}\mathcal{C}$-bimodule) M we may define the homology of $\mathcal{C}$ with respect to M as

$$H_*(\mathcal{C}, M) = \pi_* HH(\mathbf{Z}\mathcal{C}, M)$$

(notice that $\mathbf{Z}\mathcal{C}$ is flat). The standard complex $HH(\mathbf{Z}\mathcal{C}, M)$ calculating this homology, is naturally isomorphic to the complex $F(\mathcal{C}, M)$:

Definition 3.2.1 Let $\mathcal{C}$ be a category and M a $\mathbf{Z}\mathcal{C}$-bimodule. Then the homology of $\mathcal{C}$ with coefficients in M: $F(\mathcal{C}, M)$ is the simplicial Abelian group which in degree q is given by

$$F_q(\mathcal{C}, M) = \bigoplus_{c_0 \leftarrow \dots \leftarrow c_q \in B_q\mathcal{C}} M(c_0, c_q) \cong \bigoplus_{c_0,\dots,c_q \in ob\mathcal{C}} M(c_0, c_q) \otimes \bigotimes_{1\leq i\leq q} \mathbf{Z}\mathcal{C}(c_i, c_{i-1})$$

and with simplicial structure defined as follows. We write elements of $F_q(\mathcal{C}, M)$ as sums of elements of the form (x, α) where $x \in M(c_0, c_q)$ and

$$\alpha = c_0 \overset{\alpha_1}{\longleftarrow} \dots \overset{\alpha_q}{\longleftarrow} c_q \in B_q\mathcal{C}.$$

Then

$$d_i(x,\alpha) = \begin{cases} (M(\alpha_1,1)x, d_0\alpha) & \text{if } i=0 \\ (x, d_i\alpha) & \text{if } 0<i<q \\ (M(1,\alpha_q)x, d_q\alpha) & \text{if } i=q \end{cases}$$

and $s_i(x,\alpha) = (x, s_i\alpha)$.

Remark 3.2.2 *The homology of $\mathcal{C}$, or rather $F(\mathcal{C},-)\colon \mathcal{A}b^{\mathbf{Z}\mathcal{C}^o\otimes\mathbf{Z}\mathcal{C}} \to s\mathcal{A}b$ is characterized up to equivalence by the three properties*

1. *If $M \in ob\mathcal{A}b^{\mathbf{Z}\mathcal{C}^o\otimes\mathbf{Z}\mathcal{C}}$ is projective, then $F(\mathcal{C},M)\to H_0(\mathcal{C},M)$ is an equivalence.*
2. *$F(\mathcal{C},-)\colon \mathcal{A}b^{\mathbf{Z}\mathcal{C}^o\otimes\mathbf{Z}\mathcal{C}} \to s\mathcal{A}b$ take short exact sequences to fiber sequences, and*
3. *The values $H_0(\mathcal{C},M)$.*

In particular, this means that if we have a map to or from some other theory satisfying 1. and 2, and inducing an isomorphism on π_0, then this map is an equivalence.

3.3 Incorporating the S-construction

In order to compare with K-theory, we will incorporate the S-construction into the source of the homology functor.

Let $\mathcal{C}$ be a small category, and M a $\mathbf{Z}\mathcal{C}$-bimodule (i.e. a functor from $\mathcal{C}^o\times\mathcal{C}$ to abelian groups). Recall how bimodules are extended to diagram categories (see B NBNBref for the general situation).

If $\mathfrak{C}$ be an exact category, consider the full subcategory $S_q\mathfrak{C}\subseteq[\mathcal{A}r[q],\mathfrak{C}]$. Let M be a $\mathfrak{C}$-bimodule, then S_qM is defined, and is given by

$$S_qM(c,d) = \{\{m_{ij}\}\in\prod_{0\le i\le j\le q} M(c_{ij},d_{ij}) | M(1,d_{ij}\to d_{kl})m_{ij} = M(c_{ij}\to c_{kl},1)m_{kl}\}$$

Note that, if M is not pointed (i.e. a $\tilde{\mathbf{Z}}\mathfrak{C}$-bimodule) we may have elements in the groups $M(c_{ii},d_{ii}) = M(0,0)$, but these are uniquely determined by the values in the other groups. (In fact, if $\mathfrak{C}$ is split exact, then the projection $S_qM(c,d)\to M(c_{0q},d_{0,q})$ is a split monomorphism – a retract is constructed using a choice of splittings).

The construction $q\mapsto S_qM$ is functorial in q in the sense that for every map $\phi\colon[p]\to[q]\in\Delta$ there are natural maps $\phi^*\colon S_pM\to\phi^*S_qM$.

We may also iterate the S construction.

Let $\mathfrak{C}$ be an exact category, and M a pointed $\mathfrak{C}$-bimodule. Note that, since for every $\phi\colon[p]\to[q]\in\Delta$ we have a bimodule map $\phi^*\colon S_pM\to\phi^*S_qM$

$$F(S\mathfrak{C},SM) = \{[p],[q]\mapsto F_p(S_q\mathfrak{C},S_qM)\}$$

becomes a bisimplicial abelian group. Again we get a map $S^1\wedge F(\mathfrak{C},M)\to F(S\mathfrak{C},SM)$ making

$$\mathbf{F}(\mathfrak{C},M) = \{n\mapsto F(S^{(n)}\mathfrak{C},S^{(n)}M)\}$$

a spectrum. In the special case $\mathfrak{C} = \mathcal{P}_A$, and $M(c,d) = Hom_A(c, d \otimes_A P)$ for some A-bimodule P, we define

$$\mathbf{F}(A,P) = \mathbf{F}(\mathcal{P}_A, Hom_A(-, - \otimes_A P))$$

Note that this can not cause any confusion as the $\mathbf{F}(\mathfrak{C}, M)$ spectrum is only defined for additive categories (and not for rings). We will also consider the associated spectra $\mathbf{F}_q$ for $q \geq 0$ (with the obvious definition).

Lemma 3.3.1 *Let $\mathfrak{C}$ be additive category and $M \in \mathfrak{B}_{\mathfrak{C}}$ be pointed. Let*

$$\eta\colon F_q(\mathfrak{C}, M) \to \Omega F_q(S\mathfrak{C}, SM)$$

denote the (adjoint of the) structure map. Then the two composites in the noncommutative diagram

$$\begin{array}{ccc} F_q(\mathfrak{C}, M) & \xrightarrow{d_0^q} & F_0(\mathfrak{C}, M) \\ \eta\big\downarrow & & s_0^q\big\downarrow \\ \Omega F_q(S\mathfrak{C}, SM) & \xleftarrow{\eta} & F_q(\mathfrak{C}, M) \end{array}$$

are homotopic.

Proof: There are three maps $d_0, d_1, d_2\colon F_q(S_2\mathcal{C}, S_2M) \to F_q(\mathcal{C}, M)$ induced by the structure maps $S_2\mathcal{C} \to S_1\mathcal{C} = \mathcal{C}$. The two maps

$$\eta d_1 \text{ and } \eta d_0 * \eta d_2\colon F_q(S_2\mathcal{C}, S_2M) \to F_q(\mathcal{C}, M) \to \Omega F_q(S\mathcal{C}, SM)$$

are homotopic, where $\eta d_0 * \eta d_2$ denotes the loop product. This is so for general reasons: if X is a reduced simplicial set, then the two maps ηd_1 and $\eta d_0 * \eta d_2$ are homotopic as maps

$$X_2 \to X_1 \xrightarrow{\ \eta\ } \Omega X$$

where the latter map is induced by the adjoint of the canonical map $S^1 \wedge X_1 \to X$.

We define two maps

$$E, D\colon F_q(\mathfrak{C}, M) \to F_q(S_2\mathfrak{C}, S_2M)$$

by sending $(\alpha_0, \{\alpha_i\}) = (\alpha_0 \in M(c_0, c_q), \{c_{i-1} \xleftarrow{\alpha_i} c_i\})$ to $E(\alpha_0, \{\alpha_i\}) =$

$$\left(\begin{pmatrix} 0 \\ M(pr_2, \Delta)\alpha_0 \\ \alpha_0 \end{pmatrix} \in S_2M \begin{pmatrix} c_q & c_q \\ i_1\big\downarrow & i_1\big\downarrow \\ c_q \oplus c_0, & c_q \oplus c_q \\ pr_2\big\downarrow & pr_2\big\downarrow \\ c_0 & c_q \end{pmatrix}, \quad \left\{ \begin{array}{ccc} c_q & = & c_q \\ i_1\big\downarrow & & i_1\big\downarrow \\ c_q \oplus c_{i-1} & \xleftarrow{1\oplus\alpha_i} & c_q \oplus c_i \\ pr_2\big\downarrow & & pr_2\big\downarrow \\ c_{i-1} & \xleftarrow{\alpha_i} & c_i \end{array} \right\} \right)$$

and $D(\alpha_0, \{\alpha_i\}) =$

$$\left(\begin{pmatrix} M(\beta_1,1)\alpha_0 \\ M(pr_2,\Delta)\alpha_0 \\ 0 \end{pmatrix} \in S_2M\begin{pmatrix} c_q & & c_q \\ {\scriptstyle\Delta_1}\downarrow & & {\scriptstyle\Delta}\downarrow \\ c_q\oplus c_0, & & c_q\oplus c_q \\ {\scriptstyle\nabla_1}\downarrow & & {\scriptstyle\nabla}\downarrow \\ c_0 & & c_q \end{pmatrix}, \quad \left\{\begin{matrix} c_q & = & c_q \\ {\scriptstyle\Delta_i}\downarrow & & {\scriptstyle\Delta_{i+1}}\downarrow \\ c_q\oplus c_{i-1} & \xleftarrow{1\oplus\alpha_i} & c_q\oplus c_i \\ {\scriptstyle\nabla_i}\downarrow & & {\scriptstyle\nabla_{i+1}}\downarrow \\ c_{i-1} & \xleftarrow{\alpha_i} & c_i \end{matrix}\right\}\right)$$

where

1. i_1 the inclusion into the first summand, pr_2 the second projection, Δ the diagonal and $\nabla\colon c\oplus c\to c$ the difference $(a,b)\mapsto a-b$,

2. $\beta_i = \alpha_i\dots\alpha_q\colon c_q \to c_{i-1}$, $\Delta_i = (1\oplus\beta_i)\Delta$, and $\nabla_i = \nabla(1\oplus\beta_i)$.

(exercise: check that the claimed elements of $S_2M(-,-)$ are well defined).

Since $d_2E = d_0D = 0$ we get that

$$\eta = \eta d_0E \simeq \eta d_1E = \eta d_1D \simeq \eta d_2D = \eta s_0^q d_0^q$$

∎

{cor:3.3.2}

Corollary 3.3.2 *In the situation of the lemma, the inclusion of degeneracies induces a stable equivalence of spectra*

$$\mathbf{F}_0(\mathfrak{C}, M) \xrightarrow{\sim} \mathbf{F}(\mathfrak{C}, M)$$

and in particular, if A is a ring and P an A bimodule, then

$$\mathbf{F}_0(A, P) \xrightarrow{\sim} \mathbf{F}(A, P)$$

Proof: It is enough to show that for every q the map $\mathbf{F}_0(\mathfrak{C}, M) \to \mathbf{F}_q(\mathfrak{C}, M)$ induced by the degeneracy is a stable equivalence (since loops of simplicial spaces may be performed in each degree, see A.1.5.0.5, and since a degreewise equivalence of simplicial spaces induces an equivalence on the diagonal, see A.1.5.0.2). In other words, we must show that for every q and k

$$\pi_0 \lim_{m\to\infty} \Omega^{m+k} F_0(S^{(m)}\mathfrak{C}, S^{(m)}M) \xrightarrow{s_0^q} \pi_0 \lim_{m\to\infty} \Omega^{m+k} F_q(S^{(m)}\mathfrak{C}, S^{(m)}M)$$

is an isomorphism. It is an injection by definition, and a surjection by the lemma. ∎

3.4 K-theory as a theory of bimodules

Let A be a ring and let $A \ltimes P \to A$ be any split radical extension Recall the $\tilde{\mathbf{K}}_A$ construction of definition 2.5.3. The last part of theorem 2.5.4 says that

$$\tilde{\mathbf{K}}(A \ltimes P) \simeq \tilde{\mathbf{K}}_A(A \ltimes P) = \{n \mapsto \bigvee_{m \in S^{(n)}\mathcal{P}_A} B\left(S^{(n)}\mathcal{M}_A(m, m \otimes_A P)\right)\}$$

Notice the striking similarity with

$$\mathbf{F}_0(\mathcal{P}_A, M) = \{n \mapsto \bigoplus_{m \in obS^{(n)}\mathcal{P}_A} M(m,m)\}$$

In the special case where $P^2 = 0$ the group structure on $Hom_A(c, c \otimes_A P)$ for $c \in S_q^{(n)}\mathcal{P}_A$ is just the summation of maps: let $f, g \in Hom_A(c, c \otimes_A P)$, then $f \cdot g = (1+f)(1+g) - 1 = f + g + f \circ g$, where $f \circ g$ is the composite

$$c \xrightarrow{g} c \otimes_A P \xrightarrow{f \otimes 1} c \otimes_A P \otimes_A P \to c \otimes_A P$$

where the last map is induced by the multiplication in $P \subseteq A \ltimes P$, which is trivial. So $f \cdot g = f + g$. This means that the isomorphism

$$\begin{aligned} B_q Hom_A(c, c \otimes_A P) =& Hom_A(c, c \otimes_A P)^{\times q} \cong Hom_A(c, (c \otimes_A P)^{\times q}) \\ \cong& Hom_A(c, c \otimes_A P^{\times q}) = Hom_A(c, c \otimes_A B_q P) \end{aligned}$$

induces a simplicial isomorphism. Hence

$$M = B\left(S^{(n)}\mathcal{M}_A(-, - \otimes_A P)\right) \cong S^{(n)}\mathcal{M}_A(-, - \otimes_A BP)$$

is a (simplicial) $\mathcal{P}_A$-bimodule, and the only difference between $\tilde{\mathbf{K}}_A(A \ltimes P)$ and $\mathbf{F}_0(\mathcal{P}_A, M)$ is that the first is built up of wedge summands, whereas the second is built up of direct sums.

Here stable homotopy enters. Recall that a space X is 0-connected if $\pi_0 X$ is a point, and if it is connected it is *k-connected* for a $k > 0$ if for all vertices $x \in X_0$ we have that $\pi_q(X, x) = 0$ for $0 \leq q \leq k$. A space is -1-connected by definition if it is nonempty. A map $X \to Y$ is *k-connected* if its homotopy fiber is $(k-1)$-connected. We use the same convention for simplicial rings and modules.

The difference between wedge and direct sum vanishes stably, which accounts for

{theo:3.4

Theorem 3.4.1 *Let A be a ring and P a m-connected simplicial A-bimodule, the inclusion $\bigvee \subseteq \bigoplus$ induces a $2m+2$-connected map*

$$\tilde{\mathbf{K}}_A(A \ltimes P) \to \mathbf{F}_0(A, BP) \cong B\mathbf{F}_0(A, P)$$

Proof: Corollary A.1.10.0.10 says that if X is n-connected and Y is m-connected, then the inclusion $X \vee Y \to X \times Y$ is $m+n$-connected, and so the same goes for finitely many factors. Now, finite sums of modules is the same as products of underlying sets, and infinite sums are filtered colimits of the finite sub-sums. Since the functors in question commute with filtered colimits, the result follows. ∎

3.4.2 Removing the bar

What is the rôle of the bar construction in theorem 3.4.1? Removing it on the K-theory side, that is in $\mathbf{K}_A(A \ltimes P)$, we are invited to look at

$$\{n \mapsto \coprod_{c \in obS^{(n)}\mathcal{P}_A} Hom_A(c, c \otimes_A P)\} \tag{3.4.2}$$ {eq:EAP}

We identify this as follows. Let $\mathcal{E}_A P$ be the exact category with objects pairs (c, f) with $c \in ob\mathcal{P}_A$ and $f \in Hom_A(c, c \otimes_A P)$, and morphisms $(c, f) \to (d, g)$ commutative diagrams of A-modules

$$\begin{array}{ccc} c & \xrightarrow{h} & d \\ f\downarrow & & g\downarrow \\ c \otimes_A P & \xrightarrow{h\otimes 1} & d \otimes_A P \end{array}$$

We have a functor $\mathcal{E}_A P \to \mathcal{P}_A$ given by $(c, f) \mapsto c$, and a sequence in $\mathcal{E}_A P$ is exact if it is sent to an exact sequence in $\mathcal{P}_A$. As examples we have that $\mathcal{E}_A 0 = \mathcal{P}A$, and $\mathcal{E}_A A$ is what is usually called the category of endomorphisms on $\mathcal{P}_A$. We see that the expression 3.4.2 is just the K-theory spectrum $\mathbf{K}(\mathcal{E}_A P) = \{x \mapsto obS^{(n)}\mathcal{E}_A P\}$.

Definition 3.4.3 Let A be a unital ring. Set C_A to be the functor from A bimodules to spectra given by

$$\mathsf{C}_A(P) = \mathbf{K}(\mathcal{E}_A P)/\mathbf{K}(A) = \{n \mapsto \bigvee_{c \in S^{(n)}\mathcal{P}_A} S^{(n)}\mathcal{M}_A(c, c \otimes_A P)\}$$

(the homomorphism groups $S^{(n)}\mathcal{M}_A(c, c \otimes_A P)$ are pointed in the zero map).

With this definition we can restate theorem 2.5.4 for the square zero case as

$$\mathsf{C}_A(BP) \simeq fib\{\mathbf{K}(A \ltimes P) \to \mathbf{K}(A)\}$$

Note that, in the language of definition 2.5.3, yet another way of writing $\mathsf{C}_A P$ is as the spectrum $\{n \mapsto N_0^{cy} t\mathcal{D}_A^n(A \ltimes P)/N_0^{cy} t\mathcal{D}_A^n(A) = N_0^{cy} t\mathcal{D}_A^n(A \ltimes P)/obS^{(n)}\mathcal{P}_A\}$.

We are free to introduce yet another spectrum direction in $\mathsf{C}_A P$ by observing that we have natural maps $S^1 \wedge \mathsf{C}_A P \to \mathsf{C}_A(BP)$ given by $S^1 \wedge \bigvee M \cong \bigvee(S^1 \wedge M) \to \bigvee(\tilde{\mathbf{Z}}[S^1] \otimes M) \cong \bigvee BM$.

Aside 3.4.4 *There are* **two** *natural maps* $\mathbf{K}(A) \to \mathbf{K}(\mathcal{E}_A A)$*, given by sending* $c \in obS^{(n)}\mathcal{P}_A$ *to either* $(c, 0)$ *or* $(c, 1)$ *in* $obS^{(n)}\mathcal{E}_A A$*. The first is used when forming* $\mathsf{C}_A P$*, and the latter give rise to a map*

$$\mathbf{K}(A) \to \mathsf{C}_A A$$

Composing this with $\mathsf{C}_A A \to \Omega \mathsf{C}_A(BA) = \Omega \tilde{\mathbf{K}}_A(A[t]/t^2) \to \Omega \mathbf{K}(A[t]/t^2)/\mathbf{K}(A)$*, we get a weak map*

$$\mathbf{K}(A) \to \Omega \mathbf{K}(A[t]/t^2)/\mathbf{K}(A) \xleftarrow{\sim} hofib\{\mathbf{K}(A[t]/t^2) \to \mathbf{K}(A)\}$$

(cf. [63] or [129]) where hofib is (a functorial choice representing) the homotopy fiber.

The considerations above are related to the results of Grayson in [42]. Let A be commutative and $R = S^{-1}A[t]$ where $S = 1 + tA[t]$. The theorem above says that $\tilde{\mathbf{K}}_A(A[t]/t^2) = \mathbf{K}(\mathcal{E}_A(BA))/\mathbf{K}(A)$ is equivalent to $\mathbf{K}(A[t]/t^2)/\mathbf{K}(A)$, whereas Grayson's theorem tells us that the "one-simplices" of this, i.e. $\mathsf{C}_A A = \mathbf{K}(\mathcal{E}_A A)/\mathbf{K}(A)$ is equivalent to the loop of $\tilde{\mathbf{K}}_A(R) \simeq \mathbf{K}(R)/\mathbf{K}(A)$.

3.4.5 More general bimodules

Before we go on to reformulate theorem 3.4.1 in the more fashionable form "stable K-theory is homology" we will allow our K-functor more general bimodules so that we have symmetry between the input.

Definition 3.4.6 Let $\mathfrak{C}$ be an exact category and M a pointed $\mathbf{Z}\mathfrak{C}$ bimodule. Then we define the spectrum

$$\mathsf{C}_{\mathfrak{C}}(M) = \{n \mapsto \bigvee_{c \in obS^{(n)}\mathfrak{C}} S^{(n)}M(c,c)\}$$

The structure maps

$$S^1 \wedge \bigvee_{c \in ob\mathfrak{C}} M(c,c) \to \bigvee_{c \in obS\mathfrak{C}} SM(c,c)$$

are well defined, because $\bigvee_{c \in obS_0\mathfrak{C}} S_0M(c,c) = M(0,0) = 0$ since we have demanded that M is pointed

The notation should not cause confusion, although $\mathsf{C}_A P = \mathsf{C}_{\mathcal{P}_A} Hom_A(-, c \otimes_A P)$, since the ring A is not an exact category (except when $A = 0$, and then it doesn't matter).

If M is bilinear, this is the K-theory spectrum of the following category, which we will call $\mathcal{E}_{\mathfrak{C}}(M)$. The objects are pairs (c, f) with $c \in ob\mathfrak{C}$ and $f \in M(c,c)$ and a morphism from (c,m) to (c',m') is an $f \in \mathfrak{C}(c,c')$ such that $M(f,1)m' = M(1,f)m$. A sequence $(c',m') \to (c,m) \to (c'',m'')$ is exact if the underlying sequence $c' \to c \to c''$ is exact.

3.5 Stable K-theory

{sec:IKS}

Recall that, when considered as a functor from A bimodules, $P \mapsto \tilde{\mathbf{K}}(A \ltimes P)$ is not additive 3.1. If F is a pointed (simplicial) functor from A bimodules to spectra, we define it's *first differential*, D_1F, as

$$D_1F(P) = \lim_{\overrightarrow{k}} \Omega^k F(B^k P),$$

where F is degreewise applied to the k fold bar construction. We have a transformation $F \to D_1F$. If F already were additive, then $F \to D_1F$ is a weak equivalence. This means that D_1F is initial (in the homotopy category) among additive functors under F, and is a left adjoint (in the homotopy categories) to the inclusion of the additive functors into all functors from A bimodules to spectra.

Definition 3.5.1 Let A be a simplicial ring and P an A bimodule. Then

$$\mathbf{K}^S(A,P) = D_1\mathsf{C}_A(P) = \lim_{\overrightarrow{k}} \Omega^k \mathsf{C}_A(B^k P)$$

If $\mathfrak{C}$ is an exact category and M a $\mathfrak{C}$ bimodule, then

$$\mathbf{K}^S(\mathfrak{C},M) = D_1\mathsf{C}_{\mathfrak{C}}(M) = \lim_{\overrightarrow{k}} \Omega^k \mathsf{C}_{\mathfrak{C}}(M\otimes S^k)\}$$

where for a finite pointed set X, $M\otimes X$ is the bimodule sending c, d to $M(c,d)\otimes \tilde{\mathbf{Z}}X$.

Again, the equivalence $\mathbf{K}^S(A,P)\simeq \mathbf{K}^S(\mathcal{P}_A, Hom_A(-,-\otimes_A P))$ should cause no confusion. If M is a pointed simplicial $\mathfrak{C}$-bimodule, we apply $\mathsf{C}_{\mathfrak{C}}$ degreewise.

We note that

$$\begin{array}{c}
\mathbf{K}^S(A,P) = D_1\Omega\mathsf{C}_A(BP) \\
\simeq\big\downarrow \\
D_1\Omega\left(\mathbf{K}(\mathcal{P}_{A\ltimes P}, i)/\mathbf{K}(A)\right) \\
\big\uparrow\simeq \\
D_1\Omega\left(\mathbf{K}(A\ltimes P)/\mathbf{K}(A)\right) \\
\simeq\big\downarrow \\
D_1\Omega\mathrm{hofib}\{\mathbf{K}(A\ltimes P)\to\mathbf{K}(A)\} = \operatorname{holim}_{\overrightarrow{k}}\Omega^k\mathrm{hofib}\{\mathbf{K}(A\ltimes B^{k-1}P)\to\mathbf{K}(A)\}
\end{array}$$

and the latter is the (spectrum version of the) usual definition of stable K-theory, c.f. [63] and [129].

In the rational case Goodwillie proved in [38] that stable K-theory was equivalent to Hochschild homology (see later). In general this is not true, and we now turn to the necessary modification.

theo:KSisF}

Theorem 3.5.2 *Let $\mathfrak{C}$ be an exact category and M an m-connected pointed simplicial $\mathfrak{C}$-bimodule. The inclusion $\bigvee\subseteq\bigoplus$ induces a $2m$-connected map*

$$\mathsf{C}_{\mathfrak{C}}M\to \mathbf{F}_0(\mathfrak{C},M)$$

and

$$D_1\mathsf{C}_{\mathfrak{C}} \xrightarrow{\simeq} D_1\mathbf{F}_0(\mathfrak{C},-) \xleftarrow{\simeq} \mathbf{F}_0(\mathfrak{C},-)$$

are equivalences. Hence

$$\mathbf{K}^S(\mathfrak{C},M)\simeq \mathbf{F}_0(\mathfrak{C},M) \xrightarrow{\sim} \mathbf{F}(\mathfrak{C},M)$$

In particular, for A a ring and P an A-bimodule, the map $\mathsf{C}_AP\to\mathbf{F}_0(A,P)$ give rise to natural equivalences

$$\mathbf{K}^S(A,P) = D_1\mathsf{C}_A \xrightarrow{\simeq} D_1\mathbf{F}_0(A,-) \xleftarrow{\simeq} \mathbf{F}_0(A,-)\to\mathbf{F}(A,P)$$

Proof: The equivalence
$$D_1\mathbf{F}_0(\mathfrak{C},-) \xleftarrow{\simeq} \mathbf{F}_0(\mathfrak{C},-)$$
follows since by corollary 3.3.2 the inclusion by the degeneracies $\mathbf{F}_0(\mathfrak{C},-) \to \mathbf{F}(\mathfrak{C},-)$ is an equivalence, and the fact that $\mathbf{F}(\mathfrak{C},-)$ is additive, and so unaffected by the differential. The rest of the argument follows as before. ∎

Adding up the results, we get the announced theorem:

Corollary 3.5.3 *Let $\mathfrak{C}$ be an additive category, and M a bilinear $\mathfrak{C}$ bimodule. Then we have natural isomorphisms*
$$\pi_*\mathbf{K}^S(\mathfrak{C},M) \cong H_*(\mathfrak{C},M)$$
and in particular
$$\pi_*\mathbf{K}^S(A,P) \cong H_*(\mathcal{P}_A, \mathcal{M}_A(-,-\otimes_A P))$$

Proof: The calculations of homotopy groups follows from the fact that $\mathbf{F}(\mathfrak{C},M)$ is a Ω-spectrum (and so $\pi_*\mathbf{F}(\mathfrak{C},M) \cong \pi_*F(\mathfrak{C},M) = H_*(\mathfrak{C},M)$). This follows from theequivalence
$$F(\mathfrak{C},M) \sim THH(\mathfrak{C},M)$$
and results on topological Hochschild homology in chapter IV. However, for the readers who do not plan to cover this material, we provide a proof showing that $\mathbf{F}$ is an Ω spectrum directly without use of stabilizations at the end of this section, see proposition 3.6.5.

∎

3.6 A direct proof of "F is an Ω-spectrum" {subsec:I

Much of what is to follow makes sense in a linear category setting. For convenience, we work in the setting of additive categories, and we choose zero objects which always will be denoted 0.

Definition 3.6.1 Let $G\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. We let the "twisted" product category $\mathcal{A} \times_G \mathcal{B}$ be the linear category with objects $\mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{B})$ and
$$\mathcal{A} \times_G \mathcal{B}((a,b),(a',b')) = \mathcal{A}(a,a') \oplus \mathcal{B}(b,b') \oplus \mathcal{B}(G(a),b')$$
with composition given by
$$(f,g,h)\circ(f',g',h') = (f\circ f', g\circ g', h\circ G(f') + g\circ h').$$
If M is an $\mathcal{A}$-bimodule and N is a $\mathcal{B}$-bimodule, with an $\mathcal{A}$-bimodule map $G^*\colon M \to G^*N$ we define the $\mathcal{A}\times_G\mathcal{B}$-bimodule $M \times_G N$ by
$$M \times_G N((a,b),(a',b')) = M(a,a') \oplus N(b,b') \oplus N(G(a),b')$$
with bimodule action defined by
$$(M\times_G N)((f,g,h),(f',g',h'))(m,n,n_G)$$
$$= (M(f,f')m, N(g,g'), N(Gf,h')G^*m + N(h,g')n + N(Gf,g')n_G)$$

From now on, we assume for convenience that M and N are pointed (i.e. takes zero in either coordinate to zero).

We have an inclusion

$$\mathcal{A} \xrightarrow{in_{\mathcal{A}}} \mathcal{A} \times_G \mathcal{B}$$

sending $f\colon a \to a'$ $ob\mathcal{A}$ to $(f,0,0)\colon (a,0) \to (a',0)$, and an $\mathcal{A}$-bimodule map $M \to in_{\mathcal{A}}^*(M \times_G N)$; and a projection

$$\mathcal{A} \times_G \mathcal{B} \xrightarrow{pr_{\mathcal{A}}} \mathcal{A}$$

and an $\mathcal{A} \times_G \mathcal{B}$-bimodule map $M \times_G N \to pr_{\mathcal{A}}^* M$. Likewise for $\mathcal{B}$. The composite

$$F(\mathcal{A},M) \oplus F(\mathcal{B},N) \xrightarrow{in_{\mathcal{A}}+in_{\mathcal{B}}} F(\mathcal{A} \times_G \mathcal{B}, M \times_G N) \xrightarrow{(pr_{\mathcal{A}} \oplus pr_{\mathcal{B}})\Delta} F(\mathcal{A},M) \oplus F(\mathcal{B},N) \tag{3.6.1}$$

:composite} is the identity.

Lemma 3.6.2 (F is "additive") *With the notation as above*

$$F(\mathcal{A},M) \oplus F(\mathcal{B},N) \xrightarrow{in_{\mathcal{A}}+in_{\mathcal{B}}} F(\mathcal{A} \times_G \mathcal{B}, M \times_G N)$$

is an equivalence.

Proof: We will show that the other composite in 3.6 is homotopic to the identity. Let $\mathbf{x} = (x_0; x_1, \dots, x_q) \in F_q(\mathcal{A} \times_G \mathcal{B}, M \times_G N)$, where

$$x_0 = (m, n, n_G) \in M \times_G N((a_0,b_0),(a_q,b_q)), \text{ and}$$

$$x_i = (f_i, g_i, h_i) \in \mathcal{A} \times_G \mathcal{B}((a_i,b_i),(a_{i-1},b_{i-1})), \text{ for} i > 0.$$

Then x is sent to

$$J(x) = ((m,0,0); in_{\mathcal{A}}pr_{\mathcal{A}}x_1, \dots in_{\mathcal{A}}pr_{\mathcal{A}}x_q) + ((0,n,0); in_{\mathcal{B}}pr_{\mathcal{B}}x_1, \dots in_{\mathcal{B}}pr_{\mathcal{B}}x_q)$$

We define a homotopy between the identity and J as follows. Let $x_i^1 = (f_i,0,0) \in (\mathcal{A} \times_G \mathcal{B})((a_i,b_i),(a_{i-1},0))$ and $x_i^2 = (0,g_i,0) \in (\mathcal{A} \times_G \mathcal{B})((0,b_i),(a_{i-1},b_{i-1}))$. If $\phi_i \in \Delta([q],[1])$ is the map with inverse image of 0 of cardinality i, we define

$$H\colon F(\mathcal{A} \times_G \mathcal{B}, M \times_G N) \times \Delta \to F(\mathcal{A} \times_G \mathcal{B}, M \times_G N)$$

by the formula

$$\begin{aligned} H(\mathbf{x},\phi_i) =& ((m,0,0); in_{\mathcal{A}}pr_{\mathcal{A}}x_1, \dots in_{\mathcal{A}}pr_{\mathcal{A}}x_{i-1}, x_i^1, x_{i+1}, \dots x_q) \\ &- ((0,n,n_G); x_1, \dots, x_{i-1}, x_i^2, in_{\mathcal{B}}pr_{\mathcal{B}}x_{i+1}, \dots in_{\mathcal{B}}pr_{\mathcal{B}}x_q) \\ &+ ((0,n,0); in_{\mathcal{B}}pr_{\mathcal{B}}x_1, \dots in_{\mathcal{B}}pr_{\mathcal{B}}x_q) \\ &+ ((0,n,n_G); x_1, \dots, x_q) \end{aligned}$$

(note that in the negative summand, it is implicit that n_G is taken away when $i = 0$).

∎

Lemma 3.6.3 *Let $\mathfrak{C}$ be an additive category and M a bilinear bimodule. Then the natural map*

$$S_q\mathfrak{C} \xrightarrow{c \mapsto (c_{0,1},\dots,c_{q-1,q})} \mathfrak{C}^{\times q}$$

induces an equivalence

$$F(S_q\mathfrak{C}, S_qM) \xrightarrow{\sim} F(\mathfrak{C}^{\times q}, M^{\times q}) \xrightarrow{\sim} F(\mathfrak{C}, M)^{\times q}$$

Proof: Recall the equivalence $\psi_q\colon T_q\mathfrak{C} \to S_q\mathfrak{C}$ of lemma 2.2.5, and note that if $G_q\colon \mathfrak{C} \to T_q\mathfrak{C}$ is defined by $c \mapsto G_q(c) = (0,\dots,0,c)$ then we have an isomorphism $T_{q+1}\mathfrak{C} \cong \mathfrak{C} \times_{G_q} T_q\mathfrak{C}$. Furthermore, if M is a linear bimodule, then we define $T_qM = \psi_q^* S_qM$, and we have that $T_{q+1}M \cong M \times_G T_qM$.

Hence

$$\begin{aligned} F(S_q\mathfrak{C}, S_qM) &\xleftarrow{\sim} F(T_q\mathfrak{C}, T_qM) \cong F(\mathfrak{C} \times_G T_{q-1}\mathfrak{C}, M \times_G T_{q-1}M) \\ &\xleftarrow{\sim} \qquad F(\mathfrak{C}, M) \oplus F(T_{q-1}\mathfrak{C}, T_{q-1}M) \end{aligned}$$

and by induction we get that

$$F(S_q\mathfrak{C}, S_qM) \xleftarrow{\sim} F(\mathfrak{C}, M)^{\times q}$$

and this map is a right inverse to the map in the statement. ∎

Definition 3.6.4 For any simplicial category $\mathcal{D}$ we may define the path category $P\mathcal{D}$ by setting $P_q\mathcal{D} = \mathcal{D}_{q+1}$ and letting the face and degeneracy functors be given by raising all indices by one. The unused d_0 defines a functor $P\mathcal{D} \to \mathcal{D}$, and we have a map $\mathcal{D}_1 = P_0\mathcal{D} \to P\mathcal{D}$ given by the degeneracies.

Then $\mathcal{D}_0 \to P\mathcal{D}$ (given by degeneracies in $\mathcal{D}$) defines a simplicial homotopy equivalence, with inverse given by $\prod_{1\le i\le q+1} d_i\colon P_q\mathcal{D} \to \mathcal{D}_0$ (see [131, 1.5.1]).

{prop:FOm

Proposition 3.6.5 *Let $\mathfrak{C}$ be an additive category, and M a bilinear bimodule, then*

$$F(\mathfrak{C}, M) \to \Omega F(S\mathfrak{C}, SM)$$

is an equivalences.

Proof: Consider

$$F(\mathfrak{C}, M) \to F(PS\mathfrak{C}, PSM) \to F(S\mathfrak{C}, SM) \tag{3.6.5}$$ {eq:FOmega

For every q we have equivalences

$$\begin{array}{ccccc} F(\mathfrak{C}, M) & \longrightarrow & F(P_qS\mathfrak{C}, P_qSM) & \longrightarrow & F(S_q\mathfrak{C}, S_qM) \\ \| & & \wr\downarrow & & \wr\downarrow \\ F(\mathfrak{C}, M) & \longrightarrow & F(\mathfrak{C}, M)^{\times q+1} & \longrightarrow & F(\mathfrak{C}, M)^{\times q} \end{array}$$

where the lower sequence is the trivial split fibration. As all terms are bisimplicial abelian groups the sequence 3.6.5 must be a fiber sequence (see A.1.5.0.4) where the total space is contractible. ∎

Chapter II

Γ-spaces and S-algebras

{II}

In this chapter we will introduce the so-called Γ-spaces. The reader can think of these as (very slight) generalizations of (simplicial) abelian groups. The surprising fact is that this minor generalization is big enough to encompass a wide and exotic variety of new examples.

The use of Γ-spaces also fixes another disparity. Quillen defined algebraic K-theory to be a functor from things with abelian group structure (such as rings or exact categories) to abelian groups. We have taken the view that K-theory takes values in spectra, and although spectra are almost as good as abelian groups, this is somehow unsatisfactory. The introduction of Γ-spaces evens this out, in that K-theory now takes things with a Γ-space structure (such as **S**-algebras, or the Γ-space analog of exact categories) to Γ-spaces.

Furthermore, this generalization enables us to include new fields of study, such as the K-theory of spaces, into serious consideration. It is also an aid – almost a prerequisite – when trying to understand the theories to be introduced in later chapters.

To be quite honest, Γ-spaces should not be thought of as a generalization of simplicial abelian groups, but rather of simplicial abelian (symmetric) monoids, since there need not be anything resembling inverses in the setting we use the term (as opposed to Segal's original approach). On the other hand, it is very easy to "group complete": it is a stabilization process.

1 Algebraic structure

1.1 Γ-objects

A gamma-object in a category is a functor from the category of finite sets. We need to be quite precise about this, and the details follow.

1.1.1 The Category Γ^o

Roughly, Γ^o is the category of pointed finite sets – the mother of all mathematics. To be more precise, we choose a skeleton, and let Γ^o be the category with one object $k_+ =$

$\{0, 1, \ldots, k\}$ for every natural number k, and with morphism sets $\Gamma^o(m_+, n_+)$ the set of functions $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that $f(0) = 0$. In [109] Segal considered, the opposite category and called it Γ, and this accounts for the awkward situation where we call the most fundamental object in mathematics the opposite of something. Some people object to this so strongly that write Γ when Segal writes Γ^o. We follow Segal's convention.

1.1.2 Motivation

A *symmetric monoid* is a set M together with a multiplication and a unit element so that any map $M^{\times j} \to M$ gotten by composing maps in the diagram

$$* \xrightarrow{\text{unit}} M \begin{array}{c} \xrightarrow{m\mapsto(1,m)} \\ \xleftarrow{\text{multiplication}} \\ \xrightarrow{m\mapsto(m,1)} \end{array} M \times M \circlearrowleft^{\text{twist}} \begin{array}{c} \xleftarrow{(m_1m_2,m_3)\leftarrow(m_1,m_2,m_3)} \\ \xleftarrow{(m_1,m_2m_3)\leftarrow(m_1,m_2,m_3)} \end{array} M \times M \times M$$

are equal. Thinking of multiplication as "two things coming together" as in the map $2_+ \to 1_+$ given by

$$\begin{array}{lccc} 2_+ = \{ & 0 & 1 & 2\} \\ & \downarrow & \downarrow & \swarrow \\ 1_+ = \{ & 0 & 1 & \} \end{array}$$

we see that the diagram is mirrored by the diagram

$$0_+ \longrightarrow 1_+ \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} 2_+ \circlearrowleft \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} 3_+$$

in Γ^o where the two arrows $1_+ \to 2_+$ are given by

$$\begin{array}{ccc} \{0 & 1\} & \\ \downarrow & \downarrow & \\ \{0 & 1 & 2\} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \{0 & 1\} & \\ \downarrow & & \searrow \\ \{0 & 1 & 2\} \end{array}$$

and the maps $3_+ \to 2_+$ are

$$\begin{array}{cccc} \{0 & 1 & 2 & 3\} \\ \downarrow & \downarrow & \swarrow & \swarrow \\ \{0 & 1 & 2\} & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} \{0 & 1 & 2 & \} \\ \downarrow & \downarrow & \downarrow & \swarrow \\ \{0 & 1 & 2\} & \end{array}$$

(there are more maps in Γ^o, but these suffice for the moment). So we could say that a symmetric monoid is a functor from this part of Γ^o to pointed sets sending 0_+ to the one-point set and sending wedge sum to product (e.g., $3_+ = 2_+ \vee 1_+$ must be sent to the product of the values at 2_+ and 1_+,i.e., the triple product of the value at 1_+).

This doesn't seem very helpful until one notices that this extends to all of Γ^o, and the requirement of sending 1_+ to the one-point set and wedge sum to product fixes the behavior in the sense that there is a one-to-one correspondence between such functors from Γ^o to sets and symmetric monoids, see example 1.2.1.1 below for more details.

The reason for introducing this new perspective is that we can model multiplicative structures functorially, and relaxing the requirement that the functor sends wedge to product is just the trick needed to study more general multiplicative structures. For instance one could imagine situations where the multiplication is not naturally defined on $M \times M$, but on some bigger space like $M \times M \times X$ giving an entire family of multiplications varying over the set X. This is exactly what we need when we are going to study objects that are, say, commutative only up to homotopy.

1.1.3 Γ-objects

If $\mathcal{C}$ is a pointed category one may consider pointed functors $\Gamma^o \to \mathcal{C}$ and natural transformations between such functor. This defines a category we call $\Gamma\mathcal{C}$. Most notably we have the category

$$\Gamma\mathcal{S}_*$$

of Γ-spaces, that is pointed functors from Γ^o to pointed simplicial sets, or equivalently, of simplicial Γ-objects in the category of pointed sets. If $\mathcal{A} = s\mathcal{A}b$ is the category of simplicial abelian groups, we may define

$$\Gamma\mathcal{A}$$

the category of simplicial Γ-objects in abelian groups. Likewise for other module categories. Another example is the category of Γ-categories, i.e., pointed functors from Γ^o to categories. These must not be confused with the notion of $\Gamma\mathcal{S}_*$-categories (see section 1.6).

1.2 The category $\Gamma\mathcal{S}_*$ of Γ-spaces

We start with some examples of Γ-spaces.

{ex:gs}

Example 1.2.1 1. Let M be an abelian group. If we consider M as a mere pointed set, we can not reconstruct the abelian group structure. However, if we consider M as a Γ-pointed set, HM, as follows, there is no loss of structure. Send k_+ to

$$HM(k_+) = M \otimes \tilde{\mathbf{Z}}[k_+] \cong M^{\times k}$$

and a map $f \in \Gamma^o(k_+, n_+)$ gives rise to a map $HM(k_+) \to HM(n_+)$ sending $(m_1, \dots, m_k) \in M^{\times k}$ to $\left((\sum_{j\in f^{-1}(1)} m_j), \dots, (\sum_{j\in f^{-1}(n)} m_j)\right)$ (where $m_0 = 0$). (alternative description: $HM(X) = \mathcal{E}ns_*(X, M)$, and if $f\colon X \to Y \in \Gamma^o$, then $f_*\colon HM(X) \to HM(Y)$ sends ϕ to $y \mapsto f_*\phi(y) = \sum_{x\in f^{-1}(y)} \phi(x)$.) In effect, this defines a functor {Def:barH

$$\bar{H}\colon s\mathcal{A}b = \mathcal{A} \to \Gamma\mathcal{A},$$

and we follow by the forgetful functor $U\colon \Gamma\mathcal{A} \to \Gamma\mathcal{S}_*$, so that

$$H = U\bar{H}.$$

Both HM and $\bar{H}M$ will be referred to as the *Eilenberg-MacLane* objects associated with M. The reason is that these Γ-objects naturally give rise to the so-called Eilenberg-MacLane spectra.

{ex:gs2}

2. The inclusion $\Gamma^o \subset \mathcal{E}ns_* \subset \mathcal{S}_*$ is called in varying sources, $\mathbf{S}$, Id, "the sphere spectrum" etc. We will call it $\mathbf{S}$. Curiously, the Barrat-Priddy-Quillen theorem states that $\mathbf{S} \sim K(\Gamma^o)$, the K-theory of the category Γ^o (see e.g., [109].

{ex:gs3}

3. If X is a pointed simplicial set and M is a Γ-space, then $M \wedge X$ is the Γ-space sending $Y \in ob\Gamma^o$ to $M(Y) \wedge X$. Dually, we let $\underline{\mathcal{S}_*}(X, M)$ be the Γ-space $Y \mapsto \underline{\mathcal{S}_*}(X, M(Y))$. Note that $\Gamma\mathcal{S}_*(M \wedge X, N)$ is naturally isomorphic to $\Gamma\mathcal{S}_*(M, \underline{\mathcal{S}_*}(X, N))$. For any simplicial set X, we let $\mathbf{S}[X] = \mathbf{S} \wedge X_+$, and we see that this is a left adjoint to the functor $R\colon \Gamma\mathcal{S}_* \to \mathcal{S}_*$ evaluating at 1_+.

{def:R}
{ex:gs4}

4. For $X \in ob\Gamma^o$, let $\Gamma^X \in ob\Gamma\mathcal{S}_*$ be given by

$$\Gamma^X(Y) = \Gamma^o(X, Y)$$

Note that $\mathbf{S} = \Gamma^{1_+}$.

The notion of Γ-spaces we are working with is slightly more general than Segal's, [109]. It is usual to call Segal's Γ-spaces *special*:

{def:special}

Definition 1.2.2 A Γ-space M is said to be *special* if the canonical maps

$$M(k_+) \to \prod_k M(1_+)$$

are equivalences for all $k_+ \in ob\Gamma$. This induces an abelian monoid structure on $\pi_0 M(1_+)$, and we say that M is *very special* if this is an abelian group structure.

The difference between Γ-spaces and very special Γ-spaces is not really important. Any Γ-space M gives rise to a very special Γ-space, say FM, in one of many functorial ways, such that there is a "stable equivalence" $M \xrightarrow{\sim} FM$ (see 2.1.6). However, the larger category of all Γ-spaces is nicer for formal reasons, and the very special Γ-spaces are just nice representatives in each stable homotopy class.

1.2.3 The smash product

{Gammasmash}

There is a close connection between Γ-spaces and spectra (there is a functor defined in 2.1.12 that induces an equivalence on homotopy categories), and so the question of what the smash product of two Γ-spaces should be could be expected to be a complicated issue. M. Lydakis [76] realized that this was not the case: the simplest candidate works just beautifully.

If we have two Γ-spaces M and N, we may consider the "external smash", i.e., the functor $\Gamma^o \times \Gamma^o \to \mathcal{S}_*$ which sends (X, Y) to $M(X)\wedge N(Y)$. The category Γ its own smash product and we want some "universal filler" in

$$\begin{array}{ccc} \Gamma^o \times \Gamma^o & \xrightarrow{(X,Y)\mapsto M(X)\wedge N(Y)} & \mathcal{S}_* \\ \wedge\Big\downarrow & & \\ \Gamma^o & & \end{array}$$

The solutions to these kinds of questions are called "left Kan extensions" [79], and in our case it takes the following form:

Let $Z \in \Gamma^o$ and let $\wedge/Z$ be the over category, i.e., the category whose objects are tuples (X, Y, v) where $(X, Y) \in \Gamma^o \times \Gamma^o$ and $v\colon X\wedge Y \to Z \in \Gamma^o$, and where a morphism $(X, Y, v) \to (X', Y', v')$ is a pair of functions $f\colon X \to X'$ and $g\colon Y \to Y'$ in Γ^o such that $v = v' \circ (f\wedge g)$.

Then the smash product $M\wedge N)(Z)$ is defined as the colimit of the composite

$$\wedge/Z \xrightarrow{(X,Y,v)\mapsto(X,Y)} \Gamma^o \times \Gamma^o \xrightarrow{(X,Y)\mapsto M(X)\wedge N(Y)} \mathcal{S}_*,$$

that is

$$(M\wedge N)(Z) = \varinjlim_{(X,Y,v)\in\wedge/Z} M(X)\wedge N(Y)$$

In the language of coends, this becomes particularly perceptive:

$$(M\wedge N)(Z) = \int^{(X,Y)} (M(X)\wedge N(Y))\wedge\Gamma^o(X\wedge Y, Z)$$

the "weighted average of all the handicrafted smash products $M(X)\wedge N(Y)$".

Remark 1.2.4 *Note that a map from a smash product $M\wedge M' \to N \in \Gamma\mathcal{S}_*$ is uniquely described by giving a map $M(X)\wedge M'(Y) \to N(X\wedge Y)$ which is natural in $X, Y \in ob\Gamma^o$.*

1.2.5 The closed structure

The theorem 1.2.6 below states that the smash product endows the category of Γ-spaces with a structure of a *closed* category (which is short for closed symmetric monoidal category). For a thorough discussion see appendix B, but for now recall that it is symmetric monoidal means that the functor $\wedge\colon \Gamma\mathcal{S}_* \times \Gamma\mathcal{S}_* \to \Gamma\mathcal{S}_*$ is associative, symmetric and unital (**S** is the unit) up to coherent isomorphisms, and that it is closed means that in addition there is an "internal morphism object" with reasonable behavior.

The Γ-space of morphisms from M to N is defined by setting

$$\underline{\Gamma\mathcal{S}_*}(M, N) = \{k_+, [q] \mapsto \underline{\Gamma\mathcal{S}_*}(M, N)(k_+)_q = \Gamma\mathcal{S}_*(M\wedge\Delta[q]_+, N(k_+\wedge -))\}.$$

{theo:gsi

Theorem 1.2.6 *(Lydakis) With these definitions of smash and morphism object* $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$ *becomes a closed category.*

Proof: (Sketch: see [76] for further details) First one uses the definitions to show that there is a natural isomorphism $\underline{\Gamma\mathcal{S}_*}(M\wedge N, P) \cong \underline{\Gamma\mathcal{S}_*}(M, \underline{\Gamma\mathcal{S}_*}(N, P))$. Recall from 1.2.14 that $\Gamma^X(Y) = \Gamma^o(X, Y)$ and note that $\mathbf{S} = \Gamma^{1+}$, $\underline{\Gamma\mathcal{S}_*}(\Gamma^X, M) \cong M(X\wedge -)$ and $\Gamma^X\wedge\Gamma^Y \cong \Gamma^{X\wedge Y}$. We get that $M\wedge\mathbf{S} = M\wedge\Gamma^{1+} \cong M$ since $\underline{\Gamma\mathcal{S}_*}(M\wedge\mathbf{S}, N) \cong \underline{\Gamma\mathcal{S}_*}(M, \underline{\Gamma\mathcal{S}_*}(\mathbf{S}, N)) \cong \underline{\Gamma\mathcal{S}_*}(M, N)$ for any N. The symmetry $M\wedge N \cong N\wedge M$ follows from the construction of the smash product, and associativity follows by comparing with

$$M\wedge N\wedge P = \{V \mapsto \varinjlim_{X\wedge Y\wedge Z\to V} M(X)\wedge N(Y)\wedge P(Z)\}$$

That all diagrams that must commute actually do so follows from the crucial observation 1.2.7 below (with the obvious definition of the multiple smash product). ∎

{lem:1.2.7}

Lemma 1.2.7 *Any natural automorphism ϕ of expressions of the form*

$$M_1\wedge M_2\wedge\ldots\wedge M_n$$

must be the identity (i.e., $Aut(\bigwedge^n\colon \Gamma\mathcal{S}_*{}^{\times n} \to \Gamma\mathcal{S}_*)$ is the trivial group).

Proof: The analogous statement is true in Γ^o, since any element in $X_1\wedge X_2\wedge\ldots\wedge X_n$ is in the image of a map from $1_+\wedge 1_+\wedge\ldots\wedge 1_+$, and so any natural automorphism must fix this element.

Fixing a dimension, we may assume that the M_i are discrete, and we must show that $\phi(z) = z$ for any $z \in \bigwedge M_i(Z)$. By construction, z is an equivalence class represented say by an element $(x_1, \ldots, x_m) \in \bigwedge M_i(X_i)$ in the $f\colon \bigwedge X_i \to Z$ summand of the colimit. Represent each $x_i \in M_i(X_i)$ by a map $f_i\colon \Gamma^{X_i} \to M_i$ (so that $f_i(X_i = X_i) = x_i$). Then z is the image of $\wedge id_{X_i}$ in the f summand of the composite

$$(\bigwedge \Gamma^{X_i})(Z) \xrightarrow{\wedge f_i} (\bigwedge M_i)(Z)$$

Hence it is enough to prove the lemma for $M_i = \Gamma^{X_i}$ for varying X_i. But $\bigwedge \Gamma^{X_i} \cong \Gamma^{\bigwedge X_i}$

$$\Gamma\mathcal{S}_*(\Gamma^{\bigwedge X_i}, \Gamma^{\bigwedge X_i}) \cong \Gamma^o(\bigwedge X_i, \bigwedge X_i)$$

and we are done. ∎

Theorem 1.2.6 also follows from a much more general theorem of Day [21], not relying on the special situation in lemma 1.2.7.

1.3 Variants

The proof that $\Gamma\mathcal{S}_*$ is a closed category works if $\mathcal{S}_*$ is exchanged for other suitable closed categories with colimits. In particular $\Gamma\mathcal{A}$, the category of Γ-objects in $\mathcal{A} = s\mathcal{A}b$, is a closed category. The unit is $\bar{H}\mathbf{Z} = \{X \mapsto \tilde{\mathbf{Z}}[X]\}$ (it is $H\mathbf{Z}$, but we remember the group structure, see example 1.2.1.1, the tensor is given by

$$(M \otimes N)(Z) = \varinjlim_{X \wedge Y \to Z} M(X) \otimes N(Y)$$

and the internal function object is given by

$$\underline{\Gamma\mathcal{A}}(M, N) = \{X, [q] \mapsto \Gamma\mathcal{A}(M \otimes \mathbf{Z}[\Delta[q]], N(-\wedge X))\}$$

1.3.1 $\Gamma\mathcal{S}_*$ vs. $\Gamma\mathcal{A}$

{gsvsgab}

The adjoint functor pair between abelian groups and pointed sets

$$\mathcal{E}ns_* \underset{U}{\overset{\tilde{\mathbf{Z}}}{\rightleftarrows}} \mathcal{A}b$$

where U is the forgetful functor induces an adjoint functor pair

$$\Gamma\mathcal{S}_* \underset{U}{\overset{\tilde{\mathbf{Z}}}{\rightleftarrows}} \Gamma\mathcal{A}$$

and since $\tilde{\mathbf{Z}} \colon (\mathcal{E}ns_*, \wedge, S^0) \to (\mathcal{A}b, \otimes, \mathbf{Z})$ is a *(strong) symmetric monoidal functor*, so is $\tilde{\mathbf{Z}} \colon (\Gamma\mathcal{S}_*, \wedge, \mathbf{S}) \to (\Gamma\mathcal{A}, \otimes, \bar{H}\mathbf{Z})$ (a strong monoidal functor is a monoidal functor F such that the structure maps $F(a) \otimes F(b) \to F(a \otimes b)$ and $1 \to F(1)$ are isomorphisms) In particular $\tilde{\mathbf{Z}}\mathbf{S} \cong \bar{H}\mathbf{Z}$,

$$\tilde{\mathbf{Z}}(M \wedge N) \cong \tilde{\mathbf{Z}}M \otimes \tilde{\mathbf{Z}}N$$

and

$$\underline{\Gamma\mathcal{S}_*}(M, UP) \cong U\underline{\Gamma\mathcal{A}}(\tilde{\mathbf{Z}}M, P)$$

satisfying the necessary associativity, commutativity and unit conditions.

Later, we will see that the category $\Gamma\mathcal{A}$, for all practical (homotopical) purposes can be exchanged for $s\mathcal{A}b = \mathcal{A}$. The comparison functors come from the adjoint pair

$$\mathcal{A} \underset{R}{\overset{\bar{H}}{\rightleftarrows}} \Gamma\mathcal{A}$$

where $\bar{H}P(X) = P \otimes \tilde{\mathbf{Z}}[X]$ and $RM = M(1_+)$. We see that $R\bar{H} = id_{\mathcal{A}}$. The other adjunction, $\bar{H}R \to id_{\Gamma\mathcal{A}}$, is discussed in lemma 1.3.3 below. Both $\bar{H}$ and R are symmetric monoidal functors.

1.3.2 Special objects

{Def:spec

We say that $M \in ob\Gamma\mathcal{A}$ is *special* if $UM \in ob\Gamma\mathcal{S}_*$ is special, i.e., if

$$UM(X \vee Y) \xrightarrow{\sim} UM(X) \times UM(Y)$$

is a weak equivalence in $\mathcal{S}_*$. The following lemma has the consequence that all special objects in $\Gamma\mathcal{A}$ can be considered to be in the image of $\bar{H} \colon s\mathcal{A}b = \mathcal{A} \to \Gamma\mathcal{A}$:

{lem:1.4.3

Lemma 1.3.3 *Let $M \in ob\Gamma\mathcal{A}$ be special. Then the unit of adjunction $(\bar{H}RM)(k_+) \to M(k_+)$ is an equivalence.*

Proof: Since M is special, we have that $M(k_+) \to \prod_k M(1_+)$ is an equivalence. On the other hand, if we precompose this map with the unit of adjunction

$$(\bar{H}RM)(k_+) = M(1_+) \otimes \tilde{\mathbf{Z}}[k_+] \to M(k_+)$$

we get an isomorphism. ∎

{Def:IIL}

1.3.4 Additivization

There is also a Dold-Puppe–type construction: $L\colon \Gamma\mathcal{A} \to \mathcal{A}$ which is **left** adjoint to $\bar{H}$, and given by

$$LM = coker\{pr_1 - \nabla + pr_2\colon M(2_+) \to M(1_+)\}$$

where ∇ is the fold map. This functor is intimately connected with the subcategory of $\Gamma\mathcal{A}$ consisting of "additive", or coproduct preserving functors $\Gamma^o \to \mathcal{A}$.

The additive objects are uniquely defined by their value at 1_+, and we get an isomorphism $M \cong \bar{H}(M(1_+)) = \bar{H}RM$. Using this we may identify $\mathcal{A}$ with the full subcategory of additive objects in $\Gamma\mathcal{A}$, and the inclusion into $\Gamma\mathcal{A}$ has a left adjoint given by $\bar{H}L$.

Note that all the functors L, R and $\bar{H}$ between $\mathcal{A}$ and $\Gamma\mathcal{A}$ are strong symmetric monoidal.

Just the same considerations could be made with $\mathcal{A}b$ exchanged for the category of k-modules for any commutative ring k.

1.4 S-algebras

In any monoidal category there is a notion of a *monoid* (see e.g., appendix B). The reason for the name is that a monoid in the usual sense is a monoid in $(\mathcal{E}ns, \times, *)$. Furthermore, the axioms for a ring is nothing but the statement that it is a monoid in $(\mathcal{A}b, \otimes, \mathbf{Z})$. For a commutative ring k, a k-algebra is no more than a monoid in $(k-mod, \otimes_k, k)$, and so it is natural to define **S**-algebras the same way:

{Def:Salg}

Definition 1.4.1 An **S**-algebra A is a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$.

This means that A is a Γ-space together with maps $\mu = \mu^A\colon A\wedge A \to A$ and $1\colon \mathbf{S} \to A$ such that the diagrams

$$\begin{array}{ccccc} A\wedge(A\wedge A) & \xrightarrow{\cong} & (A\wedge A)\wedge A & \xrightarrow{\mu\wedge id} & A\wedge A \\ \downarrow{\scriptstyle id\wedge\mu} & & & & \downarrow{\scriptstyle \mu} \\ A\wedge A & & \xrightarrow{\mu} & & A \end{array}$$

and

$$\begin{array}{ccccc} \mathbf{S}\wedge A & \xrightarrow{1\wedge id} & A\wedge A & \xleftarrow{id\wedge 1} & A\wedge\mathbf{S} \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle \mu} & \underset{\cong}{\swarrow} & \\ & & A & & \end{array}$$

commute, where the diagonal maps are the natural isomorphisms.

We say that an $\mathbf{S}$-algebra is *commutative* if $\mu = \mu \circ tw$ where

$$A\wedge A \xrightarrow[\cong]{tw} A\wedge A$$

is the twist isomorphism.

{remark:15

Remark 1.4.2 *In the definition of an* $\mathbf{S}$*-algebra, the symmetric monoidal category structure is actually never needed, since maps* $M\wedge N \to P$ **out of** *the smash products is uniquely characterized by a map* $M(X)\wedge N(Y) \to P(X\wedge Y)$ *natural in* $X, Y \in ob\Gamma^o$*. So, since the multiplication is a map from the smash* $A\wedge A \to A$*, it can alternatively be defined as a map* $A(X)\wedge A(Y) \to A(X\wedge Y)$ *natural in both* X *and* Y*.*

This was the approach of Bökstedt [9] when he defined FSPs . These are simplicial functors from finite spaces to spaces with multiplication and unit, such that the natural diagrams commute, plus some stability conditions. These stability conditions are automatically satisfied if we start out with functors from Γ^o *(and then apply degreewise and diagonalize if we want* $X \in s\Gamma^o$ *as input), see lemma 2.1.4. On the other hand, we shall later see that there is no loss of generality to consider only* $\mathbf{S}$*-algebras.*

1.4.3 Variants

{II153}

An $\bar{H}\mathbf{Z}$-algebra is a monoid in $(\Gamma\mathcal{A}, \otimes, \bar{H}\mathbf{Z})$. This is, for all practical purposes, equivalent to the more sophisticated notion of $H\mathbf{Z} = U\bar{H}\mathbf{Z}$-algebras arising from the fact that there is a closed category $(H\mathbf{Z} - mod, \wedge_{H\mathbf{Z}}, H\mathbf{Z})$, see below 1.5.6). Since the functors

$$\Gamma\mathcal{S}_* \underset{U}{\overset{\tilde{\mathbf{Z}}}{\rightleftarrows}} \Gamma\mathcal{A} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{\bar{H}} \\ \xrightarrow[R]{} \end{array} \mathcal{A}$$

all are monoidal they send monoids to monoids. For instance, if A is a simplicial ring, then $\bar{H}A$ is an $\bar{H}\mathbf{Z}$-algebra and HA is an $\mathbf{S}$-algebra (it even is an $H\mathbf{Z}$-algebra):

{EMofring
{ex:IIess

Example 1.4.4 1. Let A be a simplicial ring, then HA is an $\mathbf{S}$-algebra with multiplication

$$HA\wedge HA \to H(A\otimes A) \to HA$$

and unit $\mathbf{S} \to \tilde{\mathbf{Z}}\mathbf{S} \cong H\mathbf{Z} \to HA$. In particular, note the $\mathbf{S}$-algebra $H\mathbf{Z}$. It is given by $X \mapsto \tilde{\mathbf{Z}}[X]$, the "integral homology", and the unit map $X = \mathbf{S}(X) \to H\mathbf{Z}(X) = \tilde{\mathbf{Z}}[X]$ is the Hurewicz map A.

{spherica

2. Of course, $\mathbf{S}$ is the initial $\mathbf{S}$-algebra. If M is a simplicial monoid, the monoid algebra $\mathbf{S}[M]$ is given by

$$\mathbf{S}[M](X) = M_+ \wedge X$$

with obvious unit and with multiplication coming from the monoid structure. Note that $R\tilde{\mathbf{Z}}\mathbf{S}[M] \cong \mathbf{Z}[M]$.

3. If A is an $\mathbf{S}$-algebra, then A^o, the opposite of A, is the $\mathbf{S}$-algebra given by A, but with the twisted multiplication

$$A \wedge A \xrightarrow[\cong]{tw} A \wedge A \xrightarrow{\mu} A.$$

4. If A and B are $\mathbf{S}$-algebras, their smash $A \wedge B$ is a new $\mathbf{S}$-algebra with multiplication

$$(A \wedge B) \wedge (A \wedge B) \xrightarrow{id \wedge tw \wedge id} (A \wedge A) \wedge (B \wedge B) \to A \wedge B,$$

and unit $S \cong \mathbf{S} \wedge \mathbf{S} \to A \wedge B$.

5. If A and B are two $\mathbf{S}$-algebras, the product $A \times B$ is formed pointwise: $(A \times B)(X) = A(X) \times B(X)$ and with componentwise multiplication and diagonal unit. The co-product also exist, but is more involved.

ssalgebras}

6. Matrices: If A is an $\mathbf{S}$-algebra, we define the $n \times n$ matrices $Mat_n A$ by

$$Mat_n A(X) = \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(X)) \cong \prod_n \bigvee_n A(X)$$

– the matrices with only "one entry in every coloumn". The unit is the diagonal, whereas the multiplication is determined by

$$\begin{array}{c} Mat_n A(X) \wedge Mat_n A(Y) = \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(X)) \wedge \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(Y)) \\ \downarrow \\ \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(X)) \wedge \underline{\mathcal{S}_*}(n_+ \wedge A(X), n_+ \wedge A(X) \wedge A(Y)) \\ \downarrow \scriptstyle{\text{composition}} \\ \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(X) \wedge A(Y)) \\ \downarrow \scriptstyle{\text{multiplication}} \\ \underline{\mathcal{S}_*}(n_+, n_+ \wedge A(X \wedge Y)) = Mat_n A(X \wedge Y) \end{array}$$

We note that for a simplicial ring B, there is a natural map of $\mathbf{S}$-algebras (sending some wedges to products, and rearranging the order)

$$Mat_n HB \to HM_n B$$

where M_nB are the ordinary matrix ring. This map is a stable eqivalence as defined in 2.1.6. We also have a "Whitehead sum"

$$Mat_n(A) \times Mat_m(A) \xrightarrow{\vee} Mat_{n+m}(A)$$

which is the block sum listing the first matrix in the upper left hand corner and the second matrix in the lower right hand corner. This sum is sent to the ordinary Whitehead sum under the map $Mat_nHB \to HM_nB$.

1.5 A-modules

If A is a ring, we define a left A-module to be an abelian group M together with a map $A \otimes M \to M$ satisfying certain properties. In other words, it is a "$(A \otimes -)$-algebra" where $(A \otimes -)$ is the triple on abelian groups sending P to $A \otimes P$. Likewise

{Def:Amod}

Definition 1.5.1 Let A be an $\mathbf{S}$-algebra. A (left) A-module is an $(A\wedge -)$-algebra.

To be more explicit, it is a pair (M, μ^M) where $M \in ob\Gamma\mathcal{S}_*$ and

$$A\wedge M \xrightarrow{\mu^M} M \in \Gamma\mathcal{S}_*$$

such that

$$\begin{array}{ccc} A\wedge A\wedge M & \xrightarrow{id\wedge\mu^M} & A\wedge M \\ {\scriptstyle \mu^A\wedge id}\downarrow & & {\scriptstyle \mu^M}\downarrow \\ A\wedge M & \xrightarrow{\mu^M} & M \end{array}$$

commutes and such that the composite

$$M \cong \mathbf{S}\wedge M \xrightarrow{1\wedge id} A\wedge M \xrightarrow{\mu^M} M$$

is the identity.

If M and N are A-modules, an A-module map $M \to N$ is a map of Γ-spaces compatible with the A-module structure (an "$(A\wedge -)$-algebra morphism").

Remark 1.5.2 *1. Note that, as remarked for* $\mathbf{S}$*-algebras in 1.4.2, the structure maps defining A-modules could again be defined directly without reference to the internal smash in $\Gamma\mathcal{S}_*$.*

2. One defines right A-modules and A-bimodules similarly as A^o-modules and $A^o\wedge A$-modules.

3. Note that an $\mathbf{S}$*-module is no more than a Γ-space. In general, if A is a commutative* $\mathbf{S}$*-algebra, then the concepts of left or right modules agree.*

4. If A is a simplicial ring, then an HA-module does not need to be of the sort HP for an A-module P, but we shall see that the difference between A-modules and HA-modules is for most applications irrelevant.

{Def:smas

Definition 1.5.3 Let A be an **S**-algebra. Let M be an A-module and M' an A^o-module. The smash product $M' \wedge_A M$ is the Γ-space given by the coequalizer

$$M' \wedge_A M = \lim_{\rightarrow} \{M' \wedge A \wedge M \rightrightarrows M' \wedge M\}$$

where the two maps represent the two actions.

{Def:HomA}

Definition 1.5.4 Let A be an **S**-algebra and let M, N be A-modules. The Γ-space of A-module maps is defined as the equalizer

$$\underline{\mathcal{M}}_A(M, N) = \lim_{\leftarrow} \{\underline{\Gamma\mathcal{S}_*}(M, N) \rightrightarrows \underline{\Gamma\mathcal{S}_*}(A \wedge M, N)\}$$

where the first map is induced by the action of A on M, and the second is

$$\underline{\Gamma\mathcal{S}_*}(M, N) \to \underline{\Gamma\mathcal{S}_*}(A \wedge M, A \wedge N) \to \underline{\Gamma\mathcal{S}_*}(A \wedge M, N)$$

induced by the action of A on N.

From these definitions, the following proposition is immediate.

{prop:166}

Proposition 1.5.5 *Let k be a commutative* **S**-*algebra. Then the smash product and morphism object over k endows the category $\mathcal{M}_k$ of k-modules with the structure of a closed category.*

[illegible]

Example 1.5.6 (k-algebras) If k is a commutative **S**-algebra, the monoids in the closed monoidal category $(k - mod, \wedge_k, k)$ are called k-algebras. The most important example to us are the $H\mathbf{Z}$-algebras. A crucial point we shall return to later is that the homotopy categories of $H\mathbf{Z}$-algebras and simplicial rings are equivalent.

1.6 $\Gamma\mathcal{S}_*$-categories

{sec:gscat}

Since $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$ is a (symmetric monoidal) closed category it makes sense to talk of a $\Gamma\mathcal{S}_*$-*category*, i.e., a collection of objects $ob\mathcal{C}$ and for each pair of objects $c, d \in ob\mathcal{C}$ a Γ-space $\mathcal{C}(c, d)$ of morphisms with multiplication

$$\mathcal{C}(c, d) \wedge \mathcal{C}(b, c) \longrightarrow \mathcal{C}(b, d)$$

and unit

$$\mathbf{S} \longrightarrow \mathcal{C}(c, c)$$

satisfying the usual identites analogous to the notion of an **S**-algebra (as a matter of fact: an **S** algebra is precisely a $\Gamma\mathcal{S}_*$-category with one object). See appendix B for more details on enriched category theory.

In particular, $\Gamma\mathcal{S}_*$ is itself a $\Gamma\mathcal{S}_*$-category. As another example; from the definition 1.5.4 of the $\Gamma\mathcal{S}_*$ of A-module morphisms the following fact follows immediately.

Proposition 1.6.1 *Let A be an* **S***-algebra. Then the category of A-modules is a $\Gamma\mathcal{S}_*$-category.*

Further examples of $\Gamma\mathcal{S}_*$-categories:

{130}

Example 1.6.2 1. Any $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ has an underlying $\mathcal{S}_*$-category $R\mathcal{C}$, or just $\mathcal{C}$ again for short, with function spaces $(R\mathcal{C})(c,d) = R(\underline{\mathcal{C}}(c,d)) = \underline{\mathcal{C}}(c,d)(1_+)$ (see 3). The prime example being $\Gamma\mathcal{S}_*$ itself, where we always drop the R from the notation.

A $\Gamma\mathcal{S}_*$-category with only one object is what we call an **S**-algebra (just as a k-*mod*-category with only one object is a k-algebra), and this is closely connected to Bökstedt's notion of an FSP. In fact, a "ring functor" in the sense of [27] is the same as a $\Gamma\mathcal{S}_*$-category when restricted to $\Gamma^o \subseteq \mathcal{S}_*$, and conversely, any $\Gamma\mathcal{S}_*$-category is a ring functor when extended degreewise to simplicial Γ-spaces. {1301}

2. Just as the Eilenberg-MacLane construction takes rings to **S**-algebras 1, it takes $\mathcal{A}b$-categories to $\Gamma\mathcal{S}_*$-categories. Let $\mathcal{E}$ be an $\mathcal{A}b$-category (i.e., enriched in abelian groups). Then using the Eilenberg-MacLane-construction of 1 on the morphism groups gives a $\Gamma\mathcal{S}_*$-category which we will call $\tilde{\mathcal{E}}$ (it could be argued that it ought to be called $H\mathcal{E}$, but somewhere there has got to be a conflict of notation, and we choose to sin here). To be precise: if $c, d \in ob\mathcal{E}$, then $\tilde{\mathcal{E}}(c,d)$ is the Γ-space which sends $X \in ob\Gamma^o$ to $\mathcal{E}(c,d) \otimes \tilde{\mathbf{Z}}[X]$. {1302}

3. Let $\mathcal{C}$ be a pointed $\mathcal{S}_*$-category. The category $\Gamma\mathcal{C}$ of pointed functors $\Gamma^o \to \mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category by declaring that

$$\underline{\Gamma\mathcal{C}}(c,d)(X) = \Gamma\mathcal{C}(c, d(X\wedge -)) \in ob\mathcal{S}_*$$

{1303}

4. Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. An augmented symmetric monoid in $\mathcal{C}$ is an object c together with maps $c \sqcup c \to c$, $e \to c \to e$ satisfying the usual identities. A slick way of encoding all the identities of an augmented symmetric monoid c is to identify it with its bar complex (Eilenberg-Mac Lane spectrum) $\bar{H}c\colon \Gamma^o \to \mathcal{C}$ where

$$\bar{H}c(k_+) = \sqcup^{k_+} c = \overbrace{c \sqcup \ldots \sqcup c}^{k \text{ times}}, \qquad (\sqcup^{0_+} c = e)$$

That is: an augmented symmetric monoid is a rigid kind of Γ-object in $\mathcal{C}$; it is an Eilenberg-Mac Lane spectrum.

5. Adding 3 and 4 together we get a functor from symmetric monoidal categories to $\Gamma\mathcal{S}_*$-categories, sending $(\mathcal{C}, \sqcup, e)$ to the $\Gamma\mathcal{S}_*$-category with objects the augmented symmetric monoids, and with morphism objects

$$\Gamma\mathcal{C}(\bar{H}c, \bar{H}d(X\wedge -))$$

{1305}

6. Important special case: If $(\mathcal{C}, \vee, e)$ is a *category with sum* (i.e., e is both final and initial in $\mathcal{C}$, and $\vee$ is a coproduct), then **all** objects are augmented symmetric monoids and {Def:catw

$$\Gamma\mathcal{C}(\bar{H}c, \bar{H}d(X\wedge -)) \cong \mathcal{C}(c, \bigvee^X d)$$

1.6.3 The $\Gamma\mathcal{S}_*$-category $\mathcal{C}^\vee$

{Ceevee}

The last example (1.6.2.6) is so important that we introduce the following notation. Let $(\mathcal{C}, \vee, e)$ be a category with sum, then $\mathcal{C}^\vee$ is the $\Gamma\mathcal{S}_*$-category with $ob\mathcal{C}^\vee = ob\mathcal{C}$ and

$$\mathcal{C}^\vee(c,d)(X) = \mathcal{C}(c, \bigvee^X d).$$

If $(\mathcal{E}, \oplus, 0)$ is an $\mathcal{A}b$-category with sum (what is often called an *additive category*), then the $\tilde{\mathcal{E}}$ of 1.6.2.2 and $\mathcal{E}^\oplus$ coincide:

$$\tilde{\mathcal{E}}(c,d)(n_+) \cong \mathcal{E}(c,d)^{\times n} \cong \mathcal{E}(c, d^{\oplus n}) = \mathcal{E}^\oplus(c,d)(n_+).$$

It is worth noting that the structure of 1.6.2.6 when applied to $(\Gamma\mathcal{S}_*, \vee, 0_+)$ is different from the $\Gamma\mathcal{S}_*$-enrichment we have given to $\Gamma\mathcal{S}_*$ when declaring it to be a symmetric monoidal **closed** category under the smash product. Then $\underline{\Gamma\mathcal{S}_*}(M,N)(X) = \Gamma\mathcal{S}_*(M, N(X\wedge -))$. However, $\vee^X N \cong X\wedge N \to N(X\wedge -)$ is a stable equivalence (see definition 2.1.6), and in some cases this is enough to ensure that

$$\underline{\Gamma\mathcal{S}_*}^\vee(M,N)(X) \cong \Gamma\mathcal{S}_*(M, X\wedge N) \to \Gamma\mathcal{S}_*(M, N(X\wedge -)) = \underline{\Gamma\mathcal{S}_*}(M,N)(X)$$

is a stable equivalence.

1.6.4 A reformulation

When talking in the language of $\mathcal{A}b$-categories (linear categories), a ring is just an $\mathcal{A}b$-category with one object, and an A-module is a functor from A to $\mathcal{A}b$. In the setting of $\Gamma\mathcal{S}_*$-categories, we can reinterpret **S**-algebras and their modules. An **S**-algebra A is simply a $\Gamma\mathcal{S}_*$-category with only one object, and an A-module is a $\Gamma\mathcal{S}_*$-functor from A to $\Gamma\mathcal{S}_*$.

Thinking of A-modules as $\Gamma\mathcal{S}_*$-functors $A \to \Gamma\mathcal{S}_*$ the definitions of smash and morphism objects can be elegantly expressed as

$$M'\wedge_A M = \int^A M'\wedge M$$

and

$$\underline{Hom}_A(M,N) = \int_A \underline{\Gamma\mathcal{S}_*}(M,N)$$

If B is another **S**-algebra, M' an $B\wedge A^o$-module we get $\Gamma\mathcal{S}_*$-adjoint functors

$$\mathcal{M}_A \underset{\mathcal{M}_B(M',-)}{\overset{M'\wedge_A -}{\leftrightarrows}} \mathcal{M}_B$$

due to the canonical isomorphism

$$\underline{\mathcal{M}}_B(M'\wedge_A N, P) \cong \underline{\mathcal{M}}_A(N, \underline{\mathcal{M}}_B(M', P)$$

which follows from playing with the definitions in the usual manner ($P \in ob\mathcal{M}_B$).

2 Stable structures

In this section we will discuss the homotopical properties of Γ-spaces and **S**-algebras. Historically Γ-spaces are nice representations of spectra and the choice of equivalences reflects this. That is, in addition to the obvious pointwise equivalences, we have the so-called stable equivalences. The functors of **S**-algebras we will define, such as K-theory, should respect stable equivalences. Any **S**-algebra can, up to a canonical stable equivalence, be replaced by a very special one.

2.1 The homotopy theory of Γ-spaces

To define the stable structure we need to take a different view to Γ-spaces.

2.1.1 Γ-spaces as simplicial functors

Let $M \in ob\Gamma\mathcal{S}_*$. It is a (pointed) functor $M\colon \Gamma^o \to \mathcal{S}_*$, and by extension by colimits and degreewise application followed by the diagonal we may think of it as a functor $\mathcal{S}_* \to \mathcal{S}_*$. To be precise, if X is a pointed set, we define

$$M(X) = \varinjlim_{\text{finite } Y\subseteq X} M(Y)$$

and so M is a (pointed) functor $\mathcal{E}ns_* \to \mathcal{S}_*$. Finally, if $X \in ob\mathcal{S}_*$, we set

$$M(X) = diag^*\{[q] \mapsto M(X_q)\}$$

Aside 2.1.2 *For those familiar with the language of coends, the extensions of a* Γ*-space* M *to an endofunctor on spaces can be done all at once: if* X *is a space, then*

$$M(X) = \int^{k_+} X^{\times k_+} \wedge M(k_+).$$

The fact that these functors come from degreewise applications of a functor on (discrete) sets make them "simplicial" (more precisely: they are $\mathcal{S}_*$-functors), i.e., they give rise to simplicial maps

$$\underline{\mathcal{S}_*}(X,Y) \to \underline{\mathcal{S}_*}(M(X),N(Y))$$

which results in natural maps

$$Y\wedge M(X) \to M(X\wedge Y)$$

coming from the identity on $X\wedge Y$ through the composite

$$\begin{aligned}\mathcal{S}_*(X\wedge Y, X\wedge Y) &\cong \mathcal{S}_*(Y, \underline{\mathcal{S}_*}(X, X\wedge Y)) \\ &\to \mathcal{S}_*(Y, \underline{\mathcal{S}_*}(M(X), M(X\wedge Y))) \cong \mathcal{S}_*(Y\wedge M(X), M(X\wedge Y))\end{aligned}$$

In particular this means that Γ-spaces define spectra: the n-th term is given by $M(S^n)$, and the structure map is $S^1\wedge M(S^n) \to M(S^{n+1})$ where S^n is $S^1 = \Delta[1]/\partial\Delta[1]$ smashed with itself n times.

:piofGamma}

Definition 2.1.3 If $M \in ob\Gamma\mathcal{S}_*$, then the *homotopy groups* are defined as

$$\pi_q M = \lim_{\overrightarrow{k}} \pi_{k+q} M(S^k).$$

Note that $\pi_q M = 0$ for $q < 0$, by the following lemma.

{lem:213}

{lem:2131} **Lemma 2.1.4** *Let $M \in \Gamma\mathcal{S}_*$.*

{lem:2132}

1. *If $Y \xrightarrow{\sim} Y' \in \mathcal{S}_*$ is an equivalence then $M(Y) \xrightarrow{\sim} M(Y')$ is an equivalence also.*

{lem:2133}

2. *If $X \in ob\mathcal{S}_*$ is n-connected then $M(X)$ is n-connected also.*
3. *If $X \in \mathcal{S}_*$ is n-connected then the canonical map $Y\wedge M(X) \to M(Y\wedge X)$ is $2n$-connected.*

Proof: Let LM be the simplicial Γ-space given by

$$LM(X)_p = \bigvee_{Z_0,\dots,Z_p\in(\Gamma^o)^{\times p+1}} M(Z_0)\wedge\Gamma^o(Z_0,Z_1)\wedge\cdots\wedge\Gamma^o(Z_{p-1},Z_p)\wedge\Gamma^o(Z_p,X)$$

with operators determined by

$$d_i(f\wedge\alpha_1\wedge\ldots\wedge\alpha_p\wedge\beta) = \begin{cases}(M(\alpha_1)(f)\wedge\alpha_2\wedge\ldots\wedge\alpha_p\wedge\beta) & \text{if } i=0\\ (f\wedge\alpha_1\wedge\ldots\alpha_{i+1}\circ\alpha_i\ldots\wedge\beta) & \text{if } 1\le i\le p-1\\ (f\wedge\alpha_1\wedge\ldots\wedge\alpha_{p-1}\wedge(\beta\circ\alpha_p)) & \text{if } i=p\end{cases}$$

$$s_j(f\wedge\alpha_1\wedge\ldots\wedge\alpha_p\wedge\beta) = (f\wedge\ldots\alpha_j\wedge\mathrm{id}\wedge\alpha_{j+1}\ldots\wedge\beta)$$

Consider the natural transformation

$$LM \xrightarrow{\eta} M$$

determined by

$$(f\wedge\alpha_1\wedge\ldots\wedge\beta) \mapsto M(\beta\circ\alpha_p\circ\cdots\circ\alpha_1)(f).$$

For each $Z \in ob\Gamma^o$ we obtain a simplicial homotopy inverse to η_Z by sending $f \in M(Z)$ to $(f\wedge\mathrm{id}_Z\wedge\ldots\wedge\mathrm{id}_Z)$. Since LM and M both commute with filtered colimits we see that η is an equivalence on all pointed sets and so by A.1.5.0.2 η is an equivalence for all pointed simplicial sets because LM and M are applied degreewise. Thus, for all pointed simplicial sets X:

$$LM(X) \xrightarrow{\sim} M(X).$$

(1) If $Y \xrightarrow{\sim} Y'$ is an equivalence then $\mathcal{S}_*(k_+, Y) \cong Y^{\times k} \xrightarrow{\sim} (Y')^{\times k} \cong \mathcal{S}_*(k_+, Y')$ is an equivalence for all k. But this implies that $LM(Y)_p \xrightarrow{\sim} LM(Y')_p$ for all p and hence by A1.6.1 that $LM(Y) \xrightarrow{\sim} M(Y)$.

(2) If X is n-connected then $\mathcal{S}_*(k_+, X) \cong X^{\times k}$ is n-connected for all k and hence $LM(X)_p$ is n-connected for all p. Thus, by A.6.4 we see that $LM(X)$ is n-connected also.

(3) If X is n-connected and X' is m-connected then $X \vee X' \to X \times X'$ is $(m+n)$-connected and so $Y\wedge(X\times X') \to (Y\wedge X')\times(Y\wedge X')$ is $(m+n)$-connected also by the commuting diagram

$$\begin{array}{ccc} Y\wedge(X\vee X') & \longrightarrow & Y\wedge(X\times X') \\ \cong\big\downarrow & & \big\downarrow \\ (Y\wedge X)\vee(Y\wedge X') & \longrightarrow & (Y\wedge X)\times(Y\wedge X') \end{array}$$

since both horizontal maps are $(m+n)$-connected. By induction we see that

$$Y\wedge\mathcal{S}_*(k_+, X) \to \mathcal{S}_*(k_+, Y\wedge X)$$

is $2n$-connectd for all k and so $Y\wedge LM(X)_p \to LM(Y\wedge X)_p$ is $2n$-connected for all p. By A1.6.3 and A1.6.4 we can conclude that $Y\wedge LM(X) \to LM(Y\wedge X)$ is $2n$-connected also.

∎

Following Schwede we now define two closed model category structures on $\Gamma\mathcal{S}_*$. We will call these the "pointwise" and the "stable" structures":

{pointwis

Definition 2.1.5 Pointwise structure: A map $M \to N \in \Gamma\mathcal{S}_*$ a pointwise fibration (resp. pointwise equivalence) if $M(X) \to N(X) \in \mathcal{S}_*$ is a fibration (resp. equivalence) for every $X \in ob\Gamma$. The map is a (pointwise) cofibration if it has the lifting property with respect to maps that are both pointwise fibrations and pointwise equivalences.

From this one constructs the stable structure. Note that the cofibrations in the two structures are the same! Because of this we often omit the words "pointwise" and "stable" when referring to cofibrations.

{def:stst

Definition 2.1.6 Stable structure: A map of Γ-spaces is a stable equivalence if it induces an isomorphism on homotopy groups (defined in 2.1.3). It is a (stable) cofibration if it is a (pointwise) cofibration, and it is a stable fibration if it has the lifting property with respect to all maps that are both stable equivalences and cofibrations.

As opposed to simplicial sets, not all Γ-spaces are cofibrant. Examples of cofibrant objects are the Γ-spaces Γ^X of 1.2.1.4 (and so the simplicial Γ-spaces LM defined in the proof of lemma 2.1.4 are cofibrant in every degree, so that $LM \to M$ can be thought of as a cofibrant resolution).

We shall see in 2.1.9 that the stably fibrant objects are the very special Γ-spaces which are pointwise fibrant.

2.1.7 Important convention

The stable structure will by far be the most important to us, and so when we occasionally forget the qualification "stable", and say that a map of Γ-spaces is a fibration, a cofibration or an equivalence this is short for it being a **stable** fibration, cofibration or equivalence. We will say "pointwise" when appropriate.

Theorem 2.1.8 *Both the pointwise and the stable structures define closed model category structures (see A.1.3.2) on* $\Gamma\mathcal{S}_*$. *Furthermore, these structures are compatible with the* $\Gamma\mathcal{S}_*$*-category structure. More precisely: If* $M \overset{i}{\rightarrowtail} N$ *is a cofibration and* $P \overset{p}{\twoheadrightarrow} Q$ *is a pointwise (resp. stable) fibration, then the canonical map*

$$\underline{\Gamma\mathcal{S}_*}(N,P) \to \underline{\Gamma\mathcal{S}_*}(M,P) \prod_{\underline{\Gamma\mathcal{S}_*}(M,Q)} \underline{\Gamma\mathcal{S}_*}(N,Q)$$

is a pointwise (resp. stable) fibration, and if in addition i *or* p *is a pointwise (resp. stable) equivalence, then it is a pointwise (resp. stable) equivalence.*

Proof: (Outline of proof, cf. Schwede [107]) That the pointwise structure is a closed simplicial model category (with $\underline{\Gamma\mathcal{S}_*}^{1+}(-,-)$ as morphism spaces) is essentially an application of Quillen's basic theorem [100, II4] to $\mathcal{A}$ the category of Γ-sets. The rest of the pointwise claim follows from the definition of $\underline{\Gamma\mathcal{S}_*}(-,-)$.

As to the stable structure, all the axioms but one follows from the pointwise structure. If $f\colon M \to N \in \Gamma\mathcal{S}_*$, one must show that there is a factorization $M \overset{\sim}{\rightarrowtail} X \twoheadrightarrow N$ of f as a cofibration which is a stable equivalence, followed by a stable fibration. However, this is an axiom we will never use, so we refer the reader to [107]. We refer the reader to the same source for compatibility of the stable structure with the $\Gamma\mathcal{S}_*$-enrichment. ∎

Note that, since the cofibrations are the same in the pointwise and the stable structure, a map is both a pointwise equivalence and a pointwise fibration if and only if it is both a stable equivalence and a stable fibration.

{cor:218}

Corollary 2.1.9 *Let $M \in ob\Gamma\mathcal{S}_*$. Then M is stably fibrant (i.e., $X \to *$ is a stable fibration) if and only if it is very special and pointwise fibrant.*

Proof: If M is stably fibrant, $M \to *$ has the lifting property with respect to all maps that are stable equivalences and cofibrations, and hence also to the maps that are pointwise equivalences and cofibrations; that is, it is pointwise fibrant. Let $X, Y \in ob\Gamma^o$, then $\Gamma^X \vee \Gamma^Y \to \Gamma^{X\vee Y} \cong \Gamma^X \times \Gamma^Y$ is a cofibration and a weak equivalence. This means that if M is stably fibrant, then

$$\underline{\Gamma\mathcal{S}_*}(\Gamma^{X\vee Y}, M) \to \underline{\Gamma\mathcal{S}_*}(\Gamma^X \vee \Gamma^Y, M)$$

is a stable equivalence and a stable fibration, which is the same as saying that it is a pointwise equivalence and a pointwise fibration, which means that

$$M(X\vee Y) \cong \underline{\Gamma\mathcal{S}_*}^{1_+}(\Gamma^{X\vee Y}, M) \to \underline{\Gamma\mathcal{S}_*}^{1_+}(\Gamma^X \vee \Gamma^Y, M) \cong M(X) \times M(Y)$$

is an equivalence. Similarly, the map

$$\mathbf{S} \vee \mathbf{S} \xrightarrow{in_1 pr_1 + \Delta} \mathbf{S} \times \mathbf{S}$$

is a stable equivalence. When $\pi_0\underline{\Gamma\mathcal{S}_*}^{1_+}(-, M)$ is applied to this sequence we get $(a, b) \mapsto (a, a+b)\colon \pi_0 M(1_+)^{\times 2} \to \pi_0 M(1_+)^{\times 2}$. If M is fibrant this must be an isomorphism, and so $\pi_0 M(1_+)$ has inverses.

Conversely, suppose that M is pointwise fibrant and very special. Let $M \overset{i}{\underset{\sim}{\rightarrowtail}} N \twoheadrightarrow *$ be a factorization. Since both M and N are very special i must be a pointwise equivalence, and so has a section (from the pointwise structure), which means that M is a retract of a stably fibrant object since we must have a lifting in the diagram

$$\begin{array}{ccc} M & = & M \\ \downarrow{\scriptstyle i} & & \downarrow \\ N & \longrightarrow & * \end{array}$$

∎

2.1.10 A simple fibrant replacement functor

{subsec:na

In the approach we will follow, it is a strange fact that we will never need to replace a Γ-space with a cofibrant one, but we will constantly need to replace them by stably fibrant ones. There is a particularly easy way to do this: let M be any Γ-space, and set

$$FM(X) = \varinjlim_{k} \Omega^k M(S^k \wedge X)$$

Obviously the map $M \to FM$ is a stable equivalence, and FM is pointwise Kan and very special (use e.g., lemma 2.1.4) For various purposes, this replacement F will not be good enough. Its main deficiency is that it will not take $\mathbf{S}$-algebras to $\mathbf{S}$-algebras.

2.1.11 Comparison with spectra

We have already observed that Γ-spaces give rise to spectra:

{def:gs2s

Definition 2.1.12 Let M be a Γ-space. Then the *spectrum associated* with M is the sequence

$$\underline{M} = \{k \mapsto M(S^k)\}$$

where S^k is $S^1 = \Delta[1]/\partial\Delta[1]$ smashed with itself k times, together with the structure maps $S^1 \wedge M(S^k) \to M(S^1 \wedge S^k) = M(S^{k+1})$.

The assignment $M \mapsto \underline{M}$ is a simplicial functor

$$\Gamma\mathcal{S}_* \xrightarrow{M \mapsto \underline{M}} \mathcal{S}pt$$

(where $\mathcal{S}pt$ is the category of spectra, see appendix A1.2 for details). and it follows from the considerations in [13] that it induces an equivalence between the stable homotopy categories Γ-spaces and connective spectra.

Crucial for the general acceptance of Lydakis' definition of the smash product was the following:

Proposition 2.1.13 *Let M and N be Γ-spaces and X and Y spaces. If M is cofibrant, then the canonical map*

$$M(X) \wedge N(Y) \to (M \wedge N)(X \wedge Y)$$

is n-connected with $n = conn(X) + conn(Y) + \min(conn(X), conn(Y))$.

Proof: (Sketch, see [76] for further details). The proof goes by induction, first treating the case $M = \Gamma^o(n_+, -)$, and observing that then $M(X) \wedge N(Y) \cong X^{\times n} \wedge N(Y)$ and $(M \wedge N)(X \wedge Y) \cong N((X \wedge Y)^{\times n})$. Hence, in this case the result follows from lemma 3.

∎

Corollary 2.1.14 *Let M and N be Γ-spaces with M cofibrant. Then $\underline{M \wedge N}$ is stably equivalent to a hadicrafted smash product of spectra, e.g.,*

$$n \mapsto \{\lim_{\overrightarrow{k,l}} \Omega^{k+l}(S^n \wedge M(S^k) \wedge N(S^l))\}.$$ ☺

2.2 A fibrant replacement for S-algebras

Note that if A is a simplicial ring, then HA is a very special Γ-space, and so maps between simplicial rings are stable equivalences if and only if they are pointwise equivalences. Hence any functor respecting pointwise equivalences of **S**-algebras will have good homotopy properties when restricted to simplicial rings.

When we want to apply functors to all **S**-algebras A, we frequently need to replace our **S**-algebras by a very special **S**-algebras before feeding them to our functor, in order to ensure that the functor will preserve stable equivalences. This is a potential problem since the fibrant replacement functor F presented in 2.1.10 does not take **S**-algebras to **S**-algebras.

For this we need a gadget first explored by Bökstedt. He noted that when he wanted to extend Hochschild homology to **S**-algebras or rather FSPs (see chapter IV), the face maps were problematic as they involved the multiplication, and this was not well behaved with respect to naïve stabilization.

2.2.1 The category $\mathcal{I}$

{Bostedts

Let $\mathcal{I} \subset \Gamma^o$ be the subcategory with all objects, but only the injective maps. This has much more structure than the natural numbers considered as the subcategory where we only allow the standard inclusion $\{0,1,\dots,n-1\} \subset \{0,1,\dots,n\}$. Most importantly, the sum of two sets $x_0, x_1 \mapsto x_0 \vee x_1$ induces a natural transformation $\mathcal{I}\times\mathcal{I} \to \mathcal{I}$. To be quite precise, the sum is given by $k_+ \vee l_+ = (k+l)_+$ with inclusion maps $k_+ \to (k+l)_+$ sending $i \in k_+$ to $i \in (k+l)_+$, and $l_+ \to (k+l)_+$ sending $j > 0 \in l_+$ to $k+j \in (k+l)_+$. Note that $\vee$ is strictly associative and unital: $(x \vee y) \vee z = x \vee (y \vee z)$ and $0_+ \vee x = x = x \vee 0_+$ (but symmetric only up to isomorphism).

This results in a simplicial category $\{p \mapsto \mathcal{I}^{p+1}\}$ with structure maps given by sending $\mathbf{x} = (x_0, \dots, x_q) \in \mathcal{I}^{q+1}$ to {II:Ipsim

$$\begin{aligned} d_i(\mathbf{x}) =& \begin{cases} (x_0, \dots, x_i \vee x_{i+1}, \dots, x_q) & \text{for } 0 \le i < q, \\ (x_q \vee x_0, , x_1, \dots, x_{p-1}) & \text{for } i = q \end{cases} \\ s_i(\mathbf{x}) =& (x_0, \dots, x_i, 0_+, x_{i+1}, \dots, x_p) \quad \text{for } 0 \le i \le q \end{aligned}$$

{Def:T0}

Definition 2.2.2 If $x = k_+ \in ob\mathcal{I}$, we let $|x| = k$. We will often not distinguish notationally between x and $|x|$. For instance, an expression like S^x will mean S^1 smashed with itself $|x|$ times: $S^{0_+} = S^0$, $S^{(k+1)_+} = S^1 \wedge S^{k_+}$. Likewise Ω^x will mean $Map(S^x, -) = \mathcal{S}_*(S^x, \sin|-|)$. If M is a Γ-space we set

$$T_0 M = \{X \mapsto \underset{\overrightarrow{x \in \mathcal{I}}}{\text{holim}}\, \Omega^x M(S^x \wedge X)\}$$

The reason for the notation T_0M will become apparent in chapter IV (no, it is not because it is the tangent space of something).

We have to know that this has the right homotopy properties, i.e., we need to know that T_0M is equivalent to

$$FM = \{X \mapsto \lim_{\overrightarrow{k}} \Omega^k M(S^k \wedge X)\}.$$

One should note that, as opposed to $\mathbf{N}$, the category $\mathcal{I}$ is not filtering, so we must stick with the homotopy colimits. However, $\mathcal{I}$ possesses certain good properties which overcome this difficulty. (Bökstedt attributes in [9] the idea behind the following very important stabilization lemma to Illusie [55])

Lemma 2.2.3 *(cf. [9, 1.5]) Let $G\colon \mathcal{I}^{q+1} \to \mathcal{S}_*$ be a functor, $\mathbf{x} \in ob\mathcal{I}^{q+1}$, and assume G sends maps in the under category $\mathbf{x}/\mathcal{I}^{q+1}$ to $n = n_{\mathbf{x}}$ connected maps. Then the map* {Bokstedt

$$G(\mathbf{x}) \to \underset{\overrightarrow{\mathcal{I}^{q+1}}}{\operatorname{holim}} G$$

is n-connected.

Proof: Consider the functor

$$\mu_{\mathbf{x}}\colon \mathcal{I}^{q+1} \xrightarrow{\mathbf{y}\mapsto\mathbf{x}\vee\mathbf{y}} \mathcal{I}^{q+1}$$

The second inclusion $\mathbf{y} \subseteq \mathbf{x} \vee \mathbf{y}$ defines a natural transformation from the identity to $\mu_{\mathbf{x}}$. Hence, for every $\mathbf{y} \in ob\mathcal{I}^{q+1}$ the under category $\mathbf{y}/\mu_{\mathbf{x}}$ is contractible, and by the dual of [14, XI.9.2] (or A1.9.2.1) we have that

$$(\mu_{\mathbf{x}})_*\colon \underset{\overrightarrow{\mathcal{I}^{q+1}}}{\operatorname{holim}} G\mu_{\mathbf{x}} \xrightarrow{\simeq} \underset{\overrightarrow{\mathcal{I}^{q+1}}}{\operatorname{holim}} G$$

is an equivalence. The map $G(\mathbf{x}) \to \operatorname{holim}_{\overrightarrow{\mathcal{I}^{q+1}}} G$ factors through $(\mu_{\mathbf{x}})_*$, and so we have to show that the map $G(\mathbf{x}) \to \operatorname{holim}_{\overrightarrow{\mathcal{I}^{q+1}}} G\mu_{\mathbf{x}}$ is n-connected. Let $G(\mathbf{x})$ also denote the constant functor with value $G(\mathbf{x})$. Since $\mathcal{I}^{q+1}$ has an initial object it is contractible (in the sense that $obN(\mathcal{I}^{q+1})$ is contractible). With this notation, we have to show that the last map in the composite

$$G(\mathbf{x}) \xrightarrow{\simeq} obN(\mathcal{I}^{q+1})_+ \wedge G(\mathbf{x}) = \underset{\overrightarrow{\mathcal{I}^{q+1}}}{\operatorname{holim}} G(\mathbf{x}) \to \underset{\overrightarrow{\mathcal{I}^{q+1}}}{\operatorname{holim}} G\mu_{\mathbf{x}}$$

is n-connected, which follows as homotopy colimits preserve connectivity (A.1.9.3.1). ∎

A stable equivalence of **S**-algebras is a map of **S**-algebras that is a stable equivalence when considered as a map of Γ-spaces.

T0monoidal}

Lemma 2.2.4 *The functor T_0 maps* **S***-algebras to* **S***-algebras, and the natural transformation $id \to T_0$ is a stable equivalence of* **S***-algebras.*

Proof: Let A be an **S**-algebra. We have to define the multiplication and the unit of T_0A. The unit is obvious: $\mathbf{S} \to T_0\mathbf{S} \to T_0A$, and the multiplication is

$$\begin{aligned}
T_0A(X)\wedge T_0A(Y) &\longrightarrow \underset{\overrightarrow{(x,y)\in\mathcal{I}^2}}{\operatorname{holim}} \Omega^{x\vee y}\left(A(S^x\wedge X)\wedge A(S^y\wedge Y)\right) \\
&\xrightarrow{\text{mult. in } A} \underset{\overrightarrow{(x,y)\in\mathcal{I}^2}}{\operatorname{holim}} \Omega^{x\vee y} A(S^{x\vee y}\wedge X\wedge Y) \\
&\xrightarrow{\text{sum in } \mathcal{I}} \underset{\overrightarrow{z\in\mathcal{I}}}{\operatorname{holim}} \Omega^z A(S^z\wedge X\wedge Y) = T_0A(X\wedge Y)
\end{aligned}.$$

That the map $A \to T_0A$ is a map of **S**-algebras is now immediate. ∎

:bZA is HB}

Corollary 2.2.5 *Any $\bar{H}\mathbf{Z}$-algebra is functorially stably equivalent to $\bar{H}$ of a simplicial ring. In particular, if A is an $\mathbf{S}$-algebra, then $\tilde{\mathbf{Z}}A$ is functorially stably equivalent to H of a simplicial ring.*

Proof: The T_0 construction can equally well be performed in $\bar{H}\mathbf{Z}$-modules: let $\Omega^1_{\mathcal{A}b}M$ be $\underline{\mathcal{S}_*}(S^1, M)$, which is a $\bar{H}\mathbf{Z}$-modules if M is, and let the homotopy colimit be given by the usual formula except the the wedges are replaced by sums (see A.1.9.5 for further details). Let $R_0A = \operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}} \Omega^x_{\mathcal{A}b}A(S^x)$. This is an $\bar{H}\mathbf{Z}$-algebra if A is. There is a natural equivalence $R_0A \to R_0(\sin|A|)$ and a natural transformation $T_0UA \to UR_0(\sin|A|)$ (U is the forgetful functor). By lemma A.1.9.5.2 and lemma 2.1.4.2 you get that $T_0UA(S^n) \to UR_0(\sin|A|)(S^n)$ is $(2n-1)$-connected. But since both sides are special Γ-spaces, this means that $T_0UA \xrightarrow{\sim} UR_0 \sin|A| \xleftarrow{\sim} UR_0A$ is a natural chain of weak equivalences. (Alternatively, we could have adapted Bökstedt's approximation theorem to prove directly that $A \to R_0A$ is a stable equivalence.)

Consequently, if A is a $\bar{H}\mathbf{Z}$-algebra, there is a functorial stable equivalence $A \to R_0A$ of $\bar{H}\mathbf{Z}$-algebras. But R_0A is special and for such algebras the unit of adjunction $\bar{H}R \to 1$ is an equivalence by lemma 1.3.3. ∎

2.3 Homotopical algebra in the category of A-modules

Although it is not necessary for the subsequent development, we list a few facts pertaining to the homotopy structure on categories of modules over $\mathbf{S}$-algebras. The stable structure on A-modules is inherited in the usual way from the stable structure on Γ-spaces.

Definition 2.3.1 Let A be an $\mathbf{S}$-algebra. We say that an A-module map is an equivalence (resp. fibration) if it is a stable equivalence (resp. stable fibration) of Γ-spaces. The cofibrations are defined by the lifting property.

Theorem 2.3.2 *With these definitions, the category of A-modules is a closed model category compatibly enriched in $\Gamma\mathcal{S}_*$: if $M \overset{i}{\rightarrowtail} N$ is a cofibration and $P \overset{p}{\twoheadrightarrow} Q$ is a fibration, then the canonical map*

$$\underline{Hom}_A(N,P) \xrightarrow{(i^*,p_*)} \underline{Hom}_A(M,P) \prod_{\underline{Hom}_A(M,Q)} \underline{Hom}_A(N,Q)$$

is a stable fibration, and if in addition i or p is an equivalence, then (i^*, p_*) is a stable equivalence.

Proof: (For a full proof, consult [107]). For the proof of the closed model category structure, see [108, 3.1.1]. For the proof of the compatibility with the enrichment, see the proof of [108, 3.1.2] where the commutative case is treated. ∎

The smash product behaves as expected (see [76] and [107] for proofs):

Proposition 2.3.3 *Let A be an $\mathbf{S}$-algebra, and let M be a cofibrant A^o-module. Then $M\wedge_A -\colon A-mod \to \Gamma\mathcal{S}_*$ sends stable equivalences to stable equivalences. If N is an A-module there are first quadrant spectral sequences*

$$\begin{aligned}\mathrm{Tor}_p^{\pi_*A}(\pi_*M,\pi_*N)_q) &\Rightarrow \pi_{p+q}(M\wedge_A N)\\ \pi_p(M\wedge_A(H\pi_q N)) &\Rightarrow \pi_{p+q}(M\wedge_A N)\end{aligned}$$

If $A\to B$ is an stable equivalence of $\mathbf{S}$-algebras, then the derived functor of $B\wedge_A -$ induces an equivalence between the homotopy categories of A and B-modules.

2.3.4 k-algebras

In the category of k-algebras, we call a map a fibration or a weak equivalence if it is a stable fibration or stable equivalence of Γ-spaces. The cofibrations are as usual the maps with the right (right meaning correct: in this case left is right) lifting property. With these definitions the category of k-algebras becomes a closed simplicial model category [107]. We will need the analogous result for $\Gamma\mathcal{S}_*$-categories:

2.4 Homotopical algebra in the category of $\Gamma\mathcal{S}_*$-categories

To be converted to LaTeX (02-01-17). For now Appendix A1=A is prioritized due to the many references.

Included to have references:

categories}

Definition 2.4.1 A $\Gamma\mathcal{S}_*$-functor of $\Gamma\mathcal{S}_*$-categories $F\colon \mathcal{C}\to\mathcal{D}$ is a *stable equivalence* if for all $c,c'\in ob\mathcal{C}$ the map

$$\mathcal{C}(c,c')\to\mathcal{D}(Fc,Fc')\in\Gamma\mathcal{S}_*$$

is a stable equivalence, and for any $d\in ob\mathcal{D}$ there is a $c\in ob\mathcal{C}$ and an isomorphism $Fc\cong d$.

Likewise, an $\mathcal{S}$-functor of $\mathcal{S}$-categories $F\colon\mathcal{C}\to\mathcal{D}$ is a *weak equivalence* if for all $c,c'\in ob\mathcal{C}$ the map $\mathcal{C}(c,c')\to\mathcal{D}(Fc,Fc')\in\mathcal{S}$ is a weak equivalence, and for any $d\in ob\mathcal{D}$ there is a $c\in ob\mathcal{C}$ and an isomorphism $Fc\cong d$.

Recall that a $\Gamma\mathcal{S}_*$-*equivalence* is a $\Gamma\mathcal{S}_*$-functor $\mathcal{C}\xrightarrow{F}\mathcal{D}$ for which there exists a $\Gamma\mathcal{S}_*$-functor $\mathcal{C}\xleftarrow{G}\mathcal{D}$ and $\Gamma\mathcal{S}_*$-natural isomorphisms $id_{\mathcal{C}}\cong GF$ and $id_{\mathcal{D}}\cong FG$.

of gs-cats}

Lemma 2.4.2 *Every stable equivalence of $\Gamma\mathcal{S}_*$-categories can be written as a composite of a stable equivalence inducing the identity on the objects and a $\Gamma\mathcal{S}_*$-equivalence.*

Proof: Let $F\colon\mathcal{C}\to\mathcal{D}$ be a stable equivalence. let $\underline{F}$ be the $\Gamma\mathcal{S}_*$-category with the same objects as $\mathcal{C}$, but with morphisms given by $\underline{F}(c,c')=\mathcal{D}(Fc,Fc')$. Then F factors as $\mathcal{C}\to\underline{F}\to\mathcal{D}$ where the first map is the identity on objects and a stable equivalence on morphisms, and the second is induced by F on objects, and is the identity on morphisms.

The latter map is a $\Gamma\mathcal{S}_*$-equivalence: for every $d \in ob\mathcal{D}$ **choose** a $c_d \in ob\mathcal{C}$ and an isomorphism $d \cong Fc_d$. As one checks, the application $d \mapsto c_d$ defines the inverse $\Gamma\mathcal{S}_*$-equivalence. ∎

So stable equivalences are the more general, and may be characterized as composites of $\Gamma\mathcal{S}_*$-equivalences and stable equivalences that induce the identity on the set of objects. Likewise for weak equivalences of $\mathcal{S}$-categories.

3 Algebraic K-theory

{IISegalH

3.1 K-theory of symmetric monoidal categories

An abelian monoid can be viewed as a symmetric monoidal category (an SMC) with just identity morphisms. An abelian monoid M gives rise to a Γ-space HM via the formula $k_+ \mapsto M^{\times k}$ (see [109]), the Eilenberg-MacLane spectrum of M. Algebraic K-theory as in Segal's paper is an extension of this to symmetric monoidal categories (see also [111] or [121]), such that for every symmetric monoidal category $\mathcal{C}$ we have a Γ-category $\bar{H}\mathcal{C}$.

For a finite set X, let $\mathcal{P}X$ set of subsets of X. If S and T are two disjoint subsets of X, then $S\coprod T$ is again a subset of X. For a strict symmetric monoidal category $(\mathcal{C}, \sqcup, e)$ (*strict* means that all coherence isomorphisms are identities) we could define the algebraic K-theory as the Γ-category which evaluated on $k_+ \in \Gamma^o$ was the category whose objects were all functions $\mathcal{P}\{1,\dots,k\} \to ob\mathcal{C}$ sending $\coprod$ to $\sqcup$ and $\emptyset$ to e

$$\begin{pmatrix}\mathcal{P}\{1,\dots,k\}\\ \coprod \\ \emptyset\end{pmatrix} \to \begin{pmatrix}\mathcal{C}\\ \sqcup \\ \mathbf{I}\end{pmatrix}$$

Such a function is uniquely given by declaring what its values are on all subsets $\{i\} \subset \{1,\dots,k\}$ and so this is nothing but $\mathcal{C}$ times itself k times.

In the nonstrict case this loosens only up a bit. If $(\mathcal{C}, \sqcup, e)$ is a symmetric monoidal category, $\bar{H}\mathcal{C}(k_+)$ is the symmetric monoidal category whose objects are the pointed functors $\mathcal{P}\{1,\dots,k\} \to \mathcal{C}$ taking $\coprod$ to $\sqcup$ up to coherent isomorphisms. More precisely

{Def:IISeg

Definition 3.1.1 Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. Let $k_+ \in ob\Gamma^o$. An object of $\bar{H}\mathcal{C}(X)$ is a function $a\colon \mathcal{P}\{1,\dots,k\} \to ob\mathcal{C}$ together with a choice of isomorphisms

$$\alpha_{S,T}\colon a_S \sqcup a_T \to a_{S\coprod T}$$

for every pair $S, T \subseteq \{1,\dots,k\}$ such that $S \cap T = \emptyset$ satisfying the following conditions:

1. $a_\emptyset = e$

2. $a_{S,\emptyset}\colon e \sqcup a_S \to a_{\emptyset \coprod S} = a_S$ and $a_{\emptyset,S}\colon a_S \sqcup e \to a_{S\coprod\emptyset} = a_S$ are the inverses to the corresponding structure isomorphism in $\mathcal{C}$

3.

$$\begin{array}{ccccc}
(a_S \sqcup a_T) \sqcup a_U & & \longrightarrow & & a_S \sqcup (a_T \sqcup a_U) \\
\downarrow{\scriptstyle \alpha_{S,T} \sqcup id} & & & & \downarrow{\scriptstyle id \sqcup \alpha_{T,U}} \\
a_{S \coprod T} \sqcup a_U & \xrightarrow[\alpha_{S \coprod T, U}]{} & a_{S \coprod T \coprod U} & \xleftarrow[\alpha_{S, T \coprod U}]{} & a_S \sqcup a_{T \coprod U}
\end{array}$$

where the unlabelled arrow is the corresponding structure isomorphism in $\mathcal{C}$

4.

$$\begin{array}{ccc}
a_S \sqcup a_T & \longrightarrow & a_T \sqcup a_S \\
& \searrow{\scriptstyle \alpha_{S,T}} & \downarrow{\scriptstyle \alpha_{T,S}} \\
& & a_{S \coprod T} = a_{T \coprod S}
\end{array}$$

where the unlabelled arrow is the corresponding structure isomorphism in $\mathcal{C}$

A morphism $f\colon (a, \alpha) \to (b, \beta) \in \bar{H}\mathcal{C}(X)$ is a collection of morphisms

$$f_S\colon a_S \to b_S \in \mathcal{C}$$

such that

1. $f_\emptyset = id_e$

2.

$$\begin{array}{ccc}
a_S \sqcup a_T & \xrightarrow{f_S \sqcup f_T} & b_S \sqcup b_T \\
{\scriptstyle \alpha_{S,T}}\downarrow & & {\scriptstyle \beta_{S,T}}\downarrow \\
a_{S \coprod T} & \xrightarrow{f_{S \coprod T}} & b_{S \coprod T}
\end{array}$$

If $\phi\colon k_+ \to l_+ \in \Gamma^o$ then $\bar{H}\mathcal{C}(k_+) \to \bar{H}\mathcal{C}(l_+)$ is defined by sending $a\colon \mathcal{P}\{1, \dots, k\} \to \mathcal{C}$ to

$$\mathcal{P}\{1, \dots, l\} \xrightarrow{\phi^{-1}} \mathcal{P}\{1, \dots, k\} \xrightarrow{a} \mathcal{C}$$

(this makes sense as ϕ was pointed at 0).

This defines the Γ-category $\bar{H}\mathcal{C}$, which again is obviously functorial in $\mathcal{C}$, giving the functor

$$\bar{H}\colon \text{symmetric monoidal categories} \to \Gamma\text{-categories}$$

{KofSMC} The nerve $obN\bar{H}\mathcal{C}$ forms a Γ–space which is often called the *algebraic K-theory of* $\mathcal{C}$.

If $\mathcal{C}$ is discrete, or in other words, $\mathcal{C} = ob\mathcal{C}$ is an abelian monoid, then this is exactly the Eilenberg–Mac Lane spectrum of $ob\mathcal{C}$.

Note that $\bar{H}\mathcal{C}$ becomes a **special** Γ-category in the sense that

Lemma 3.1.2 *Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. The canonical map*

$$\bar{H}\mathcal{C}(k_+) \to \bar{H}\mathcal{C}(1_+) \times \cdots \times \bar{H}\mathcal{C}(1_+)$$

is an equivalence of categories.

Proof: We do this by producing an equivalence $E_k\colon \mathcal{C}^{\times k} \to \bar{H}\mathcal{C}(k_+)$ such that

$$\begin{array}{ccc} \mathcal{C}^{\times k} & = & \mathcal{C}^{\times k} \\ {\scriptstyle E_k}\downarrow & & \downarrow{\scriptstyle E_1^{\times k}} \\ \bar{H}\mathcal{C}(k_+) & \longrightarrow & \bar{H}\mathcal{C}(1_+)^{\times k} \end{array}$$

commutes. The equivalence E_k is given by sending $(c_1, \ldots, c_k) \in ob\mathcal{C}^{\times k}$ to $E_k(c_1, \ldots, c_k) = \{(a_S, \alpha_{S,T})\}$ where

$$a_{\{i_1,\ldots,i_j\}} = c_{i_1} \sqcup (c_{i_2} \sqcup \ldots \sqcup (c_{i_{k-1}} \sqcup c_{i_k}) \ldots)$$

and $\alpha_{S,T}$ is the unique isomorphism we can write up using only the structure isomorphisms in $\mathcal{C}$. Likewise for morphisms. A quick check reveals that this is an equivalence (check the case $k = 1$ first), and that the diagram commutes. ∎

3.1.3 Enrichment in $\Gamma\mathcal{S}_*$

The definitions above makes perfect sense also in the $\Gamma\mathcal{S}_*$-enriched world, and we may speak about symmetric monoidal $\Gamma\mathcal{S}_*$-categories $\mathcal{C}$.

A bit more explicitly: a *symmetric monoidal* $\Gamma\mathcal{S}_*$*-category* is a tuple $(\mathcal{C}, \sqcup, e, \alpha, \lambda, \rho, \gamma)$ {Def:SMgsC such that $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category, $\sqcup\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a $\Gamma\mathcal{S}_*$-functor, $e \in ob\mathcal{C}$ and α, λ, ρ and γ are $\Gamma\mathcal{S}_*$-natural transformations satisfying the usual requrements listed in appendix B.??.

The definition of $\bar{H}\mathcal{C}$ at this generality is as follows: the objects in $\bar{H}\mathcal{C}(k_+)$ are the same as before, and the Γ-space $\bar{H}\mathcal{C}((a, \alpha), (b, \beta))$ is defined as the equalizer

$$\bar{H}\mathcal{C}((a, \alpha), (b, \beta)) \longrightarrow \prod_{S \subseteq \{1,\ldots,k\}} \mathcal{C}(a_S, b_S) \rightrightarrows \prod_{\substack{S,T \subseteq \{1,\ldots,k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_S \sqcup a_T, b_{S \coprod T})$$

where the upper map is

$$\prod_{S \subseteq \{1,\ldots,k\}} \mathcal{C}(a_S, b_S) \xrightarrow{\sqcup} \prod_{\substack{S,T \subseteq \{1,\ldots,k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_S \sqcup a_T, b_S \sqcup b_T) \xrightarrow{(\beta_{S,T})_\emptyset} \prod_{\substack{S,T \subseteq \{1,\ldots,k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_S \sqcup a_T, b_{S \coprod T})$$

and the lower map is

$$\prod_{S \subseteq \{1,\ldots,k\}} \mathcal{C}(a_S, b_S) \xrightarrow{\text{proj.}} \prod_{\substack{S,T \subseteq \{1,\ldots,k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_{S \coprod T}, b_{S \coprod T}) \xrightarrow{(\alpha_{S,T})^\emptyset} \prod_{\substack{S,T \subseteq \{1,\ldots,k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_S \sqcup a_T, b_{S \coprod T})$$

3.1.4 Categories with sum

{subsec:I

The simplest example of symmetric monoidal $\Gamma\mathcal{S}_*$-categories comes from categories with sum (i.e., $\mathcal{C}$ is pointed and has a coproduct $\vee$). If $\mathcal{C}$ is a category with sum we consider it as a $\Gamma\mathcal{S}_*$-category via the enrichment

$$\mathcal{C}^\vee(c, d)(k_+) = \mathcal{C}(c, \bigvee^k d)$$

(see 1.6.3).

The sum structure survives to give $\mathcal{C}^\vee$ the structure of a symmetric $\Gamma\mathcal{S}_*$-monoidal category:

$$(\mathcal{C}^\vee \times \mathcal{C}^\vee)((c_1, c_2), (d_1, d_2))(k_+) = \mathcal{C}(c_1, \bigvee^k d_1) \times \mathcal{C}(c_2, \bigvee^k d_2)$$
$$\to \mathcal{C}(c_1 \vee c_2, \left(\bigvee^k d_1\right) \vee \left(\bigvee^k d_2\right)) \cong \mathcal{C}(c_1 \vee c_2, \bigvee^k (d_1 \vee d_2)) = \mathcal{C}^\vee(c_1 \vee c_2, d_1 \vee d_2)(k_+)$$

Categories with sum also have a particular transparent K-theory. The data for a symmetric monoidal category above simplifies in this case to $\bar{H}\mathcal{C}(k_+)$ having as objects functors from the pointed category of subsets and inclusions of $k_+ = \{0, 1, \ldots, k\}$, sending 0_+ to 0 and pushout squares to pushout squares, see also section III.2.1.1.

3.2 Quite special Γ-objects

Let $\mathcal{C}$ be a Γ-$\Gamma\mathcal{S}_*$-category, i.e., a functor $\mathcal{C}\colon \Gamma^o \to \Gamma\mathcal{S}_*$-categories.Consider a functor

$$\mathcal{D}\colon \Gamma^o \to \Gamma\mathcal{S}_*\text{-categories.}$$

We say that $\mathcal{D}$ is *special* if for all finite pointed sets X and Y the canonical $\Gamma\mathcal{S}_*$-functor $\mathcal{C}(X \vee Y) \to \mathcal{C}(X) \times \mathcal{C}(Y)$ are $\Gamma\mathcal{S}_*$-equivalences of $\Gamma\mathcal{S}_*$-categories. So, for instance, if $\mathcal{C}$ is a symmetric monoidal category, then $\bar{H}\mathcal{C}$ is special. We need a slightly weaker notion.

te special}

Definition 3.2.1 Let $\mathcal{D}$ be a Γ-$\Gamma\mathcal{S}_*$-category. We say that $\mathcal{D}$ is *quite special* if for all $X, Y \in ob\Gamma^o$ the canonical map $\mathcal{D}(X \vee Y) \to \mathcal{D}(X) \times \mathcal{D}(Y)$ is a stable equivalenceof $\Gamma\mathcal{S}_*$-categories (see 2.4.1 for definition).

Likewise, a functor $\mathcal{D}\colon \Gamma^o \to \mathcal{S}$-categories is quite special if $\mathcal{D}(X \vee Y) \to \mathcal{D}(X) \times \mathcal{D}(Y)$ is a weak equivalence of $\mathcal{S}$-categories 2.4.1.

Typically, theorems about special $\mathcal{D}$ remain valid for quite special $\mathcal{D}$.

Lemma 3.2.2 *Let $\mathcal{D}\colon \Gamma^o \to \mathcal{S}$-categories be quite special. Then $obN\mathcal{D}$ is special.*

Proof: This follows since the nerve functor obN preserves products and by [29] takes weak equivalences of $\mathcal{S}$-categories to weak equivalences of simplicial sets. ∎

Recall the fibrant replacement functor T_0 of 2.2.2. The same proof as in lemma 2.2.4 gives that if we use T_0 on all the morphism objects in a $\Gamma\mathcal{S}_*$-category we get a new category where the morphism objects now are stbly fibrant.

Lemma 3.2.3 *Let $\mathcal{C}\colon \Gamma^o \to \Gamma\mathcal{S}_*$-categories be quite special. Then $T_0\mathcal{C}$ is quite special.*

Proof: This follows since T_0 preserves stable equivalences, and since

$$T_0(M \times N) \xrightarrow{\sim} T_0 M \times T_0 N$$

is a stable equivalence for any $M, N \in ob\Gamma\mathcal{S}_*$. Both these facts follow from the definition of T_0 and Bökstedt's approximation lemma 2.2.3. ∎

3.3 A uniform choice of weak equivalences

sec:IIunif}

In the discrete case, algebraic K-theory focuses on the isomorphisms. In the general case we still have a canonical choice of weak equivalences, which is good enough for the applications we have in mind, but might be modified in more complex situations where we must be free to choose our weak equivalences. For a trivial example of this, see note 3.3.3 below.

Define the functor

$$\omega\colon \Gamma\mathcal{S}_* - \text{categories} \to \mathcal{S} - \text{categories}$$

by means of the pullback

$$\begin{array}{ccc} \omega\mathcal{C} & \xrightarrow{w_{\mathcal{C}}} & RT_0\mathcal{C} \\ \downarrow & & \downarrow \\ i\pi_0\mathcal{C} & \longrightarrow & \pi_0\mathcal{C} \end{array}$$

where $i\pi_0\mathcal{C}$ is the subcategory of isomorphisms in $\pi_0\mathcal{C}$, and R was the forgetful functor from $\Gamma\mathcal{S}_*$ to $\mathcal{S}$ applied to all morphism objects (see 3). Note that $\pi_0\mathcal{C} \cong \pi_0 RT_0\mathcal{C}$.

Lemma 3.3.1 *Let $\mathcal{C}$ be a quite special Γ-$\Gamma\mathcal{S}_*$-category. Then $\omega\mathcal{C}$ is a quite special Γ-$\mathcal{S}$-category*

Proof: That $\mathcal{C}$ is quite special implies that $RT_0\mathcal{C}$ is quite special since stable equivalences of stably fibrant Γ-spaces are pointwise equivalences, and hence taken to weak equivalences by R. The map $RT_0\mathcal{C} \to \pi_0 RT_0\mathcal{C} \cong \pi_0\mathcal{C}$ is a (pointwise) fibration since R takes fibrant Γ-spaces to fibrant spaces.

Furthermore $\pi_0\mathcal{C}$ is special since π_0 takes stable equivalences of $\Gamma\mathcal{S}_*$-spaces to isomorphisms. The subcategory of isomorphisms in a special Γ-category is always special (since the isomorphism category in a product category is the product of the isomorphism categories), so $i\pi_0\mathcal{C}$ is special too.

We have to know that the pullback behaves nicely with respect to this structure. The map $RT_0\mathcal{C}(X \vee Y) \to RT_0\mathcal{C}(X) \times RT_0\mathcal{C}(Y)$ is a weak equivalence. Hence it is enough to show that if $\mathcal{A} \to \mathcal{B}$ is a weak equivalence of $\mathcal{S}$-categories with fibrant morphism spaces, then $i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A} \to i\pi_0\mathcal{B} \times_{\pi_0\mathcal{B}} \mathcal{B}$ is a weak equivalence. Notice that $ob\mathcal{A} \cong ob(i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A})$ and that there is a surjection from the set of isomorphisms of $\mathcal{A}$ to the set of isomorphisms of $i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A}$ (and likewise for $\mathcal{B}$). Hence we only have to show that the map induces a weak equivalence on morphism spaces, which is clear since pullbacks along fibrations are equivalent to homotopy pullbacks. ∎

{lemma:II

Lemma 3.3.2 *Let $\mathcal{E}$ be an $\mathcal{A}b$-category with subcategory $i\mathcal{E}$ of isomorphisms, and let $\tilde{\mathcal{E}}$ be the associated $\Gamma\mathcal{S}_*$-category (see 1.6.2.2) . Then natural map $i\mathcal{E} \to \omega\tilde{\mathcal{E}}$ is a stable equivalence.*

Proof: Since $\tilde{\mathcal{E}}$ has stably fibrant morphism objects $RT_0\tilde{\mathcal{E}} \xleftarrow{\sim} \tilde{\mathcal{E}}$ and by construction $R\tilde{\mathcal{E}} = \mathcal{E}$ (considered as an $\mathcal{S}$-category). This means also that $\pi_0\tilde{\mathcal{E}} \cong \mathcal{E}$, and the result follows. ∎

{II:Gamma

Note 3.3.3 So for $\mathcal{A}b$-categories our uniform choice of weak equivalences essentially just picks out the isomorphisms, which is fine since that is what we usually want. For modules over $\mathbf{S}$-algebras they also give a choice which is suitable for K-theory (more about later).

However, occasionally this construction will not pick out the weak equivalences you had in mind. As an example, consider the category Γ^o itself with its monoidal structure coming form the sum. It turns out that the category of isomorphisms $i\Gamma^o = \coprod_{n\geq 0}\Sigma_n$ is an extremely interesting category: its algebraic K-theory is equivalent to the sphere spectrum by the Barratt-Priddy-Quillen theorem (see e.g., [109, proposition 3.5]).

However, since Γ^o is a category with sum, by 3.1.4 it comes with a natural enrichment $(\Gamma^o)^\vee$. We get that $(\Gamma^o)^\vee(m_+, n_+)(k_+) \cong \Gamma^o(m_+, k_+ \wedge n_+)$. But in the language of 6, this is nothing but the n by m matrices over the sphere spectrum. Hence $(\Gamma^o)^\vee$ is isomorphic to the $\Gamma\mathcal{S}_*$-category whose objects are the natural numbers, and where the Γ-space of morphisms from n to m is $Mat_{n,m}\mathbf{S} = \prod_m \vee_n \mathbf{S}$. The associated uniform choice of weak equivalences are exactly the "homotopy invertible matrices" $\widehat{GL}_n(\mathbf{S})$ of III.2.3.1, and the associated algebraic K-theory is the algebraic K-theory of $\mathbf{S}$ - also known as Waldhausen's algebraic K-theory of a point $A(*)$, see III.2.3.

Chapter III

Reductions

{III}

In this chapter we will perform two important reductions and clean up some of the mess due to our use of varying defintions along the way.

The first reduction takes place in section 1 and tells that our handling of simplicial rings in chapter I is not in conflict with the usual conventions of algebraic K-theory, and in particular the one we get from chapter II. This is of importance even if one is only interested in ordinary rings: there are certain points (in chapter V) where even the statements for ordinary rings relies on functoriality in the category of simplicial rings.

Together with section 2 which tells us that all the various definitions of K-theory agrees, the ones only interested in ordinary rings are then free to pass on to chapter IV.

The second reduction, which you will find in section 3 is the fact that for most practical purposes, theorems that are true for simplicial rings are true in general for **S**-algebras. One may think of this as a sort of denseness property, coupled with the fact that the requirement that a functor is "continuous" is rather weak.

1 Degreewise K-theory.

{sec:III1}

Algebraic K-theory is on one hand a group-completion device, which is apparent from the definition of K_0. When looking at K_1 we can also view it is also an "abelianization" device. You kill off the commutator of the general linear group to get K_1. To get K_2 you "kill off" yet another piece where some homology group vanish. The procedure of killing off stuff for which homology is blind ends in group-theory at this point, but if you are willing to go into spaces, you may continue, and that is just what Quillen's plus-construction is all about.

When studying the stable K-theory, we had to introduce simplicial rings into the picture, and it turned out that we could be really naïve about it: we just applied our constructions in every dimension. That this works is quite surprising. When one wants to study K-theory of simplicial rings, the degreewise application of the K-functor only rarely gives anything interesting. One way to get an interesting K-theory would be to take the S-construction of some suitable category of modules, but instead of isomorphisms use weak equivalences. Another, and simpler way is to use Quillen's plus construction on a nice space similar to the classifying space of the general linear group. This is what we will do in this section, but it will not be proven until the next section that the two approaches are equivalent (by means of a yet another approach to K-theory due to Segal, see II.3). The plus construction has the advantage that the comparison between the "correct" and degreewise definitions is particularly simple.

1.1 K-theory of simplicial rings

Def:gplike} A simplicial monoid M is called *group-like* if $\pi_0 M$ is a group. This has the nice consequences that we may form a good classifying space. That is, if BM is (the diagonal of) the space you get by taking the nerve degreewise, then $\Omega BM \simeq M$ (see corollary A.1.5.0.12

If A is a simplicial (associative and unital) ring, Waldhausen [128] defined $\widehat{GL}_n(A)$ as the pullback of the diagram

$$\begin{array}{ccc} \widehat{GL}_n(A) & \longrightarrow & M_n(A) \\ \downarrow & & \downarrow \\ GL_n(\pi_0 A) & \longrightarrow & M_n(\pi_0 A) \end{array}$$

Similar to the discrete case, $\widehat{GL}_n(A)$ sits inside $\widehat{GL}_{n+1}(A)$ via $m \mapsto m \oplus 1$, and we let $\widehat{GL}(A)$ be the union of the $\widehat{GL}_n(A)$. As $\pi_0 \widehat{GL}_n(A) = GL_n(\pi_0(A))$ we get that $\widehat{GL}_n(A)$, and hence also $\widehat{GL}(A)$, is group-like.

Recall Quillen's plus construction (see I.1.6.1, or more thoroughly in appendix A.1.6). In analogywith the definition of the algebraic K-theory space I.1.6.6 of a ring Waldhausen suggested the following definition.

spacesring}

Definition 1.1.1 If A is a simplicial ring, then the algebraic K-theory space of A is

$$K(A) = B\widehat{GL}(A)^+$$

We note that

$$\pi_1 K(A) = \pi_1 B\widehat{GL}(A)/P(\pi_1 B\widehat{GL}(A)) = GL(\pi_0 A)/P(GL(\pi_0 A)) = K_1(\pi_0 A)$$

($P()$ denotes the maximal perfect subgroup, I.1.2.1). This pattern does not continue, the fiber of $K(A) \to K(\pi_0 A)$ has in general highly nontrivial homotopy groups. Waldhausen proves in [128, proposition 1.2] that if k is the first **positive** number for which $\pi_k A$ is nonzero, the first nonvanishing group of the fiber sits in dimension $k+1$ and is $HH_0(\pi_0 A, \pi_k A)$. We shall not prove this now, but settle for the weaker

{lem:1.2.

Lemma 1.1.2 *If $B \to A$ is a $k > 0$-connected map of simplicial rings, then $K(B) \to K(A)$ is $(k+1)$-connected.*

Proof: Obviously $M_n A \to M_n B$ is k-connected. As $k > 0$, we have $\pi_0 B \cong \pi_0 A$, and so $\widehat{GL}_n(B) \to \widehat{GL}_n(A)$ is also k-connected. Hence $B\widehat{GL}(B) \to B\widehat{GL}(A)$ is $k+1$ connected, and we are done as the plus construction preserves connectivity of maps (A.1.6.3.2). ∎

1.2 Degreewise K-theory

Waldhausen's construction is very different from what we get if we apply Quillen's definition to A degreewise, i.e.,

$$K^{deg}(A) = diag^*\{[q] \mapsto K(A_q)\}$$

This is also a useful definition. For instance, we know by [35] that if A is a regular and right Noetherian ring, then $K(A)$ agrees with the Karoubi-Villamajor K-theory of A, which may be defined to be the degreewise K-theory of a simplicial ring $\Delta A = \{[q] \mapsto A[t_0, \dots, t_q]/\sum t_i = 1\}$ with

$$d_i t_j = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } i = j \\ t_{j-1} & \text{if } j > i \end{cases}$$

On the other hand, the homotopy groups of ΔA vanish, and so, by 1.1.2, $K(\Delta A)$ is contractible, and so in this case, Waldhausen's functor give very little information.

The inclusion $GL(A) \subset \widehat{GL}(A)$ induces a map

$$BGL(A)^+ \to B\widehat{GL}(A)^+ = K(A)$$

As we will see in 1.3, the first space is equivalent to $K^{\mathrm{deg}}(A)$, and it is of interest to know when the map preserves information.

{ex:degis

Example 1.2.1 The following example is rather degenerate, but still of great importance. For instance, it was the example we considered when talking about stable algebraic K-theory in section I.3.5.

Let A be a discrete ring, and let P be a reduced A bimodule (in the sense that it is a simplicial bimodule, and $P_0 = 0$). Then we may form the square zero extension $A \ltimes P$ (that is $A \ltimes P = A \oplus P$ as a simplicial bimodule, and the multiplication is given by $(a_1, p_1) \cdot (a_2, p_2) = (a_1a_2, a_1p_2 + p_1a_2)$). Then one sees that $GL(A \ltimes P)$ is actually equal to $\widehat{GL}(A \ltimes P)$: as P is reduced and A discrete $GL(\pi_0(A \ltimes P)) = GL(A)$ and as P is square zero $ker\{GL(A \ltimes P) \to GL(A)\} = (1 + M(P))^\times \cong M(P)$. Hence $GL(A) = \widehat{GL}(A)$.

If you count the number of occurrences of the comparison of degreewise and ordinary K-theory in what is to come, it is this trivial example that will pop up most often. However, we have profound need of the more general cases too. We are content with only an equivalence, and even more so, only an equivalence in relative K-theory. In order to extend this example to cases where A might not be discrete and P not reduced, we have to do some preliminary work.

1.3 The plus construction on simplicial spaces

sec:III13}

The "plus" construction on the diagonal of a simplicial space (bisimplicial set) may be performed degreewise in the following sense. Remember, I.1.2.1, that a quasi-perfect group is a group G in which the maximal perfect subgroup is the commutator: $PG = [G, G]$.

mma:1.4.1}

Lemma 1.3.1 *Let $\{[q] \mapsto X_q\}$ be a simplicial space such that X_q is connected for every $q \geq 0$, and let $X^1 = diag^*\{[q] \mapsto X_q^+\}$. Consider the diagram*

$$\begin{array}{ccc} diag^*X & \longrightarrow & X^1 \\ \downarrow & & \downarrow \\ (diag^*X)^+ & \longrightarrow & (X^1)^+ \end{array}$$

The lower horizontal map is always an equivalence, and the right vertical map is an equivalence if and only if $\pi_1 X^1$ has no nontrivial perfect subgroup. This is true if e.g. $\pi_1(X_0^+)$ is abelian which again follows if $\pi_1(X_0)$ is quasi-perfect.

Proof: Let $A(X_q) = fiber\{X_q \to X_q^+\}$, and consider the sequence

$$\{[q] \mapsto A(X_q)\} \longrightarrow \{[q] \mapsto X_q\} = X \xrightarrow{\{[q] \mapsto q_{X_q}\}} \{[q] \mapsto X_q^+\}$$

As X_q and X_q^+ are connected, theorem A.1.5.0.4 gives that

$$diag^*\{[q] \mapsto A(X_q)\} \longrightarrow diag^*X \xrightarrow{Q_X = diag^*\{[q] \mapsto q_{X_q}\}} X^1$$

is a fiber sequence. But as each $A(X_q)$ is acyclic, the spectral sequence A.1.5.0.6 calculating the homology of a bisimplicial set gives that $\tilde{H}_*(diag^*\{[q] \mapsto A(X_q)\}) = 0$, and so

$Q_X\colon diag^*X \to X^1$ is acyclic. The lower horizontal map is thus the plus of an acyclic map, and hence acyclic itself. But $P\pi_1((diag^*X)^+) = *$, so this map must be an equivalence.

The right vertical map is the plus construction applied to X^1, and so is an equivalence if and only if it induces an equivalence on π_1, i.e., if $P\pi_1(X^1) = *$. If $\pi_1(X_0)$ is quasi-perfect, then $\pi_1(X_0^+) = \pi_1(X_0)/P\pi_1(X_0) = H_1(X_0)$ is abelian, and so the quotient $\pi_1 X^1$ is also abelian, and hence has no perfect (nontrivial) subgroups. ∎

Remark 1.3.2 *Note that some condition is needed to ensure that $\pi_1 X^1$ is without nontrivial perfect subgroups, for let $X_q = BF_q$ where $F \xrightarrow{\sim} P$ is a free resolution of a perfect group P. Then $X^1 \simeq BP \not\simeq X^2 \simeq BP^+$.*

Since the Whitehead lemma I.1.2.2 states that $K_1(A_0)$ is abelian, we get that

{cor:1.4.3

Corollary 1.3.3 *Let A be a simplicial ring. There is a natural chain of weak equivalences*

$$K^{deg}(A) \xrightarrow{\sim} K^{deg}(A)^+ \xleftarrow{\sim} BGL(A)^+. \quad ☺$$

1.4 Nilpotent fibrations and the plus construction

Let π and G be groups, and let π act on G. The action is *nilpotent* if there exists a finite filtration {Def:nilp

$$* = G_{n+1} \subseteq G_n \subseteq \cdots \subseteq G_2 \subseteq G_1 = G$$

respected by the action, such that each $G_{i+1} \subset G_i$ is a normal subgroup and such that the quotients G_i/G_{i+1} are abelian with induced trivial action.

A group G is said to be *nilpotent* if the self action via inner automorphisms is nilpotent. {Def:nilp

{Def:nilp

Definition 1.4.1 If $f\colon E \to B$ is a fibration of connected spaces with connected fiber F, then $\pi_1(E)$ acts on each $\pi_i(F)$ (see A.1.4.1), and we say that f is *nilpotent* if these actions are nilpotent.

Generally, we will say that a map of connected spaces $X \to Y$ is nilpotent if the associated fibration is.

{lem:1.5.

Lemma 1.4.2 *$F \to E \to B$ is any fiber sequence of connected spaces where $\pi_1 E$ acts trivially on $\pi_* F$, then the fibration is nilpotent.*

Proof: Since $\pi_q F$ is abelian for $q > 1$, a trivial action is by definition nilpotent, and the only thing we have to show is that the action of $\pi_1 E$ on $\pi_1 F$ is nilpotent. Let $A' = ker\{\pi_1 F \to \pi_1 E\}$ and $A'' = ker\{\pi_1 E \to \pi_1 B\}$. Since $\pi_1 E$ acts trivially on A' and A'', and both are abelian (the former as it is the cokernel of $\pi_2 E \to \pi_2 B$, and the latter as it is in the center of $\pi_1 E$), $\pi_1 E$ acts nilpotently on $\pi_1 F$. ∎

{lemma:1.

Lemma 1.4.3 *Let $f\colon X \to Y$ be a map of connected spaces. If either*

1. *f fits in a fiber sequence $X \xrightarrow{f} Y \longrightarrow Z$ where Z is connected and $P\pi_1(Z) = *$, or*

2. f is a nilpotent,

then

$$\begin{array}{ccc} X & \longrightarrow & X^+ \\ {\scriptstyle f}\downarrow & & {\scriptstyle f^+}\downarrow \\ Y & \longrightarrow & Y^+ \end{array}$$

is (homotopy) cartesian.

Proof: Part 1. Since $P\pi_1 Z$ is trivial, $Z \to Z^+$ is an equivalence. We may assume that both maps in $Y \to Y^+ \to Z^+$ are fibrations and that X is the fiber of the composite. Consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & P = Y^+ \prod_{Z^+} * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y^+ & \longrightarrow & Z^+ \end{array}$$

As both the rightmost and the outer squares are cartesian, the leftmost must be cartesian. This means that $X \to P$ is acyclic (it has the same fiber as $Y \to Y^+$). We check that $\pi_1 X \to \pi_1 P$ is surjective (the left square), and that $\pi_1 P$ is without perfect subgroups ($P\pi_1 P$ is in $ker\{\pi_1 P \to \pi_1 Y^+\}$ which is isomorphic to $coker\{\pi_2 Y^+ \to \pi_2 Z^+\}$ which is abelian). Hence the leftmost square is the square of the lemma.

Part 2. That f is nilpotent is equivalent, up to homotopy, to the statement that f factors as a tower of fibrations

$$Y = Y_0 \xleftarrow{f_1} Y_1 \xleftarrow{f_2} \dots \xleftarrow{f_k} Y_k = X$$

where each f_i fits in a fiber sequence

$$Y_i \xrightarrow{f_i} Y_{i-1} \longrightarrow K(G_i, n_i)$$

with $n_i > 1$ (see e.g. [14, page 61]). But statement 1 tells us that this implies that

$$\begin{array}{ccc} Y_i & \longrightarrow & Y_i^+ \\ \downarrow & & \downarrow \\ Y_{i-1} & \longrightarrow & Y_{i-1}^+ \end{array}$$

is cartesian, and by induction on k, the statement follows. ∎

1.5 Degreewise vs. ordinary K-theory of simplicial rings

Recall the definition of the subgroup of elementary matrices $E \subseteq GL$. For this section, we reserve the symbol $K_1(A)$ for the quotient $\{[q] \mapsto K_1(A_q)\} = GL(A)/E(A)$, which must not be confused with $\pi_1 K(A) \cong K_1(\pi_0 A)$. Let $\widehat{E}(A) \subset \widehat{GL}(A)$ consist of the components belonging to $E(\pi_0 A) \subseteq GL(\pi_0 A)$.

tdegv$W&ld}

Theorem 1.5.1 *Let A be an associative (simplicial) ring. Then*

$$\begin{array}{ccc} BGL(A) & \longrightarrow & BGL(A)^+ \\ \downarrow & & \downarrow \\ B\widehat{GL}(A) & \longrightarrow & B\widehat{GL}(A)^+ \end{array}$$

is (homotopy) cartesian.

Proof: Note that both horizontal maps in the left square of

$$\begin{array}{ccccc} BE(A) & \longrightarrow & BGL(A) & \longrightarrow & BK_1(A) \\ \downarrow & & \downarrow & & \downarrow \\ B\widehat{E}(A) & \longrightarrow & B\widehat{GL}(A) & \longrightarrow & BK_1(\pi_0 A) \end{array}$$

satisfy the conditions in lemma 1.4.3.1, since both rows are fiber sequences with base space simplicial abelian groups.

So we are left with proving that

$$\begin{array}{ccc} BE(A) & \longrightarrow & B\widehat{E}(A) \\ \downarrow & & \downarrow \\ BE(A)^+ & \longrightarrow & B\widehat{E}(A)^+ \end{array}$$

is cartesian, but by lemma 1.4.3.2 this follows from the lemma below. ∎

{lem:1.6.1

Lemma 1.5.2 *(c.f. [31] or [116]) The map $BE(A) \to B\widehat{E}(A)$ is nilpotent.*

Proof: Let F_k (resp. F) be the homotopy fiber of $BE_k(A) \to B\widehat{E}_k(A)$ (resp. $E(A) \to \widehat{E}(A)$). Instead of showing that the action of $\pi_0 E(A) \cong \pi_1 BE(A)$ on $\pi_*(F)$ is nilpotent, we show that it is trivial. In view of 1.4.2 this is sufficient, and it is in fact an equivalent statement since $\pi_0 E(A)$ is perfect (being a quotient of $E(A_0)$) and any nilpotent action of a perfect group is trivial.

Let $j_k \colon E_k(A) \to \widehat{E}_k(A)$ for $1 \leq k \leq \infty$ be the inclusions (with $j = j_\infty$). Consider the simplicial categories $j_k/1$ with objects $\widehat{E}_k(A)$ and where a morphism in degree q from m to n is a $g \in E_k(A_q)$ such that $m = n \cdot g$. The nerve $N(j_k/1)$ is isomorphic to the bar construction $B(\widehat{E}_k(A), E_k(A), *) = \{[q] \mapsto \widehat{E}_k(A) \times E_k(A)^{\times q}\}$. The forgetful functor $j_k/1 \to E_k(A)$ induces an equivalence $N(j_k/1) \xrightarrow{\sim} F_k$ (see e.g. [130, p, 166]) compatible with stablization $E_k(A) \xrightarrow{t} E_{k+1}(A)$. By A.1.4.2.1, the action on the fiber

$$N(j_k/1) \times E_k(A) \xrightarrow{\sim} N(j_k/1) \times \Omega BE_k(A) \to N(j_k/1)$$

is induced by the simplicial functor $j_k/1 \times E_k(A) \xrightarrow{(m,g) \mapsto i_g(m)} j_k/1$ (where $E_k(A)$ now is considered as a simplicial discrete category with one object for every element in $E_k(A)$ and only identity morphisms) sending (m, g) to $i_g(m) = gmg^{-1}$.

Dually, there is an under category $1/j_k$ (with objects $\widehat{E}_k(A)$ and where a morphism in degree q from m to n is a $g \in E_k(A_q)$ such that $n = g \cdot m$), and an equivalence $N(1/j_k) \to F_k$ over $E_k(A)$. Also here the action on the fiber is induced by conjugation $1/j_k \times E_k(A) \xrightarrow{(m,g)\mapsto i_g(m)} 1/j_k$.

We will show that the induced map

$$\pi_0 E(A) \to \pi_0 Map_*(F,F) \to End(\pi_*(F))$$

is trivial. Note that

$$Map_*(F,F) \cong \lim_{\leftarrow} Map_*(F_k, F)$$

and

$$End(\pi_*(F)) \cong \lim_{\leftarrow} Hom(\pi_*(F_k), \pi(F))$$

where the limits are over $F_1 \to \dots \to F_k \to F_{k+1} \to \dots$ induced by stabilization. Hence it is enough to show that that for each k

$$E(A_0) \twoheadrightarrow \pi_0 E(A) \to \lim_{\leftarrow} \pi_0 Map_*(F_k, F) \to \pi_0 Map_*(F_k, F) \to Hom(\pi_*(F_k), \pi_*(F))$$

is trivial. Note that this map factors in two interesting ways:

$$\begin{array}{ccc} E(A_0) & \longrightarrow & \pi_0 Map_*(N(j_k/1), N(j/1)) \\ \downarrow & & \downarrow \\ \pi_0 Map_*(N(1/j_k), N(1/j)) & \longrightarrow & Hom(\pi_*(F_k), \pi_*(F)) \end{array}$$

Now we fix a k. The zero simplices $E(A_0)$ are considered as simplices in $E(A)$ of arbitrary dimension by the unique inclusion which we supress from the notation. As natural transformations give rise to homotopies, we are done if we display a natural simplicial isomorphism between t and $i_x t$ in either of the categories of pointed functors $[j_k/1, j/1]_*$ or $[1/j_k, 1/j]_*$ ($t(m) = m \oplus I$ and $i_x m = xmx^{-1}$) for each given element x in a generating set of $E(A_0)$. As a generating set we may choose the set containing the elements $e^a_{k+1,i}$ and $e^a_{i,k+1}$ for $a \in A_0$ and $i \neq k+1$. To see that this set really generates $E(A_0)$, consider the Steinberg relations of I.1.5: if $i \neq j$ and neither i nor j equals $k+1$, then $e^a_{ij} = [e^a_{i,k+1}, e^1_{k+1,j}]$.

If $m = (m_{ij}) \in M_k(A)$ is any matrix we have that $e^a_{k+1,j} \cdot t(m) \cdot e^{-a}_{k+1,j} = t(m) \cdot \tau^a_{k+1,j}(m)$ where

$$\tau^a_{k+1,j}(m) = e^{-a}_{k+1,j} \cdot \prod_{i \leq k} e^{am_{ji}}_{k+1,i}$$

It is easy to check that $\tau^a_{k+1,j}(m)$ is simplicial ($\psi^* \tau^a_{ij}(m) = \tau^a_{ij}(\psi^* m)$ for $\psi \in \Delta$) and natural in $m \in j_k/1$ So $m \mapsto \tau^a_{k+1,j}(m)$ is the desired natural isomorphism between $i_{e^a_{k+1,j}} t$ and t in $[j_k/1, j/1]_*$. Likewise we have that $e^a_{i,k+1} \cdot t(m) \cdot e^{-a}_{i,k+1} = \tau^a_{i,k+1}(m) \cdot t(m)$ where

$$\tau^a_{i,k+1}(m) = e^a_{i,k+1} \cdot \prod_{j \leq k} e^{-m_{ji}a}_{j,k+1}$$

and $\tau^a_{i,k+1}$ is a natural isomorphism in $[1/j_k, 1/j]_*$. ∎

The outcome is that we are free to choose our model for the fiber of the plus construction applied to $B\widehat{GL}(A)$ among the known models for the fiber of the plus construction applied to $BGL(A)$:

{cor:1.6.3

Corollary 1.5.3 *If X is any functor from discrete rings to spaces with a natural transformation $X(-) \to BGL(-)$ such that*

$$X(A) \to BGL(A) \to BGL(A)^+$$

is a fiber sequence for any ring A, then X extended degreewise to a functor of simplicial rings is such that

$$X(A) \to B\widehat{GL}(A) \to B\widehat{GL}(A)^+$$

is a fiber sequence for any simplicial ring A.

Proof: By the theorem it is enough to show that $[q] \mapsto X(A_q)$ is equivalent to the fiber of $BGL(A) \to BGL(A)^+$, but this will follow if $\{[q] \mapsto BGL(A_q)\}^+ \to \{[q] \mapsto BGL(A_q)^+\}$ is an equivalence. But by lemma 1.3.1 this is true since $GL(A_0)$ is quasi-perfect, which is part of the Whitehead lemma I.1.2.2. ∎

Example 1.5.4 The resolving complex and Stein relativization. We have already seen in I.1.4.1 that the most naïve kind of excision fails for algebraic K-theory. Related to this is the classical method of describing relative K-theory. In Bass [4] and Milnor's [86] books on K-theory, the Stein relativization is used to describe relative K-theory. As is admitted in Milnor's book, this is not a satisfactory description, and we will give the reason why it works in low dimensions, but fails higher up. See [118] to get further examples of the failure.

Let $f\colon A \to B$ be a surjection of associative rings with unit, and define $K_i^{Stein}(f) = coker\{K_i(A) \to K_i(A \prod_B A)\}$ given by the diagonal splitting $A \to A \prod_B A$. The question is: when do we have exact sequences

$$\to K_{i+1}(A) \to K_{i+1}(B) \to K_i^{Stein}(f) \to K_i(A) \to K_i(B) \to$$

or more precisely, how far is

$$\begin{array}{ccc} K(A \prod_B A) & \xrightarrow{pr_1} & K(A) \\ {\scriptstyle pr_2}\downarrow & & {\scriptstyle f}\downarrow \\ K(A) & \xrightarrow{f} & K(B) \end{array}$$

from being cartesian?

The failure turns up for $i = 2$, but this oughtn't be considered as bad as was fashionable at the time: The Stein relativization can be viewed as a first approximation to the fiber as follows. Let S be the "resolving complex", i.e., the simplicial ring given in dimension q as the $q+1$ fold product of A over f with the various projections and diagonals as face and degeneracies

$$\dots A\prod_B A\prod_B A \rightrightarrows A\prod_B A \rightrightarrows A$$

This gives a factorization $A \to S \to B$ where the former map is inclusion of the zero skeleton, and the latter is a weak equivalence. Now, as one may check directly, GL respects products, and

$$GL(S_q) = GL(A)\prod_{GL(B)} \cdots \prod_{GL(B)} GL(A)$$

($q+1$ $GL(A)$ factors). Just as for the simplicial ring S, this simplicial group is concentrated in degree zero, but as GL does not respect surjections we see that $\pi_0(GL(S)) \cong im\{GL(A) \to GL(B)\}$. But this is fine, for as $E(-)$ respects surjections we get that $GL(B)/im\{GL(A) \to GL(B)\} \cong \bar{K}_1(B) = K_1(B)/im\{K_1(A) \to K_1(B)\}$, and we get a fiber sequence

$$BGL(S) \to B\widehat{GL}(S) \to B\bar{K}_1(B)$$

where the middle space is equivalent to $BGL(B)$. Applying theorem 1.6.1 (overkill as $\bar{K}_1(B)$ is abelian) to $BGL(S) \to B\widehat{GL}(S)$ we get that there is a fiber sequence

$$K^{deg}(S) \to K(B) \to B\bar{K}_1(B)$$

which means that $\phi(f) = fiber\{K(A) \to K^{deg}(S)\}$ is the connected cover of the fiber of $K(A) \to K(B)$.

We may regard $\phi(f)$ as a simplicial space $[q] \mapsto \phi_q(f) = fiber\{K(A) \to K(S_q)\}$. Then $\phi_0(f) = 0$ and $\pi_i(\phi_1(f)) = K^{Stein}_{i+1}(f)$. An analysis shows that $d_0 - d_1 + d_2\colon \pi_0(\phi_2(f)) \to \pi_0(\phi_1(f))$ is zero, whereas $d_0 - d_1 + d_2 - d_3\colon \pi_0(\phi_3(f)) \to \pi_0(\phi_2(f))$ is surjective, so the E_2 term of the spectral sequence associated to the simplicial space looks like

$$\begin{array}{ccccc} 0 & K_3^{Stein}(f)/? & \dots & & \\ 0 & K_2^{Stein}(f)/? & ? & \dots & \\ 0 & K_1^{Stein}(f) & 0 & ? & \dots \end{array}$$

This gives that $K_1^{Stein}(f)$ is correct, wheras $K_2^{Stein}(f)$ surjects onto π_2 of relative K-theory.

1.6 K-theory of simplicial radical extensions may be defined degreewise

If $f\colon B \to A$ is a map of simplicial (associative and unital) rings, we will let $K(f)$ denote the fiber of $K(B) \xrightarrow{f} K(A)$. If f is surjective and $I_q = ker\{f_q\colon B_q \to A_q\}$ is inside the *Jacobson radical* $Rad(B_q) \subseteq B_q$ for every $q \geq 0$ we say that f is a *radical extension*.

prop:1.7.1}

Proposition 1.6.1 *Let $f\colon B \twoheadrightarrow A$ be a radical extension of unital simplicial rings. Then the relative K-theory $K(f)$ is equivalent to $diag^*([q] \mapsto K(f_q))$.*

Proof: The proof follows closely the one given in [39] for the nilpotent case. Let $I = ker\{f\colon B \to A\}$. Since all spaces are connected we may just as well consider

$$[q] \mapsto fiber\{BGL(B_q)^+ \to BGL(A_q)^+\}.$$

As $\pi_1(BGL(-)^+)$ has values in abelian groups, we see by lemma 1.3.1. that $diag^*\{[q] \mapsto BGL(-)^+\}$ is equivalent to the plus of the diagonal $BGL(A)^+$. Hence to prove the proposition it is enough to prove that

$$\begin{array}{ccc} BGL(B)^+ & \longrightarrow & B\widehat{GL}(B)^+ \\ \downarrow & & \downarrow \\ BGL(A)^+ & \longrightarrow & B\widehat{GL}(A)^+ \end{array}$$

is homotopy cartesian.

Note that $GL_n(B_q) \to GL_n(A_q)$ is a group epimorphism with kernel $(1+M_n(I_q))^\times$, the multiplicative group of all $n\times n$ matrices of the form $1+m$ where m has entries in I. Hence $B(1+M(I)^\times)$ is the fiber of $BGL(B) \to BGL(A)$. Similarly, we see that $(1+M_n(I))^\times$ is the fiber of the map of group-like simplicial monoids $\widehat{GL}_n(B) \to \widehat{GL}_n(A)$. This follows as $J = ker\{\pi_0(B) \to \pi_0(A)\}$ is a radical ideal in $\pi_0(B)$, which implies that

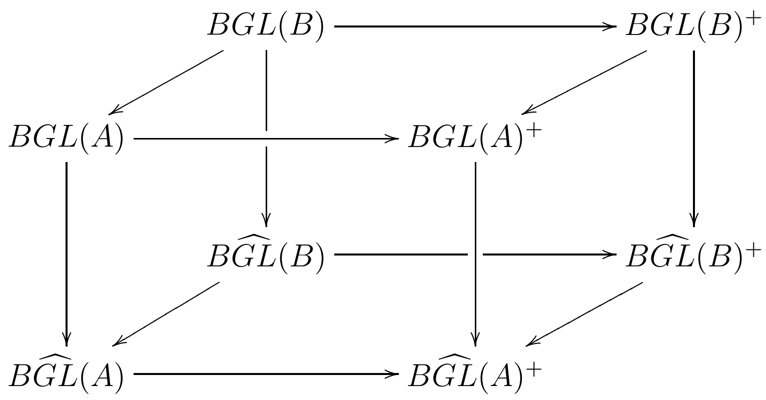
$$\begin{aligned}(1+M_n(J))^\times =& ker\{GL_n(\pi_0(B)) \to GL_n(\pi_0(A))\}\\ =& ker\{M_n(\pi_0(B)) \to M_n(\pi_0(A))\}\end{aligned}$$

and so $fiber\{B\widehat{GL}(B) \to B\widehat{GL}(A)\} = B(1+M(I))^\times$.

Hence

$$\begin{array}{ccc} BGL(B) & \longrightarrow & BGL(A) \\ \downarrow & & \downarrow \\ B\widehat{GL}(B) & \longrightarrow & B\widehat{GL}(A) \end{array}$$

is homotopy cartesian. By theorem 1.5.1 all vertical squares in



except possibly

$$\begin{array}{ccc} BGL(B)^+ & \longrightarrow & B\widehat{GL}(B)^+ \\ \downarrow & & \downarrow \\ BGL(A)^+ & \longrightarrow & B\widehat{GL}(A)^+ \end{array}$$

are homotopy cartesian, and so this square is also homotopy cartesian. ∎

2 Agreement of the various K-theories.

{sec:III2

This section aims at removing the uncertainty due to the many definitions of algebraic K-theory that we have used. In 2.1 we show that the approach of Waldhausen and Segal agree, at least for additive categories. In section 2.2 we show that Segal's machine is an infinite delooping of the plus-construction, and show how this is related to group-completion. In 2.3 we give the definition of the algebraic K-theory space of an **S**-algebra. For "spherical group rings", i.e., **S**-algebras of the form $\mathbf{S}[G]$ for G a simplicial group, we show that the algebraic K-theory space of $\mathbf{S}[G]$ is the same as Waldhausen's algebraic K-theory of the classifying space BG. Lastly, we show that the definition of the algebraic K-theory of an **S**-algebra as defined in chapter II is the infinite delooping of the plus-construction.

2.1 The agreement of Waldhausen and Segal's approach

{WaldSega

We give a quick proof of the fact that the S-construction of I and the $\bar{H}$-construction of chapter II coincide on additive categories. This fact is much more general, and applies to a large class of categories with categories with cofibrations and weak equivalences where the cofibrations are "splittable up to weak equivalences", see Waldhausen's [131, section 1.8].

2.1.1 Segal's construction applied to categories with cofibrations

{subsec:H

Let $\mathfrak{C}$ be a category with cofibration. By forgetting structure, we may consider it as a category with sum, and apply Segal's Γ-space machine II.3 to it, or we may apply Waldhausen's S-construction I.2.2.1.

Note that Segal's Γ-space machine could be interpreted as the functor $\bar{H}\mathfrak{C}$ from the category Γ^o of finite pointed sets to categories with sum whose value at $k_+ = \{0, \dots, k\}$ was the category $\bar{H}\mathfrak{C}$ described as follows. Its objects are functors from the pointed category of subsets and inclusions of $k_+ = \{0, 1, \dots, k\}$, sending 0 to 0 and pushout squares to pushout squares. The morphisms are simpy natural transformations of such diagrams. For instance, $\bar{H}\mathfrak{C}(1_+)$ is isomorphic to $\mathfrak{C}$, whereas $\bar{H}\mathfrak{C}(2_+)$ consists of pushout diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & c_{\{0,1\}} \\ \big\downarrow & & \big\downarrow \\ c_{\{0,2\}} & \longrightarrow & c_{\{0,1,2\}} \end{array} .$$

We see that $\bar{H}\mathfrak{C}(k_+)$ is equivalent as a category to $\mathfrak{C}^{\times k}$ via the map sending a functor $c \in ob\bar{H}\mathfrak{C}(k_+)$ to $c_{\{0,1\}}, \dots, c_{\{0,k\}}$. However, $\mathfrak{C}^{\times k}$ is not necessarily functorial in k, making $\bar{H}\mathfrak{C}$ the preferred model for the bar construction of $\mathfrak{C}$.

Also, this formulation of $\bar{H}\mathfrak{C}$ is clearly isomorphic to the one we gave in II.3, the advantage is that it is easier to compare with Waldhausen's construction.

Any functor from Γ^o is naturally a simplicial object by precomposing with a the circle $S^1\colon \Delta^o \to \Gamma^o$ (after all, the circle is a simplicial **finite pointed** set). We could of course precompose with any other simplicial finite pointed set, and part of the point about Γ-spaces was that if M was a functor from Γ^o to sets, then $\{m \mapsto M(S^m)\}$ is a spectrum.

2.1.2 The relative $\bar{H}$-construction.

If $\mathcal{C} \to \mathcal{D}$ is an exact functor of categories with sum (or more generally, a monoidal functor of symmetric monoidal categories), we define the simplicial Γ-category $C_{\mathcal{C}\to\mathcal{D}}$ by the pullback

$$\begin{array}{ccc} C_{\mathcal{C}\to\mathcal{D}}(X) & \longrightarrow & \bar{H}\mathcal{D}(PS^1\wedge X) \\ \downarrow & & \downarrow \\ \bar{H}\mathcal{C}(S^1\wedge X) & \longrightarrow & \bar{H}\mathcal{D}(S^1\wedge X) \end{array}.$$

The point of this construction is lemma 2.1.5 which displays it as a relative version of the $\bar{H}$-construction, much like the usual construction involving the path space in topological spaces.

Usually categorical pullbacks are of little value, but in this case it turns out that it is equivalent to the *fiber product.*

er product}

Definition 2.1.3 Let $\mathcal{C}_1 \xrightarrow{f_1} \mathcal{C}_0 \xleftarrow{f_2} \mathcal{C}_2$ be a diagram of categories. The *fiber product* $\prod(f_1, f_2)$ is the category whose objects are tuples (c_1, c_2, α) where $c_i \in ob\mathcal{C}_i$ for $i = 1, 2$ and α is an isomorphism in $\mathcal{C}_0$ from f_1c_1 to f_2c_2; and where a morphism from (c_1, c_2, α) to (d_1, d_2, β) is a pair of morphisms $g_i\colon c_i \to d_i$ for $i = 1, 2$ such that

$$\begin{array}{ccc} f_1c_1 & \xrightarrow{\alpha} & f_2c_2 \\ {\scriptstyle f_1g_1}\downarrow & & \downarrow{\scriptstyle f_2g_2} \\ f_1d_1 & \xrightarrow{\beta} & f_2d_2 \end{array}$$

commutes.

Fiber products (as homotopy pullbacks) are good because their invariance: if you have a diagram

$$\begin{array}{ccccc} \mathcal{C}_1 & \xrightarrow{f_1} & \mathcal{C}_0 & \xleftarrow{f_2} & \mathcal{C}_2 \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \mathcal{C}'_1 & \xrightarrow{f'_1} & \mathcal{C}'_0 & \xleftarrow{f'_2} & \mathcal{C}'_2 \end{array}$$

where the vertical maps are equivalences, you get an equivalence $\prod(f_1, f_2) \to \prod(f'_1, f'_2)$. Note also the natural map $F\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2 \to \prod(f_1, f_2)$ sending (c_1, c_2) to $(c_1, c_2, 1_{f_1c_1})$.

This map is occasionally an equivalence, as is exemplified in the following lemma. If $\mathcal{C}$ is a category, then $Iso\mathcal{C}$ is the class of isomorphisms, and if f is a morphism, then sf is its source and tf is its target.

ackisfiber}

Lemma 2.1.4 *Let* $\mathcal{C}_1 \xrightarrow{f_1} \mathcal{C}_0 \xleftarrow{f_2} \mathcal{C}_2$ *be a diagram of categories, and assume that the map of classes*

$$Iso\mathcal{C}_1 \xrightarrow{g\mapsto(f_1g, sg)} Iso\mathcal{C}_0 \times_{ob\mathcal{C}_0} ob\mathcal{C}_1$$

has a section (the pullback is taken along source and f_1). Then the natural map $F\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ and $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2 \to \prod(f_1, f_2)$ is an equivalence.

Proof: Let $\sigma\colon Iso\mathcal{C}_0 \times_{ob\mathcal{C}_0} ob\mathcal{C}_1 \to Iso\mathcal{C}_1$ be a section, and define $G\colon \prod(f_1, f_2) \to \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ by $G(c_1, c_2, \alpha) = (t\sigma(\alpha, c_1), c_2)$ and $G(g_1, g_2) = (\sigma(d_1, \beta)g_1\sigma(c_1, \alpha)^{-1}, g_2)$. Checking the diagrams proves that F and G are inverses up to natural isomorphisms built out of σ.

∎

One should think about the condition as a categorical equivalent of the Kan-condition in simplicial sets. This being one of the very few places you can find an error (even tiny and in the end totally irrelevant) in Waldhausen's papers, it is cherished by his fans since in [131] he seems to claim that the pullback is equivalenct to the fiber products if f_1 has a section (which is false). At this point there is even a small error in [45, page 257], where it seems that they claim that the relevant map in lemma 2.1.4 factors through f_1.

Now, since $i\bar{H}\mathcal{D}(PS^1 \wedge X) \to i\bar{H}\mathcal{D}(S^1 \wedge X) \times_{ob\bar{H}\mathcal{D}(S^1 \wedge X)} ob\bar{H}\mathcal{D}(PS^1 \wedge X)$ has a section (given by pushouts in the relevant diagrams) $C_{\mathcal{C}\to\mathcal{D}}(X)$ is equivalent to the fiber product category, and as such is invariant under equivalences, so the natural map

$$C_{\mathfrak{C}\to\mathcal{D}}(k_+)_q \longrightarrow \mathcal{C}^{\times qk} \times_{\mathcal{D}^{\times qk}} \mathcal{D}^{\times (q+1)k} \cong \mathcal{C}^{\times qk} \times \mathcal{D}^{\times k}$$

is an equivalence. If the consider categories with sum and weak equivalences, we get a structure of sum and weak equivalence on $C_{\mathcal{C}\to\mathcal{D}}$ as well with

$$wC_{\mathcal{C}\to\mathcal{D}} = w\bar{H}\mathcal{C}(S^1 \wedge X) \times_{w\bar{H}\mathcal{D}(S^1 \wedge X)} w\bar{H}\mathcal{D}(PS^1 \wedge X).$$

Notice also that the construction is natural: if you have a commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{D}' \end{array}$$

you get an induced map $C_{\mathcal{C}\to\mathcal{D}} \to C_{\mathcal{C}'\to\mathcal{D}'}$ such that

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & C_{\mathcal{C}\to\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & C_{\mathcal{C}'\to\mathcal{D}'} \end{array}$$

commutes. Furthermore $C_{*\to\mathcal{E}}(1_+)$ is isomorphic to $\mathcal{E}$, so if we have maps $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ whose composite is trivial, we get a map $C_{\mathcal{C}\to\mathcal{D}}(1_+) \to \mathcal{E}$.

{cor:1.4.3

Lemma 2.1.5 *Let $\mathcal{C} \to \mathcal{D}$ be an exact functor of categories with sum and weak equivalences. Then there is a stable fiber sequence*

$$w\bar{H}\mathcal{C} \to w\bar{H}\mathcal{D} \to w\bar{H}(C_{\mathcal{C}\to\mathcal{D}}(1_+))$$

Proof: It is enough to show that

$$w\bar{H}\mathcal{D}(S^1) \to w\bar{H}(C_{\mathcal{C}\to\mathcal{D}}(1_+))(S^1) \to w\bar{H}(\bar{H}(\mathcal{C})(S^1))(S^1)$$

is a fiber sequence, and this follows since in each degree n

$$w\bar{H}\mathcal{D}(S^1) \to w\bar{H}(C_{\mathcal{C}\to\mathcal{D}}(1_+)_n)(S^1) \to w\bar{H}(\bar{H}(\mathcal{C})(S^1)_n)(S^1)$$

is equivalent to the product fiber sequence

$$w\bar{H}\mathcal{D}(S^1) \to w\bar{H}(\mathcal{D}\times\mathcal{C}^{\times n})(S^1) \to w\bar{H}(\mathcal{C}^{\times n})(S^1)$$

and all spaces involved are connected. ∎

We have a canonical map

$$\bar{H}\mathfrak{C}(S^1) \to S\mathfrak{C}$$

which in dimension q is induced by sending the sum diagram $C \in ob\bar{H}\mathfrak{C}(q_+)$ to $c \in obS_q\mathfrak{C}$ with $c_{ij} = C_{\{0,i+1,i+2,\ldots,j-1,j\}}$.

theo:2.1.2}

Theorem 2.1.6 *Let $\mathfrak{C}$ be an additive category. Then the map*

$$i\bar{H}\mathfrak{C}(S^1) \to iS\mathfrak{C}$$

is a weak equivalence.

Proof: Since both $Bi\bar{H}\mathfrak{C}$ and $BiS\mathfrak{C}$ are connected, the vertical maps in

$$\begin{array}{ccc} Bi\bar{H}\mathfrak{C}(S^1) & \longrightarrow & BiS\mathfrak{C} \\ \simeq\downarrow & & \simeq\downarrow \\ \Omega\left(Bi\bar{H}(\bar{H}\mathfrak{C}(S^1))(S^1)\right) & \longrightarrow & \Omega\left(B\bar{H}(iS\mathfrak{C})(S^1)\right) \end{array}$$

are equivalences by A.1.5.0.11, and so it is enough to prove that

$$Bi\bar{H}(\bar{H}\mathfrak{C}(S^1)) \to Bi\bar{H}(S\mathfrak{C})$$

is an equivalence, which again follows if we can show that for every q

$$Bi\bar{H}(\bar{H}\mathfrak{C}(q_+)) \to Bi\bar{H}(S_q\mathfrak{C})$$

is an equivalence.

Essentially this is the old triangular matrices vs. diagonal matrices question, and can presumably be proven directly by showing that $iS_q\mathcal{C} \to i\mathfrak{C}^{\times q}$ induces an isomorphism in homology after inverting $\pi_0(iS_q\mathcal{C}) \cong \pi_0(i\mathfrak{C}^{\times q})$.

Assume we have proven that the projection $i\bar{H}(S_k\mathfrak{C}) \to i\bar{H}(\mathfrak{C}^{\times k})$ is an equivalence for $k < q$ (this is trivial for $k = 0$ or $k = 1$), and we must show that it is also an equivalence for $k = q$. Consider the inclusion by zero'th degeneracies $\mathfrak{C} \to S_q\mathfrak{C}$ (sending c

to $0 \rightarrowtail 0 \rightarrowtail \ldots \rightarrowtail 0 \rightarrowtail c$), and the last face map $S_q\mathfrak{C} \to S_{q-1}\mathfrak{C}$. We want to show that we have a map of fiber sequences

$$\begin{array}{ccccc} i\bar{H}(\mathfrak{C}) & \longrightarrow & i\bar{H}(S_q\mathfrak{C}) & \longrightarrow & i\bar{H}(S_{q-1}\mathfrak{C}) \\ \| & & \downarrow & & \downarrow \\ i\bar{H}(\mathfrak{C}) & \longrightarrow & i\bar{H}(\mathfrak{C}^{\times q}) & \longrightarrow & i\bar{H}(\mathfrak{C}^{\times q-1}) \end{array}$$

We do have maps of fiber sequences

$$\begin{array}{ccccc} i\bar{H}(\mathfrak{C}) & \longrightarrow & i\bar{H}(S_q\mathfrak{C}) & \longrightarrow & i\bar{H}(C_{\mathfrak{C}\to S_{q-1}\mathfrak{C}}(1_+)) \\ \| & & \downarrow & & \downarrow \\ i\bar{H}(\mathfrak{C}) & \longrightarrow & i\bar{H}(\mathfrak{C}^{\times q}) & \longrightarrow & i\bar{H}(C_{\mathfrak{C}\to \mathfrak{C}^{\times q}}(1_+)) \end{array}$$

and the only trouble lies in identifying the base spaces of the fibrations. We have a commuting square

$$\begin{array}{ccc} i\bar{H}(C_{\mathfrak{C}\to S_q\mathfrak{C}}(1_+)) & \longrightarrow & i\bar{H}(C_{\mathfrak{C}\to \mathfrak{C}^{\times q}}(1_+)) \\ \downarrow & & \simeq\downarrow \\ i\bar{H}(S_{q-1}\mathfrak{C}) & \xrightarrow{\sim} & i\bar{H}(\mathfrak{C}^{\times q-1}) \end{array}$$

and the right vertical map is obviously an equivalence. We have to show that the left vertical map is an equivalence, and for this purpose it is enough to show that

$$iC_{\mathfrak{C}\to S_q\mathfrak{C}}(1_+) \xrightarrow{p} iS_{q-1}\mathfrak{C}$$

is an equivalence. For every $c \in obS_{q-1}\mathfrak{C}$ the over category p/c is a simplicial category. If we can show that p/c is contractible for all c we are done by theorem A (prime, ref). In dimension n p/c consists of certain sum diagrams of dimension $n+1$ of objects in $S_q\mathfrak{C}$ together with some extra data. Call the vertices of cardinality one $c_0, \ldots, c_n$. Part of the data is an isomorphism $d_q c_0 \xrightarrow{\cong} c$, and $c_1, \ldots, c_n$ only have nonzero elements in the last column (i.e., $(c_k)(i \leq j) = 0$ if $0 < k$ and $j < q$). Hence $(p/c)_n$ is equivalent to the category $iC_{\mathfrak{C}\to\mathfrak{C}_x}(n_+)$ where $x = c_{0,q-1}$ and $\mathfrak{C}_x$ is the category of split inclusions $x \rightarrowtail y \in \mathfrak{C}$ (which is a category with sum by taking pushout over the structure maps from x). The equivalence is induced by sending $c_0, \ldots c_n$ to $x \rightarrowtail (c_0)_{0,q}, (c_1)_{0,q}, \ldots, (c_n)_{0,q}$ (considered as objects in $\mathfrak{C}_x \times \mathfrak{C}^{\times n}$). The equivalence is natural in n, and so induces an equivalence $p/c \to iC_{\mathfrak{C}\to\mathfrak{C}_x}(S^1)$, and we show that the latter is contractible.

This is the group completion part: it does not matter what x we put in $\mathfrak{C}_x$. First we show that $\pi_0(iC_{\mathfrak{C}\to\mathfrak{C}_x}(S^1)) = 0$ (which implies that $iC_{\mathfrak{C}\to\mathfrak{C}_x}(S^1) \simeq \Omega i\bar{H}C_{\mathfrak{C}\to\mathfrak{C}_x}(S^1)$) and then that $i\bar{H}\mathfrak{C} \to i\bar{H}\mathfrak{C}_x$ is an equivalence.

The vertices of $iC_{\mathfrak{C}\to\mathfrak{C}_x}(S^1)$ are split inclusions $x \rightarrowtail c$; the 1-simplices in the nerve direction are isomorphisms under x, whereas the 1-simplices in the $\bar{H}$-construction are pushout diagrams

$$\begin{array}{ccc} x & \longrightarrow & c \\ {\scriptstyle in_x}\downarrow & & \downarrow \\ x \vee c'' & \longrightarrow & c \vee c'' \end{array}$$

Hence, in $\pi_0(iC_{\mathfrak{C}\to\mathfrak{C}_x}(S^1))$ the class of $x \rightarrowtail c$ is equal the class of $x \overset{in_x}{\rightarrowtail} x \rightarrowtail c/x$ (since the inclusion was splittable), which is equal to the class of the basepoint $x = x$.

Finally, consider the map $i\bar{H}\mathfrak{C} \to i\bar{H}\mathfrak{C}_x$. It is induced by $\mathfrak{C} \xrightarrow{j} \mathfrak{C}_x$ sending c to $x\overset{i}{n}_x \to\rightarrowtail x \vee c$, and it has a section $\mathfrak{C}_x \xrightarrow{q} \mathfrak{C}$ given by sending $x \rightarrowtail c$ to c/x (there is no danger in choosing quotients). We have to show that jq induces a selfmap on $i\bar{H}\mathfrak{C}_x$ homotopic to the identity. Note that there is a natural isomorphism $c \coprod_x c \to c \vee c/x \cong c \times c/x$ under x given by sending the first summand by the identity to the first factor, and the second summand to the identity on the first factor and the projection on the second factor. Hence, 2 (twice the identity) is naturally isomorphic to $1 + jq$ in $i\bar{H}\mathfrak{C}_x$, and since this is a connected H-space we have homotopy inverses, giving that jq is homotopic to the identity. ∎

2.2 Segal's machine and the plus construction

ISegvsplus}

We give a brief review of Segal's results on group completion, focusing on the examples that are important to our appliciations. There are many excellent accounts related to this issue (see e.g. [1], [36], [57], [84], [41]), but we more or less follow the approach of [109].

Let $\mathcal{C}$ be a symmetric monoidal category with weak equivalences, and consider the simplicial Γ-category $H'\mathcal{C}$ defined by the pullback

$$\begin{array}{ccc} H'\mathcal{C}(X)_q & \longrightarrow & \bar{H}\mathcal{C}(PS^1_q\wedge X) \\ \downarrow & & \downarrow \\ \bar{H}\mathcal{C}(PS^1_q\wedge X) & \longrightarrow & \bar{H}\mathcal{C}(S^1_q\wedge X) \end{array} .$$

By the same considerations as in corollary 2.1.5 (i.e., by reversal of priorities w.r.t. simplicial directions) we get a fiber sequence

$$w\bar{H}\mathcal{C}(S^1) \longrightarrow wH'\mathcal{C}(S^1) \longrightarrow wH(PS^1\wedge S^1),$$

but the last simplicial category is contractible, and so $w\bar{H}\mathcal{C}(S^1) \to wH'\mathcal{C}(S^1)$ is an equivalence.

Furthermore, the Γ-category $wH'\mathcal{C}$ is not only special, but very special: it has a homotopy inverse gotten by flipping the defining square around the diagonal.

IIIplusisH}

Lemma 2.2.1 *Let $\mathcal{C}$ be a symmetric monoidal category with weak equivalences, and let $\mu \subseteq w\mathcal{C}$ be a symmetric monoidal subcategory such that the image of $\pi_0\mu$ in $\pi_0 w\mathcal{C}$ is cofinal. Then the map*

$$w\bar{H}\mathcal{C} \to wH'\mathcal{C}$$

is a stable equivalence and $wH'\mathcal{C}$ is very special. Furthermore, if $wT_{\mathcal{C},\mu}$ is defined as the pullback

$$\begin{array}{ccc} wT_{\mathcal{C},\mu} & \longrightarrow & \bar{H}\mathcal{C}(PS^1) \\ \downarrow & & \downarrow \\ \bar{H}\mu(PS^1) & \longrightarrow & \bar{H}\mathcal{C}(S^1) \end{array} .$$

the natural map $wT_{\mathcal{C},mu} \to wH'\mathcal{C}(S^1)$ is an acyclic map.

Consequently there is a chain of natural equivalences

$$(obNwT_{\mathcal{C},mu})^+ \xrightarrow{\sim} (obNwH'\mathcal{C}(S^1))^+ \xleftarrow{\sim} obNwH'\mathcal{C}(S^1) \xleftarrow{\sim} obN\bar{H}\mathcal{C}(S^1).$$

Proof: Only the part about $wT_{\mathcal{C},\mu} \to wH'\mathcal{C}(S^1)$ being an acyclic map needs explanation. Since $wH'\mathcal{C}(S^1)$ is an H-space, this is equivalent to claiming that the map induces an isomorphism in integral homology.

By coherence theory (ref NBNB), we may assume that $w\mathcal{C}$ is strict monoidal (the symmetric structure is still free to wiggle). Hence we are reduced to the following proposition: given a simplicial monoid M (the simplicial set given by the nerve of $w\mathcal{C}$) which is commutative up to all higher homotopies and a submonoid $\mu \subseteq M$ whose image in $\pi_0 M$ is cofinal, then the map $Y \to X$ given by the pullback squares

$$\begin{array}{ccccc} Y & \longrightarrow & X & \longrightarrow & EM \\ \downarrow & & \downarrow & & \downarrow \\ E\mu & \longrightarrow & EM & \longrightarrow & BM \end{array}$$

induces an isomorphism in homology. Analyzing the structures, we see that $Y \to X$ is nothing but the map of two-sided bar-constructions $B(M\times\mu,\mu,*) \subseteq B(M\times M, M,*)$ (with the diagonal action). Segal gives an argument why this is an isomorphism in homology in [109, page 305-306] by an explicit calculation with arbitrary field coefficients.

The argument is briefly as follows: let k be a field and let $H = H_*(M;k)$ which is a graded ring since M is a monoid, and a Hopf algebra due to the diagonal map. The E^1-term of the spectral sequence for computing $H_*(B(M\times M, M,*)$ is exactly the standard complex for calculating $Tor^H_*(H\otimes_k H, k)$ (in dimension q it is $(H\otimes_k H)\otimes H^{\otimes_k q}\otimes k$). But this complex collapses: $(H\otimes_k H)\otimes_H k \cong H[\pi^{-1}]$ (where $\pi = \pi_0 M$) and $Tor^H_0(H\otimes_k H, k) = 0$. This uses that localization in the commutative case is flat. In consequence, we get that $H_*B(M\times M, M, *; k) \cong H[\pi^{-1}]$. A similar calculation gives the same result for $H_*B(M\times\mu,\mu,*;k)$, and the induced map is an isomorphism. ∎

2.2.2 Application to the category of finitely generated free modules over a discrete ring.

As an example we may consider the category of finitely generated free modules over a discrete ring A. For this purpose we use the model $\mathcal{F}_A$ of I.2.1.4 whose set of objects is the natural numbers and morphisms matrices. Assume for simplicity that A has the invariance of basis property (see I.4). Then $obNi\mathcal{F}_A$ is the simplicial monoid $\coprod_{n\in\mathbf{N}} BGL_n(A)$ under Whitney sum (block sum). If $\mu = ob\mathcal{F}_A = \mathbf{N}$ then $obNiT_{\mathcal{F}_A,\mathbf{N}} = B(obNi\mathcal{F}_A\times\mathbf{N},\mathbf{N},*)$ is a model for the homotopy colimit over the maps $\coprod_{n\in\mathbf{N}} BGL_n(A) \to \coprod_{n\in\mathbf{N}} BGL_n(A)$ given by Whitney sum (with identity matrices of varying sizes). The homotopy limit is equivalent to the homotopy limit over the natural numbers over the maps $\coprod_{n\in\mathbf{N}} BGL_n(A) \to \coprod_{n\in\mathbf{N}} BGL_n(A)$ given by Whitney sum with the rank one identity matrix. This homotpy colimit is equivalent to the corresponding categorical colimit, which simply is $\mathbf{Z} \times BGLA$. Hence lemma 2.2.1 says that there is a chain of weak equivalences beween $\mathbf{Z} \times BGL(A)^+$ and $\Omega i\bar{H}\mathcal{F}_A$. Hence, for the category of finitely generated free modules over a ring A with IBN, the approaches through S, $\bar{H}$ and $+$ are all equivalent:

$$\mathbf{Z} \times BGL(A)^+ \simeq \Omega i\bar{H}\mathcal{F}_A(S^1) \xrightarrow{\simeq} \Omega iS\mathcal{F}_A.$$

If we instead consider the category $\mathcal{P}_A$ of finitely generated projective modules over a ring A, and $\mu = ob\mathcal{F}_A \subseteq \mathcal{P}_A$, then $T_{i\mathcal{P}_A,\mu} \simeq K_0(A) \times BGL(A)$ and we get

Theorem 2.2.3 *Let A be a discrete ring. Then there is a chain of equivalences*

$$K_0(A) \times BGL(A)^+ \simeq \Omega i\bar{H}\mathcal{P}_A(S^1) \xrightarrow{\simeq} \Omega iS\mathcal{P}_A.$$

Notice that comparing the results for $\mathcal{F}_A$ (remove the IBN hypothesis) and $\mathcal{P}_A$ gives one proof of *cofinality* in the sense used in e.g., [41]: the connected cover of K-theory does not see the difference between free and projective modules.

Note 2.2.4 One should notice that the homotopy eqiuvalence $K(A) \simeq K_0(A) \times BGL(A)^+$ is not functorial in A. As an example, consider the ring $C(X)$ of continuous maps from a compact topological space X to the complex numbers. There is a functorial (in X) map

$$\Omega^\infty obNi\bar{H}\mathcal{P}_{C(X)} \to \Omega^\infty obNi\bar{H}\mathcal{P}^{top}_{C(X)} \xleftarrow{\simeq} obNi\mathcal{P}^{top}_{C(X)}$$

The superscript *top* means that we shall remember the topology and and consider $\mathcal{P}_{C(X)}$ as a topological category. The latter spectrum is by a theorem of Swan (the connective cover of) what is known as the Atiyah and Hirzebruch's topological K-theory of X (see [2]) and is represented by the spectrum $ku = obNi\bar{H}\mathcal{P}^{top}_{C(*)}$. The map from the algebraic K-theory of $C(X)$ to the topological K-theory of X is an isomorphism on path component and a surjection on the fundamental group (see [86, page 61] or [4]). Consider the map $C(B(\mathbf{Z}/2)) \to C(B(\mathbf{Z}))$ induced by the projection $\mathbf{Z} \to \mathbf{Z}/2$. Let F be the fiber of $K(C(B(\mathbf{Z}/2))) \to K(C(B(\mathbf{Z})))$, and let G be the fiber of $[B(\mathbf{Z}/2), ku] \to [B(\mathbf{Z}), ku]$. By naturality this induces a map of long exact sequences

$$\begin{array}{ccccccc}
K_1(C(B(\mathbf{Z}))) & \longrightarrow & \pi_0 F & \longrightarrow & K_0(C(B(\mathbf{Z}/2))) & \longrightarrow & K_0(C(B(\mathbf{Z}))) \\
\downarrow & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
K^1(B(\mathbf{Z})) & \longrightarrow & \pi_0 G & \longrightarrow & K^0(B(\mathbf{Z}/2)) & \longrightarrow & K^0(B(\mathbf{Z})) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 \oplus \mathbf{Z} & \xrightarrow{0+id} & \mathbf{Z}
\end{array}$$

(since $\pi_0 G \cong \tilde{K}^0(B(\mathbf{Z}/2) \coprod_{B(\mathbf{Z})} B(\mathbf{Z}) \wedge I) \cong \tilde{K}^0(S^0) \cong \mathbf{Z}$, and the map $K^1(B(\mathbf{Z})) \to K^1(B(\mathbf{Z})) \cong \tilde{K}^0(S^0)$ is induced by multiplication with 2). This means that $\pi_0 F \to$

$\tilde{K}_0 C(B(\mathbf{Z}/2)) = \mathbf{Z}/2$ is a nonsplit extension, in contrast with what you get if you consider the fiber of

$$K_0(C(B(\mathbf{Z}/2))) \times BGL(C(B(\mathbf{Z}/2)))^+ \to K_0(C(B(\mathbf{Z}))) \times BGL(C(B(\mathbf{Z})))^+$$

We are grateful to D. Grayson and J. Rognes for assistance with this argument.

2.3 The algebraic K-theory space of S-algebras

{subsec:K

The definition of K-theory space for **S**-algebras follows the idea for simplicial rings. We will later give spectrum level definitions which agree with this simple definition.

2.3.1 The general linear group like monoid $\widehat{GL}(A)$

{Def:GLha

What is to play the rôle of the general linear group? We could of course let it be the group of automorphisms of $A^{\times n}$ (mimicking degreewise K-theory), but this will be much too restrictive for our applications. Rather we must somehow capture all self-equivalences. Note that we are to perform some unfriendly operations on the monoid of self-equivalences, so we had better ensure that our input is fibrant.

Note also that if A is an **S**-algebra, then the multiplication in A gives rise to a simplicial monoid structure on $T_0A(1_+)$ where T_0 is the fibrant replacement functor of II.2.2.2. This would not be true if we had used the other fibrant replacement FA.

Consider the simplicial monoid

$$\widehat{M}_n A = T_0 Mat_n A(1_+) = \underset{\overrightarrow{x\in\mathcal{I}}}{\operatorname{holim}} \Omega^x(Mat_n(A)(S^x))$$

Its monoid of components is $\pi_0(\widehat{M}_n(A)) = M_n(\pi_0 A)$, andwe let $\widehat{GL}_n(A)$ be the grouplike simplicial monoid of homotopy units:

$$\begin{array}{ccc} \widehat{GL}_n(A) & \longrightarrow & \widehat{M}_n(A) \\ \downarrow & & \downarrow \\ GL_n(\pi_0 A) & \longrightarrow & M_n(\pi_0 A) \end{array}$$

is a pullback diagram.

This stabilises correctly, in the sense that

$$S^0 = \mathbf{S}(1_+) \to A(1_+) = Mat_1(A)(1_+) \to \Omega^n Mat_1(A)(S^n)$$

and

$$Mat_n A \times Mat_1 A \xrightarrow{\vee} Mat_{n+1} A$$

induce maps

$$\begin{array}{ccc}
\widehat{M}_n(A) & = & \underset{\overrightarrow{x\in\mathcal{I}}}{\mathrm{holim}}\,\Omega^x(Mat_n(A)(S^x)) \\
 & & \downarrow \\
 & & \underset{\overrightarrow{x\in\mathcal{I}}}{\mathrm{holim}}\,\Omega^x\left(Mat_n(A)(S^x)\times Mat_1(A)(S^x)\right) \\
 & & \downarrow\vee \\
\widehat{M}_{n+1}(A) & = & \underset{\overrightarrow{x\in\mathcal{I}}}{\mathrm{holim}}\,\Omega^x(Mat_{n+1}(A)(S^x))
\end{array}$$

which in turn induce the usual Whitehead sum $M_n(\pi_0 A) \xrightarrow{m\mapsto m\oplus 1} M_{n+1}(\pi_0 A)$. We let $\widehat{GL}(A)$ denote the colimit of the resulting directed system of $\widehat{GL}_n(A)$s.

We now can form the classifying space in the usual way and define the algebraic K-theory space just as we did for simplicial rings in 1.1.1:

spaceSalgs}

Definition 2.3.2 Let A be an **S**-algebra. Then the algebraic K-theory space of A is

$$K(A) = B\widehat{GL}(A)^+$$

From the construction we get

{lem:2.4.2}

Lemma 2.3.3 *to $k+1$ connected maps. If the* **S***-algebra A comes from a ring, then this definition is equivalent to the earlier one (and the equivalence is induced by the weak equivalence of monoids $M_nA \xrightarrow{\sim} \widehat{M}_nA$).*

2.3.4 Comparison with Waldhausen's algebraic K-theory of a connected space

A particularly important example is the K-theory of groups algebras, that is of the **S**-algebra $\mathbf{S}[G]$ coming from a simplicial group G. Then Waldhausen essentially shows that $K(\mathbf{S}[G])$ is equivalent to $A(BG)$, the algebraic K-theory of the connected space BG. (Some notes on significance). There is a slight difference between the end product in [131, theorem 2.2.1] and the present definition and we must cover this gap (see also the discussion at the bottom of page 385 in [131]). For our purposes, we may consider Waldhausen's definition of (the connected cover of) $A(BG)$ to be

$$\underset{\overrightarrow{k,m}}{\lim}\, B\mathcal{H}^k_m(G)^+$$

where $\mathcal{H}^k_m(G)$ is the simplicial monoid of pointed $|G|$-equivariant weak self-equivalences of $|\mathbf{m}_+\wedge S^k\wedge G_+|$. More precisely (see appendix CNBNB), set

$$M^k_m = \underline{G\mathcal{S}_*}(\mathbf{m}_+\wedge S^k\wedge G_+, \sin|\mathbf{m}_+\wedge S^k\wedge G_+|) \cong \sin Map_{|G|}(|\mathbf{m}_+\wedge S^k\wedge G_+|, |\mathbf{m}_+\wedge S^k\wedge G_+|)$$

This is a simplicial monoid under composition of maps $(f,g)\mapsto f\circ g$, and $\mathcal{H}^k_m(G)$ is the grouplike submonoid of invertible components. As a simplicial set M^k_m is isomorphic to $\Omega^k Mat_m\mathbf{S}[G](S^k)$. By Waldhausen's approximation theorem [131, theorem 1.6.7]

$$\lim_{\overrightarrow{k}} B\mathcal{H}^k_m(G) \xleftarrow{\simeq} \operatorname{holim}_{\overrightarrow{m}} B\mathcal{H}^k_m(G) \xrightarrow{\simeq} \operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}} B\mathcal{H}^x_m(G)$$

and we want to compare this with $B\widehat{GL}_m(\mathbf{S}[G])$.

We define a map (for convenience, we here use non-pointed homotopy colimit)

$$(\operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}}\Omega^x Mat_m(\mathbf{S}[G](S^x)))^{\times q}\cong \operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}}\prod_{i=1}^q M^{x_i}_m \to \operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}}(M^{\vee\mathbf{x}}_m)^{\times q}\to \operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}}(M^x_m)^{\times q}$$

The first map is induced by the ith inclusion $x_i\subseteq\vee\mathbf{x}$ in the ith factor, and the last map is given by composition with $\mathcal{I}^q\to\mathcal{I}$. When restricted to homotopy units, this gives by the approximation lemma the desired equivalence $B_q\widehat{GL}_m(\mathbf{S}[G])\to\operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}}B_q\mathcal{H}^x_m(G)$. We must just show that it is a simplicial map.

Note that the diagram

$$\begin{array}{ccc} M^x_m\wedge M^y_m & \xrightarrow{\mathbf{S}\text{-algebra multiplication}} & M^{x\vee y} \\ \downarrow & & \Vert \\ M^{x\vee y}\wedge M^{x\vee y} & \xrightarrow{\text{composition}} & M^{x\vee y} \end{array}$$

is commutative, where the left vertical map is induced by the first and second inclusion $x\subseteq x\vee y$ and $y\subseteq x\vee y$. Thus we see, that if $0<i<q$, then the ith face map in $\operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}}\prod_{i=1}^q M^{x_i}_m$ using the $\mathbf{S}$-algebra multiplication, corresponds to the ith face map in $\operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}}(M^x_m)^{\times q}$, since we have used the ith inclusion in the ith factor, and the $i+1$th inclusion in the $i+1$th factor. The face maps d_0, d_q just drops the first or last factor in both cases, and the degeneracies include the common unit in the appropriate factor.

2.4 Agreement of the K-theory of S-algebras through Segal's machine and the definition through the plus construction

Let $K^f_0(\pi_0 A)$ be the Grothendieck group of the category of finitely generated free $\pi_0 A$-modules. If $\pi_0 A$ has the invariance of basis number property (i.e., $(\pi_0 A)^{\times k}$ is isomorphic to $(\pi_0 A)^{\times l}$ if and only if $l=k$, which is true for most reasonable rings, and always true for commutative rings), $K^f_0(\pi_0 A)\cong\mathbf{Z}$, and otherwise it is finite cyclic.

Definition 2.4.1 Let A be an $\mathbf{S}$-algebra. Then the category $\mathcal{F}_A$ of *finitely generated free A-modules* is the $\Gamma\mathcal{S}_*$-category whose objects are the natural numbers, and where $\mathcal{F}_A(m,n)=Mat_{n,m}A\cong\prod_m\bigvee_n A$.

Note that then Segal's definition of the algebraic K-theory spectrum of A (with the uniform choice of weak equivalences II.3.3) is

$$K(A) = obN\omega\bar{H}\mathcal{F}_A.$$

{theo:2.5

Theorem 2.4.2 *There is a chain of weak equivalences*

$$\Omega^\infty K(A) \simeq K_0^f(\pi_0 A) \times B\widehat{GL}(A)^+$$

Proof: First, note that since $K(A) = obN\omega\bar{H}\mathcal{F}_A$ is special

$$\Omega^\infty K(A) \simeq \Omega obN\omega\bar{H}\mathcal{F}_A(S^1).$$

For each $n_+ \in \Gamma^o$ we have that $obN\omega\bar{H}\mathcal{F}_A(n_+) \simeq (obN\omega\mathcal{F}_A)^{\times n}$. For each $k \geq 0$, let $w\mathcal{F}^k$ be the full subcategory of $\omega\mathcal{F}_A$ whose only object is $k_+ \wedge A$. Note that by definition, this is nothing but $\widehat{GL}_k(A)$ considered as a simplicial category with only one object. Hence we are done, for by Segal [109]there is a chain of weak equivalences

$$\Omega obN\omega\bar{H}\mathcal{F}_A(S^1) \simeq K_0^f(\pi_0 A) \times \lim_{\rightarrow} k(obNw\mathcal{F}^k)^+ = K_0^f(\pi_0 A) \times B\widehat{GL}(A)^+$$

∎

If A is a discrete ring, we have a chain of weak equivalences

$$obNi\bar{H}\mathcal{F}_A \xrightarrow{\sim} obN\omega\bar{H}\widetilde{\mathcal{F}_A} \xleftarrow{\sim} obN\omega\bar{H}\mathcal{F}_{HA}$$

where $\widetilde{\mathcal{F}_A}$ is the construction of 1.6.2.2 making an $\mathcal{A}b$-category into a $\Gamma\mathcal{S}_*$-category through the Eilenberg-MacLane construction. The first weak equivalence follows by lemma II.3.3.2, whereas the second follows from the fact that the natural map $Mat_n HA \to H(M_n A)$ is a weak equivalence(wedges are stably products).

3 Simplicial rings are dense in S-algebras.

{sec:III3}

The map $\mathbf{S} \to H\mathbf{Z}$ from the sphere spectrum to the integral Eilenberg-MacLane spectrum may either be thought of as the projection onto π_0 or as the Hurewicz map. Either way, we get that it is 1-connected. This implies that there is just a very controlled difference between their module categories. The argument which we are going to give for this could equally well be considered in any setting where you have a 1-connected map $A \to B$ of $\mathbf{S}$-algebras. In fact, it is perhaps easiest to see that the result is true in this setting. Assume everything is cofibrant and that $A \to B$ is a cofibration of $\mathbf{S}$-algebras too, so as to avoid technicalities. Consider the adjoint pair

$$\mathcal{M}_B \underset{f^*}{\overset{-\otimes_B A}{\rightleftarrows}} \mathcal{M}_A$$

where f^* is restriction of scalars, which we will drop from the notation. Let M be any A-module, and consider the unit of adjunction $M \to B\wedge_A M$. This map has cofiber $B/A \wedge_A M$, and since $A \to B$ is 1-connected this gives that $M \to B \wedge_A M$ is 1-connected, and so $B \wedge_A M$ is a B-module giving a rather coarse approximation to M.

But we can continue doing this: apply $B \wedge_A -$ to $M \to B \wedge_A M$ gives the square

$$\begin{array}{ccc} M & \longrightarrow & B \wedge_A M \\ \downarrow & & \downarrow \\ B \wedge_A M & \longrightarrow & B \wedge_A \wedge_A M \end{array}$$

and a quick analyisis gives that this has iterated cofiber $B/A \wedge_A B/A \wedge_A M$, and so is "2-cartesian", meaning that M is approximated by the pullback of the rest of square, at least up to dimension two. This continues, and gives that any A-module may be approximated to any degree of accuracy by means of B-modules. However, the maps connecting the B-modules are not B-module maps. This is often not dangerous, because of the rapid convergence, functors satisfying rather weak "continuity" properties and that vanish on B-modules must vanish on all A-modules.

We will be pursuing this idea, only that we will be working non-stably, and our resolutions will in fact be resolutions of **S**-algebras (in the setup as sketched above, that would require commutativity conditions).

3.1 A resolution of S-algebras by means of simplicial rings

Recall the adjoint functor pairs of II.1.3.1

$$s\mathcal{A}b = \mathcal{A} \underset{R}{\overset{\bar{H}}{\rightleftarrows}} \Gamma\mathcal{A} \underset{U}{\overset{\tilde{\mathbf{Z}}}{\leftrightarrows}} \Gamma\mathcal{S}_*$$

(the left adjoints are on the top). All are monoidal (all but U are even strong monoidal), and so all take monoids to monoids. Furthermore, the construction T_0 of II.2.2.2 could equally well be performed in $\Gamma\mathcal{A}$, where it is called R_0 to remind us that the coproducts involved are now sums and not wedges. In particular, the approximation lemma II.2.2.3 works equally well in this setting. If A is an $\bar{H}\mathbf{Z}$-algebra, R_0A is a special $\bar{H}\mathbf{Z}$-algebra, and so by lemma II.1.3.3 the rightmost map in

$$A \xrightarrow{\sim} R_0A \xleftarrow{\sim} \bar{H}R(R_0A)$$

is a pointwise equivalence. Hence: any $\bar{H}\mathbf{Z}$-algebra is canonically stably equivalent to $\bar{H}$ of a simplicial ring (this has already been noted in II.2.2.5). This also works for (bi)modules: if P is an A-bimodule, then R_0P is an R_0A-bimodule, stably equivalent to P (as an A-bimodule); $\bar{H}(RR_0P)$ is an $\bar{H}(RR_0A)$-bimodule and pointwise equivalent to R_0P (as an $\bar{H}(RR_0A)$-bimodule). In short: $(A, P) \to (R_0A, R_0P) \longleftarrow (\bar{H}(RR_0A), \bar{H}(RR_0P))$ are stable equivalences of natural bimodules.

In particular, remembering that $H = U\bar{H}$:

{lem3.1.1

Lemma 3.1.1 *If A is any $\mathbf{S}$-algebra and P an A-bimodule, then $(U\tilde{\mathbf{Z}}A, U\tilde{\mathbf{Z}}P)$ is canonically stably equivalent to a pair (HR, HQ) where R is a simplicial ring and Q an R-bimodule:*

$$(U\tilde{\mathbf{Z}}A, U\tilde{\mathbf{Z}}P) \xrightarrow{\sim} (UR_0\tilde{\mathbf{Z}}A, UR_0\tilde{\mathbf{Z}}P) \xleftarrow{\sim} (H(RR_0\tilde{\mathbf{Z}}A), H(RR_0\tilde{\mathbf{Z}}P))$$

The adjoint pair connecting $\Gamma\mathcal{A}$ and $\Gamma\mathcal{S}_*$ defines an adjoint pair

$$\bar{H}\mathbf{Z} - \text{algebras} \underset{U}{\overset{\tilde{\mathbf{Z}}}{\leftrightarrows}} \mathbf{S} - \text{algebras}$$

(that is, $U\tilde{\mathbf{Z}}$ is a "triple" in $\mathbf{S}-$ algebras) and so we have the canonical resolution of A.0.3 (to be precise and consise in the language of A2NBNB, it is the augmented cobar resolution of the monoid $U\tilde{\mathbf{Z}}$ in the category of endofunctors of $\mathbf{S}-$ algebras)

{lem3.1.2:}

Lemma 3.1.2 *If A is an $\mathbf{S}$-algebra, then the adjoint pair gives an augmented cosimplicial object $A \to \{[q] \mapsto (U\tilde{\mathbf{Z}})^{q+1}A\}$, which is equivalent to H of a simplicial ring in each non-negative degree.*

It is fairly straightforward to see that $A \to \operatorname{holim}_{\overleftarrow{[q]\in\Delta}} (U\tilde{\mathbf{Z}})^{q+1}A$ is an equivalence, but we will not show that now, since we eventually will use the somewhat stronger Hurewicz theorem A.1.10.0.17 which tells us that this limit converges fast enough, so that the homotopy limit pass through constructions like K-theory. This has the consequence that these constructions depend on their value on simplicial rings, and on $\mathbf{S}$-algebra maps between simplicial rings. Generally this is bothersome: we would have liked the diagram we are taking the limit of to be contained wholly in the category of simplicial rings. This is of course not possible, since it would imply that all $\mathbf{S}$-algebras were stably equivalent to simplicial rings. For instance, $\mathbf{S}$ itself is not stably equivalent to a simplicial ring, but it IS the homotopy limit of a diagram

$$H\mathbf{Z} \rightrightarrows U\tilde{\mathbf{Z}}H\mathbf{Z} \overrightarrow{\rightrightarrows} U\tilde{\mathbf{Z}}U\tilde{\mathbf{Z}}H\mathbf{Z} \dots$$

Remark 3.1.3 *The categories $s\mathcal{A}b = \mathcal{A}$, $\Gamma\mathcal{A}$ and $H\mathbf{Z}$-mod, are all naturally model categories, and the functors*

$$\mathcal{A} \xrightarrow{\bar{H}} \Gamma\mathcal{A} \xrightarrow{U} H\mathbf{Z}\text{-mod}$$

induce equivalences between their homotopy categories. This uses the functor $L\colon \Gamma\mathcal{A} \to \mathcal{A}$ of II.1.3.4 to construct an adjoint functor pair (see [107]).

3.1.4 Review on cubical diagrams

We need some language in order to calculate this resolution effectively. For a more thorough discussion we refer the reader to appendix A1.9?. There the reader also will find explained why it does not matter whether we look at the cosimplicial resolutions or the cubical construction.

Let $\mathcal{P}$ be the category of finite subsets of the natural numbers $\{1, 2, \dots\}$, and inclusions. We let $\mathcal{P}n$ be the subcategory allowing only subsets of $\{1, \dots, n\}$.

{Def:ncube}

Definition 3.1.5 An n-cube is a functor $\mathcal{X}$ from the category $\mathcal{P}n$. A cubical diagram is a functor from $\mathcal{P}$.

If we adjoin the empty set $[-1] = \emptyset$ as an initial object to Δ, we get Ord, the category of finite ordered sets. A functor from Ord is what is usually called an augmented cosimplicial object. There is a functor $\mathcal{P} \to Ord$ sending a set S of cardinality n to $[n-1]$. Hence any augmented cosimplicial object gives rise to a cubical diagram. In most cases there is no loss of information in considering augmented cosimplicial objects as cubical diagrams (see appendix A1.9?).

{Def:kcar

Definition 3.1.6 Let $\mathcal{X}$ be an n-cube with values in any of the categories where homotopy (co)limits are defined. We say that $\mathcal{X}$ is k-cartesian if

$$\mathcal{X}_\emptyset \to \underset{\overleftarrow{S\neq\emptyset}}{\operatorname{holim}} \mathcal{X}_S$$

is k-connected, and k-cocartesian if

$$\underset{\overrightarrow{S\neq\{1,\dots,n\}}}{\operatorname{holim}} \mathcal{X}_S \to \mathcal{X}_{\{1,\dots,n\}}$$

is k-connected. It is homotopy cartesian if it is k-cartesian for all k, and homotopy cocartesian if it is k-cocartesian for all k.

When there is no possibility of confusing with the categorical notions, we write just cartesian and cocartesian. Homotopy (co)cartesian cubes are also called homotopy pullback cubes (resp. homotopy pushout cubes), and the initial (resp. final) vertex is then called the homotopy pullback (resp. homotopy pushout) of the rest of the diagram.

As a convention we shall say that a 0 cube is k-cartesian (resp. k-cocartesian) if $\mathcal{X}_\emptyset$ is $(k-1)$-connected (resp. k-connected).

So, a 0 cube is an object $\mathcal{X}_\emptyset$, a 1 cube is a map $\mathcal{X}_\emptyset \to \mathcal{X}_{\{1\}}$, and a 1 cube is k-(co)cartesian if it is k-connected as a map. A 2 cube is a square

$$\begin{array}{ccc} \mathcal{X}_\emptyset & \longrightarrow & \mathcal{X}_{\{1\}} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\{2\}} & \longrightarrow & \mathcal{X}_{\{1,2\}} \end{array}$$

and so on. We will regard a natural transformation of n cubes $\mathcal{X} \to \mathcal{Y}$ as an $n+1$ cube. In particular, if $F \to G$ is some natural transformation of functors of simplicial sets, and $\mathcal{X}$ is an n cube of simplicial sets, then we get an $n+1$ cube $F\mathcal{X} \to G\mathcal{X}$.

We will need the generalized Hurewicz theorem which we cite from appendix A.1.10.0.17:

{Hhrew3c2

Theorem 3.1.7 *Let $k > 1$. If $\mathcal{X}$ is an $id + k$ cartesian cube of simplicial sets, then so is $\mathcal{X} \to \tilde{\mathbf{Z}}\mathcal{X}$.* ☺

{Def:appro

Definition 3.1.8 Let A be an $\mathbf{S}$-algebra and $n > 0$. Define the n-cube of $\mathbf{S}$-algebras $(A)^n$ by applying the unit of adjunction $id \to U\tilde{\mathbf{Z}}$ n-times to A. Carrying this on indefinitly, we get a functor

$$\mathcal{P} \xrightarrow{S \mapsto (A)_S} \mathbf{S}\text{-algebras}$$

such that the restriction to $\mathcal{P}\mathbf{n} \subseteq \mathcal{P}$ is $\{S \mapsto (A)^n_S\}$.

More concretely $(A)^2$ is the 2-cube

$$\begin{array}{ccc} A & \xrightarrow{h_A} & U\tilde{\mathbf{Z}}A \\ {\scriptstyle h_A}\downarrow & & {\scriptstyle h_{U\tilde{\mathbf{Z}}A}}\downarrow \\ U\tilde{\mathbf{Z}}A & \xrightarrow{U\tilde{\mathbf{Z}}h_A} & (U\tilde{\mathbf{Z}})^2A \end{array} .$$

{cor:3.2.4}

Corollary 3.1.9 *Let $n \geq 0$. The n-cube of spectra $(A)^n$ is id-cartesian.*

Proof: For each $k > 1$, the space $A(S^k)$ is $(k-1)$-connected by 2.1.4.2 (and so $(id+k)$-cartesian as a 0-cube). Hence the Hurewicz theorem 3.1.7 says that $S \mapsto (A)^n_S(S^k)$ is $(id+k)$-cartesian, which is stronger than $S \mapsto (A)^n_S$ being id-cartesian as a spectrum. ∎

The very reason for the interest in this cosntruction stems from the following observation which follows immediately from II.2.2.5.

are rings}

Proposition 3.1.10 *Let A be an $\mathbf{S}$-algebra. Then $(A)^S$ is canonically equivalent to H of a simplicial ring for all $S \neq \emptyset$.*

3.2 K-theory is determined by its values on simplicial rings

IIIKapprox}

First note that K-theory behaves nicely with respect to id-cartesian squares (note that a square being merely highly cartesian is not treated nicely by K-theory, you need good behaviour on all subskeleta of your cube).

theo:3.3.1}

Theorem 3.2.1 *Let $\mathcal{A}$ be an id cartesian n cube of $\mathbf{S}$-algebras, $n > 0$. Then $K(\mathcal{A})$ is $n+1$ cartesian.*

Proof: Let $\mathcal{M} = Mat_m\mathcal{A}$ be the cube given by the $m \times m$ matrices in $\mathcal{A}$. This is id cartesian, and so $T_0\mathcal{M} = \operatorname{holim}_{\overrightarrow{x \in I}} \Omega^x(Mat_m\mathcal{A})(S^x)$ is an id cartesian cube of simplicial monoids. As all maps are 1-connected, we get $\mathcal{G} = \widehat{GL}_m(\mathcal{A})$ as the pullback in

$$\begin{array}{ccc} \mathcal{G}_T & \longrightarrow & T_0\mathcal{M}_T \\ \downarrow & & \downarrow \\ GL_m(\pi_0 A) & \longrightarrow & M_m(\pi_0 A) \end{array}$$

for all $T \subset \mathbf{n}$. Hence $\widehat{GL}_m(\mathcal{A})$ is id cartesian, and so $B\widehat{GL}_m(A)$ is $id+1$ cartesian. Using lemma 2.7?NBNB we get that also

$$K(\mathcal{A}) = B\widehat{GL}(\mathcal{A})^+ \cong \left(\lim_{\overrightarrow{m}} B\widehat{GL}_m(\mathcal{A})\right)^+$$

is $id+1$ cartesian. ∎

Note that with any of our other definitions of algebraic K-theory there is a non functorially equivalence to

$$B\widehat{GL}(\mathcal{A}_S)^+ \times K_0(\pi_0\mathcal{A}_S)$$

we still get that the algebraic K-theory of $\mathcal{A}$ is $n+1$-cartesian (it is not $id+1$ cartesian because it consists of non connected spaces). This is so since all the maps of **S**-algebras involved are 1-connected, and so $K_0(\pi_0\mathcal{A})$ is the constant cube $K_0(\pi_0\mathcal{A}_\emptyset)$.

{theo:3.3

Theorem 3.2.2 *Let A be an* **S***-algebra. Then*

$$K(A) \to \underset{\overleftarrow{S\in\mathcal{P}-\emptyset}}{\operatorname{holim}} K((A)_S)$$

is an equivalence.

Proof: We know there is high connectivity to any of the finite cubes: theorem 3.2.1 tells us that $K(A) \to \operatorname{holim}_{\overleftarrow{S\in\mathcal{P}\mathbf{n}-\emptyset}} K((A)^n_S)$ is $(n+1)$-connected, so we just have to know that this assembles correctly. Now, by lemma A.1.9.2.4 the map

$$\underset{\overleftarrow{S\in\mathcal{P}\mathbf{n+1}-\emptyset}}{\operatorname{holim}} K((A)^{n+1}_S) \to \underset{\overleftarrow{S\in\mathcal{P}\mathbf{n}-\emptyset}}{\operatorname{holim}} K((A)^n_S)$$

induced by restriction along $\mathcal{P}\mathbf{n} \subseteq \mathcal{P}\mathbf{n+1}$ is a fibration. By writing out explicitly the cosimplicial replacement formula of A.1.9.3 for the homotopy limit, you get that

$$\underset{\overleftarrow{J}}{\operatorname{holim}} F \cong \underset{\overleftarrow{n\in\mathbf{N}}}{\lim} \underset{\overleftarrow{J_n}}{\operatorname{holim}} F|_{J_n}.$$

Hence by lemma A.1.9.3.2 and theorem A.1.9.5.1 you get that $\operatorname{holim}_{\overleftarrow{S\in\mathcal{P}\mathbf{n}-\emptyset}} K((A)^n_S)$ approximates $\operatorname{holim}_{\overleftarrow{S\in\mathcal{P}-\emptyset}} K((A)_S)$. ∎

Chapter IV

Topological Hochschild homology

{IV}

As K-theory is hard to calculate, it is important to know theories that are related to K-theory, but that are easier to calculate. Thus, if somebody comes up with a nontrivial map between K-theory and something one thinks one can get hold on, it is considered a good thing. In 1976 (ch) R. Keith Dennis observed that there existed a map from the K-theory of a ring A to the so-called Hochschild homology $HH(A)$. This map has since been called the Dennis trace.

Waldhausen noticed in [129] that there was a connection between the sphere spectrum, stable K-theory and Hochschild homology. Although the proof appeared only much later ([132]), he also knew before 1980 that stable A-theory coincided with stable homotopy. Motivated by his machine "calculus of functors" and his study of stable pseudo isotopy theory, T. Goodwillie conjectured that there existed a theory sitting between K-theory and Hochschild homology, agreeing integrally with stable K-theory for all "rings up to homotopy", but with a Hochschild-style definition. He called the theory topological Hochschild homology (THH), and the only difference between THH and HH should be that whereas the ground ring in HH is the the ring of integers, the ground ring of THH should be the sphere spectrum $\mathbf{S}$, considered as a "ring up to homotopy". This would also be in agreement with his proof that stable K-theory and Hochschild homology agreed rationally, as the higher homotopy groups of S are all torsion. He also made some conjectural calculations of $THH(\mathbf{Z})$ and $THH(\mathbf{Z}/p\mathbf{Z})$.

The next step was taken in the mid eighties by M. Bökstedt, who was able to give a definition of THH, satisfying all of Goodwillie's conjectural properties, except possibly the equivalence with stable K-theory. To model rings up to homotopy, he defined functors with smash products which are closely related to the $\mathbf{S}$-algebras defined in chapter II).

{theo:THH

Theorem 0.2.3 (Bökstedt)

$$\pi_k THH(\mathbf{Z}) \cong \begin{cases} \mathbf{Z} & \textit{if } k=0 \\ \mathbf{Z}/i\mathbf{Z} & \textit{if } k=2i-1 \\ 0 & \textit{if } k=2i>0 \end{cases}$$

$$\pi_k THH(\mathbf{Z}/p\mathbf{Z}) \cong \begin{cases} \mathbf{Z}/p\mathbf{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$

Later it was realized that a work of Breen [15] actually calculated $THH(\mathbf{Z}/p\mathbf{Z})$. The outcome of the two papers of Jibladze, Pirashvili and Waldhausen [58], [95] was that $THH(A)$ could be thought of as the homology of $\mathcal{P}_A$ in the sense of I.3, or alternatively as "MacLane homology", [78]. This was subsequently used by Franjou, Lannes and Schwartz and Pirashvili to give purely algebraic proofs of Bökstedt's calculations, [32] and [33].

For (flat) rings A, there is a (3-connected) map $THH(A) \to HH(A)$ which should be thought of as being induced by the change of base ring $\mathbf{S} \to \mathbf{Z}$.

After it became clear that the connection between K-theory and THH was as good as could be hoped, many other calculations of THH have appeared. For instance, THH posesses localization, in the same sense as HH does, THH of group rings can be described, and so on. Many calculations has been done in this setting or in the dual MacLane cohomology, for instance Pirashvili's [93], [94] and [92]. For further calculations see Larsen and Lindenstrauss' papers [68], [72] and [67]. For A a ring of integers in a number field, A. Lindenstrauss and I. Madsen obtained in [73] the non-canonical isomorphism

$$\pi_i THH(A) \cong \begin{cases} A & \text{if } i = 0 \\ A/n\mathcal{D}_A & \text{if } i = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{D}_A$ is the different ideal. In [52] Hesselholt and Madsen give a canonical description, which we will return to later.

For concrete calculations the spectral sequence of Pirashvili and Waldhausen in [95] (see 1.3.7) is very useful. This especially so since in many cases it degenerate, a phenomenon which is partially explained in [105].

As we have already noted, the first example showing that stable K-theory and THH are equivalent is due to Waldhausen, and predates the definition of THH. He showed this in the cases coming from his K-theory of spaces; in particular, he showed the so-called "vanishing of the mystery homology": stable K-theory of the sphere spectum $\mathbf{S}$, is equivalent to $\mathbf{S} \simeq THH(\mathbf{S})$. Based upon this, [106] announced that they could prove $K^S \simeq THH$ in general, but due to unfortunate circumstances, the proof has not been completed yet.

The second example appeared in [26], and took care of the case of rings, using the interpretation of $THH(A)$ as the homology of $\mathcal{P}_A$. In [24] it was shown how this implies $K^S \simeq THH$ for all $\mathbf{S}$-algebras.

When A is a commutative $\mathbf{S}$-algebra we get by an appropriate choice of model that $THH(A)$ is also a commutative $\mathbf{S}$-algebra, and the homotopy groups become a graded commutative ring. For instance, the calculation of $\pi_* THH(\mathbf{Z}/p\mathbf{Z})$ could be summed up more elegantly by saying that it is the graded polynomial ring in $\mathbf{Z}/p\mathbf{Z}$ in one generator in degree 2.

0.2.4 Organization

In the first section we will give a definition of topological Hochschild homology for **S**-algebras, and prove some basic results w ith a special view on the ring case. In the second section, we will extend our definition to include $\Gamma\mathcal{S}_*$-categories in general as input. This is very similar, and not much more involved; but we have chosen to present the theory for **S**-algebras first so that people not interested with anything but rings can have the definition without getting confused by too much generality. However, this generality is very convenient when one wants to construct the trace map from K-theory, and also when one wants to compare with the homology of additive categories. This is particularly clear when one wants good defintions for the "trace" map from algebraic K-theory, which we present in the third section. In the fourth section we will discuss what happens in a "dual numbers" situation.

0.3 Where to read

The literature on THH is not as well developed as for K-theory; and there is a significant overlap between these notes and most of the other sources. The original paper [9] is good reading, but has unfortunately not yet appeared. The article [50] develops the ideas in [6] further, and is well worth studying to get an equivariant point of view on the matter. For the THH spectrum for exact categories, [27] is slightly more general than these notes.

For a general overview, the survey article of Madsen, [80], is recommended.

1 Topological Hochschild homology of S-algebras.

As Topological Hochschild homology is supposed to be a modelled on the idea of Hochschild homology, we recall the standard complex calculating $HH(A)$.

1.1 Hochschild homology of k-algebras

{Def:HH}

Recall the definition of Hochschild homology (see I.3.2): Let k be a commutative ring, let A be a *flat* k-algebra, and let P be an A-bimodule. Then we define the *Hochschild homology* (over k) of A with coefficients in P to be the simplicial k module

$$HH^k(A,P) = \{[q] \mapsto HH^k(A,P)_q = P \otimes_k A^{\otimes_k^q}\}$$

with face and degeneracies given by

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_q) = \begin{cases} ma_1 \otimes a_2 \cdots \otimes a_q & \text{if } i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q & \text{if } 0 < i < q \\ a_q m \otimes a_1 \otimes \cdots \otimes a_{q-1} & \text{if } i = q \end{cases}$$

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_q) = m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_q$$

Just the same definition may be applied to simplicial k-algebras, and this definition of HH^k will preserve weak equivalences. Again we either assume that our ring is flat, or else we substitute it with one that is, and so we are really defining what some call Shukla homology. To make this functorial in (A,P) we really should choose a functorial flat resolution of rings once and for all, but since our main applications are to rings that are already flat, we choose to suppress this.

1.1.1 Cyclic structure

In the case $P = A$ something interesting happens. Then $HH^k(A) = HH^k(A,A)$ is not only a simplicial object, but also a *cyclic* object (see C.C.3.2 for a more detailed discussion of cyclic objects, and section 1.2.7 below for the structure on THH). Recall that a *cyclic object* is a functor from Connes' category Λ^o, where Λ is the category containing Δ, but with an additional endomorphism for each object, satisfying certain relations. In terms of generators, this means that in addition to all maps coming from Δ for each $[q]$ there is a map $t = t_q\colon [q] \to [q]$. In our case t is sent to the map $A^{\otimes_k^{q+1}} \to A^{\otimes_k^{q+1}}$ sending $a_0 \otimes \cdots \otimes a_q$ to $a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}$.

Def:Lambda} To be precise:

$$\Lambda([p],[q]) = \Delta([p],[q]) \times C_{p+1}$$

with composition subject to the extra relations (where t_n is the generator of C_{n+1})

$$
\begin{aligned}
t_n d^i &= d^{i-1} t_{n-1} \qquad 1 \le i \le n \\
t_n d^0 &= d^n \\
t_n s^i &= s^{i-1} t_{n+1} \qquad 1 \le i \le n \\
t_n s^0 &= s^n t_{n+1}^2
\end{aligned}
$$

A *cyclic object* in some category $\mathcal{C}$ is a functor $\Lambda^o \to \mathcal{C}$ and a *cyclic map* is a natural transformation between cyclic objects. Due to the inclusion $j\colon \Delta \subset \Lambda$, any cyclic object X gives rise to a simplicial object j^*X.

Hochschild homology is just an instance of a general gadget giving cyclic objects: let M be a monoid in a symmetric monoidal category $(\mathcal{C}, \Box, e)$. Then the *cyclic bar construction* {cyclic ba is the cyclic object $B^{cy}(M) = \{[q] \mapsto M^{\Box(q+1)}\}$. Hocschild homology is then the example coming from $(k\text{-mod}, \otimes_k, k)$. The most basic example is the *cyclic bar construction* of ordinary monoids: the symmetric monoidal category of sets with cartesian product, a monoid is just an ordinary monoid, and $B^{cy}_q(M) = M^{\times(q+1)}$. Slightly more fancy are the cases $(Cat, \times, *)$: monoids are strict monoidal categories, or $(\mathcal{S}, \times, *)$: monoids are simplicial monoids. We have already seen an example of the former: the object $\{[q] \mapsto \mathcal{I}^{q+1}\}$ which appeared in II.2.2.1 was simply $B^{cy}\mathcal{I}$.

Just as for Hochschild homology, these constructions can also be applied to "bimodules", but will the give only simplicial objects.

1.2 One definition of topological Hochschild homology of S-algebras

In analogy with the above definition of HH^k, Bökstedt defined topological Hochschild homology. Of course, **S** is initial among the **S**-algebras, just as k is among k-algebras, and the idea is that we should try to substitute $(k\text{-mod}, \otimes_k, k)$ with $(\mathbf{S}\text{-mod}, \wedge, \mathbf{S})$. That is, instead of taking tensor product over k, we should take "tensor product over S", which is, smash of Γ-spaces. So we could consider

$$HP \wedge HA \wedge \ldots \wedge HA$$

(or even smashed over some other commutative **S**-algebra if desirable), and there is nothing wrong with this, except that

1. as it stands it is prone to all the nuisances of the classical case: unless we substitute HA for something fairly free in $\Gamma\mathcal{S}_*$ first, this will not preserve equivalences; and

2. without some amendment this will not have enough structure to define the goal of the next chapter: topological cyclic homology.

Inspired by spectra rather than Γ-spaces, Bökstedt defined a compact definition which takes care of both these problems. But before we give Bökstedt's definition, we note that we have already twice encountered one of the obstructions to a too naïve generalization.

Let A be a ring. The associated $\mathbf{S}$-algebra HA sending X to $HA(X) = A \otimes \tilde{\mathbf{Z}}[X]$ has a multiplication; but if we want to loop this down we have a problem: the multiplication gives a map from

$$\varinjlim_{k,l \in \mathbf{N}^2} \Omega^{k+l}((A \otimes \tilde{\mathbf{Z}}[S^k]) \wedge (A \otimes \tilde{\mathbf{Z}}[S^l]))$$

to

$$\varinjlim_{k,l \in \mathbf{N}^2} \Omega^{k+l}(A \otimes \tilde{\mathbf{Z}}[S^{k+l}])$$

which sure enough is isomorphic to $\varinjlim_{k \in \mathbf{N}} \Omega^k(A \otimes \tilde{\mathbf{Z}}[S^k])$, but not equal. The problem gets nasty when we consider associativity: we can't get the two maps from the "triple smash" to be **equal**. For Hochschild homology we want a simplicial space which in degree 0 is equivalent to $\varinjlim_{k \in \mathbf{N}} \Omega^k(A \otimes \tilde{\mathbf{Z}}[S^k])$, in degree 1 is equivalent to

$$\varinjlim_{k,l \in \mathbf{N}^2} \Omega^{k+l}((A \otimes \tilde{\mathbf{Z}}[S^k]) \wedge (A \otimes \tilde{\mathbf{Z}}[S^l]))$$

and so on, and one of the simplicial relations ($d_1^2 = d_1 d_2$) will exacly reflect associativity and it is not clear how to do this.

In [9], Bökstedt shows how one can get around this problem by using the category $\mathcal{I}$ (the subcategory of Γ^o with all objects and just injections, see II.2.2.1) instead of the natural numbers. To ensure that the resulting colimit has the right homotopy properties, we must use the homotopy colimit, see the approximation lemma II.2.2.3.

Recall that, if $x = k_+ \in ob\mathcal{I}$, then an expression like $S^x = S^k$ will mean S^1 smashed with itself k times, and $\Omega^x = \Omega^k$ will mean $Map_*(S^k, -) = \underline{\mathcal{S}_*}(S^k, \sin|-|)$.

{DefTHH{A,P}}

Definition 1.2.1 Let A be an $\mathbf{S}$-algebra, P an A bimodule and X a space, and define for every q the assignment $V(A,P)\colon ob\mathcal{I}^{q+1} \to ob\mathcal{S}_*$ by

$$(x_0, \dots, x_q) \mapsto V(A,P)(x_0, \dots, x_q) = P(S^{x_0}) \wedge \bigwedge_{1 \le i \le q} A(S^{x_i})$$

This gives rise to a functor $G_q = G(A,P,X)_q \colon \mathcal{I}^{q+1} \to \mathcal{S}_*$ given by

$$\mathbf{x} \mapsto G_q(\mathbf{x}) = \Omega^{\vee \mathbf{x}}(X \wedge V(A,P)(\mathbf{x}))$$

and

$$THH(A,P)(X)_q = \underset{\mathbf{x} \in \mathcal{I}^{q+1}}{\underrightarrow{\operatorname{holim}}} G_q(\mathbf{x})$$

1.2.2 The homotopy type

We have to know that this has the right homotopy properties, i.e., we need to know that it is equivalent to

$$\varinjlim_{(n_0, \dots n_q) \in \mathbf{N}^{q+1}} \Omega^{\sum n_i}(X \wedge P(S^{n_0}) \wedge \bigwedge_{1 \le i \le q} A(S^{n_i}))$$

By the approximation lemma of $\mathcal{I}$ II.2.2.3, this will be the case if we can show that a map $\mathbf{x} \subseteq \mathbf{y} \in \mathcal{I}^{q+1}$ will induce a map $G_q(\mathbf{x}) \to G_q(\mathbf{y})$ which gets higher and higher connected with the cardinality of $\mathbf{x}$. Maps in $\mathcal{I}^{q+1}$ can be written as compositions of an isomorphism together with a standard inclusion. The isomorphisms pose no problem, so we are left with considering the standard inclusions which again can be decomposed into successions of standard inclusions involving only one coordinate. Since the argument is rather symmetric, we may assume that we are looking at the standard inclusion

$$\mathbf{x} = (k_+, x_1, \dots, x_q) \subseteq ((k+1)_+, x_1, \dots, x_q).$$

Since P is a Γ-space, lemma II.2.1.4.3 says that $S^1 \wedge P(S^k) \to P(S^{k+1})$ is roughly $2k$-connected, and so (by the same lemma II.2.1.4.2) $S^1 \wedge P(S^k) \wedge \bigwedge A(S^{x_i}) \to P(S^{k+1}) \wedge \bigwedge A(S^{x_i})$ is roughly $2k + \vee x_i$ connected. The Freudenthal suspension theorem A.1.10.0.9 then gives the result.

1.2.3 Functoriality

We note that, when varying X in Γ^o, $THH(A,P;X)_q$ becomes a very special Γ-space which we simply call $THH(A,P)_q$ (it is "stably fibrant" in the terminology of chapter II, see corollary II.2.1.9), and so defines an Ω-spectrum. Also we see that it is a functor in the maps of pairs $(A,P) \xrightarrow{f} (B,Q)$ where $f\colon A \to B$ is a map of **S**-algebras, and $P \to f^*Q$ is a map of A-bimodules – that is, a map of $\Gamma\mathcal{S}_*$-*natural bimodules* in the sense of appendix B.B.1.4.2.

1.2.4 Simplicial structure

So far, we have not used the multiplicative structure of our **S**-algebra, but just as for ordinary Hochschild homology this enters when we want to make $[q] \mapsto THH(A,P;X)_q$ into a functor, that is, a simplicial space. The compact way of describing the face and degeneracy maps is to say that they are "just as for ordinary Hochschild homology". This is true and will suffice for all future considerations, and the pragmatic reader can stop here. However, we have seen that it is difficult to make this precise, and the setup of Bökstedt is carefully designed to make this rough definition work.

In detail: Consider the functor $G_q = G(A,P,X)_q\colon \mathcal{I}^{q+1} \to \mathcal{S}_*$ of the definition of $THH(A,P;X)_q$. Homotopy colimits are functors of "$\mathcal{S}_*$-natural modules", in this case restricted to pairs (I,F) where I is a small category and $F\colon I \to \mathcal{S}_*$ a functor. A map $(I,F) \to (J,G)$ is a functor $f\colon I \to J$ together with a natural transformation $F \to G \circ f$. So to show that $[q] \to THH(A,P;X)_q$ is a functor, we must show that $[q] \mapsto (\mathcal{I}^{q+1}, G_q)$ is a functor from Δ^o to $\mathcal{S}_*$-natural modules. Let $\phi \in \Delta([n],[q])$. The maps $\phi^*\colon \mathcal{I}^{q+1} \to \mathcal{I}^{n+1}$ comes from the fact that $\mathcal{I}$ is symmetric monoidal with respect to the pointed sum $m_+ \vee n_+ = (m+n)_+$, and even strict monoidal if you are careful. Hence $\mathcal{I}^{q+1}$ is just a disguise for the q-simplices of the cyclic bar construction $B^{cy}\mathcal{I}$ of 1.1.1, and the ϕ^* are just the structure maps for the cyclic bar construction. The maps $G_q(\mathbf{x}) \to G_n(\phi^*\mathbf{x})$ are defined as follows. The loop coordinates are mixed by the obvious isomorphisms $S^{\phi^*\mathbf{x}} \cong S^{\mathbf{x}}$, and the maps $V(A,P)(\mathbf{x}) \to V(A,P)(\phi^*\mathbf{x})$ are defined by

for $\phi \in \Lambda([q], ?)$ define $V(A,P)(\mathbf{x}) \to V(A,P)(\phi^*\mathbf{x})$ by means of..

d^0	$P(S^{x_0})\wedge A(S^{x_1}) \to P(S^{x_0 \vee x_1})$
d^i for $0 < i < q$	$A(S^{x_i})\wedge A(S^{x_{i+1}}) \to A(S^{x_i \vee x_{i+1}})$
d^q	$A(S^{x_q})\wedge P(S^{x_0}) \to P(S^{x_q \vee x_0})$
s^i for $0 \leq i \leq q$	$S^0 = \mathbf{S}(S^0) \to A(S^0)$ in the $i+1$st slot
t (when $A = P$)	cyclic permutation of smash factors

We check that these obey the usual simplicial identities. For this, use the associative and unital properties of $\mathcal{I}$, A and P.

{1.2.5}

Definition 1.2.5 Let A be an $\mathbf{S}$-algebra, P an A bimodule and X a space. Then the topological Hochschild homology is defined as

$$THH(A,P;X) = \{[q] \mapsto THH(A,P;X)_q\}$$

This gives rise to the very special Γ-space

$$THH(A,P) = \{Y \in ob\Gamma^o \mapsto THH(A,P;Y)\}$$

and the Ω-spectrum

$$\underline{T}(A,P;X) = \{m \mapsto \sin|THH(A,P;S^m\wedge X)|\}$$

The $\sin|-|$ in the definition of $\underline{T}$ will not be of any importance to us now, but will be convenient when discussing the cyclic structure in chapter IV. We also write $THH(A,P) = THH(A,P;S^0)$ and $THH(A) = THH(A;S^0)$ and so on.

Note that by lemma 1.3.1 below,

$$THH(A,P;X) \simeq diag^*\{[q] \mapsto THH(A,P;X_q)\} = THH(A,P)(X)$$

for all spaces X.

{1.2.6}

Lemma 1.2.6 *$THH(A,P;X)$ is functorial in (A,P) and X, and takes (stable) equivalences to pointwise equivalences. Likewise for THH and $\underline{T}$.*

Proof: This follows from the corresponding properties for $THH(A,P;X)_q$. ∎

1.2.7 Cyclic structure

{cyclicTHH}

In the case where $P = A$ we have that $THH(A;X) = THH(A,A;X)$ is a cyclic space. Furthermore, $THH(A) = THH(A,A)$ is a cyclic Γ-space and $\underline{T}(A;X) = \underline{T}(A,A;X)$ becomes an $\mathbf{S^1}$-spectrum (where $\mathbf{S^1} = \sin|S^1|$). This last point needs some explanation, and will become extremely important in the next chapter.

If Z is a cyclic space, then the realization $|Z|$ of the corresponding simplicial space has a natural $|S^1| \cong \mathbb{T}$-action (see VI.1.1 for further details), and so $\sin|Z|$ has a natural $\mathbf{S^1} = \sin|S^1|$-action (of course, there is no such thing as an "S^1-space", since S^1 is only an innocent space - not a group - before realizing).

In the case where $Z = THH(A,X)$ (considered as a simplicial cyclic set) the actual $\mathbf{S^1}$-fixed points are not very exciting: as we will show in more details in chapter VI,

$$\sin|THH(A;X)|^{\mathbf{S^1}} \cong \sin|X|$$

An important fact in this connection is that, considered as a $\Gamma\mathcal{S}_*$-category, A has only one object. In the next section we will consider more general situations, and get more exciting results.

In chapter VI we shall see that, although the $\mathbf{S^1}$-fixed points are not very well behaved, the finite cyclic subgroups give rise to a very interesting theory.

1.2.8 Hochschild homology over other commutative S-algebras

{IV:HHk}

Bökstedt's definition of topological Hochschild homology is very convenient, and accessible for hands on manipulations. On the other hand, it is conceptually more rewarding to view topological Hochschild homology as Hochschild homology over **S**. Let k be a commutative **S**-algebra. Then $(k\text{-mod}, \wedge_k, k)$ is a symmetric monoidal category, and we may form the cyclic bar construction, see 1.1.1, in this category: if A is a k-algebra which is cofibrant as a k-module and P is an A-bimodule, then $HH^k(A,P)$ is the simplicial k-module

$$HH^k(A,P) = \{[q] \to P\wedge_k A\wedge_k \ldots \wedge_k A\}$$

By the results of the previous chapter, we see that $HH^{\mathbf{S}}$ and THH have stably equivalent values (the smash product has the right homotopy type when applied to cofibrant Γ-spaces, and so $HH^{\mathbf{S}}(A,P)$ and $THH(A,P)$ are equivalent in every degree). Many of the results we prove in the following section have more natural interpretations in this setting.

If we want to talk about Hochschild homology of k-algebras that are not cofibrant as k-modules, we should use that the category of k-algebras is a model category and apply a functorial cofibrant replacement before using the construction of HH^k above.

{ex:IVTHHS

Example 1.2.9 (THH of spherical group rings) Let G be a simplicial group, and consider the spherical group ring $\mathbf{S}[G]$ of II.1.4.4.2 given by sending a finite pointed set X to $\mathbf{S}[G](X) = X\wedge G_+$. Then $THH(\mathbf{S}[G])_q$ has the homotopy type of $\mathbf{S}[G]$ smashed with itself $q+1$ ($\mathbf{S}[G]$ is a cofibrant Γ-space, so one does not have to worry about cofibrant replacements), with face and degeneracy maps as in Hochschild homology. Hence

$THH(\mathbf{S}[G])$ is equivalent to $\mathbf{S}[B^{cy}(G)]$, whose associated infinite loopspace calculates the stable homotopy of the cyclic bar construction of G.

A particularly nice interpretation is gotten if we set $X = |BG|$, because there is a natural equivalence $|B^{cy}G| \simeq \Lambda X$ between the cyclic nerve of the loop group and the free loop space (see e.g., [37, proof of V.1.1]), because then

$$|THH(\mathbf{S}[G])(1_+)| \simeq \Omega^\infty \Sigma^\infty \Lambda X.$$

1.3 Simple properties of topological Hochschild homology

An important example is THH of an $\mathbf{S}$-algebra coming from a (simplicial) ring. We consider THH as a functor of rings and bimodules, and when there is no danger of confusion, we write $THH(A, P; X)$, even though we actually mean $THH(HA, HP; X)$ and so on. Whether the ring is discrete or truly simplicial is of less importance as

{1.3.1}

Lemma 1.3.1 *Let A be a simplicial $\mathbf{S}$-algebra, P an A-bimodule and X a space. Then*

$$THH(diag^* A, diag^* P; X) \simeq diag^*\{[q] \mapsto THH(A_q, P_q; X_q)\}.$$

Proof: Let $\mathbf{x} \in \mathcal{I}^{n+1}$. Using that smash product is formed degreewise, we get that

$$|X \wedge V(diag^* A, diag^* P)(\mathbf{x})| \simeq |[q] \mapsto X_q \wedge V(A_q, P_q)(\mathbf{x})|$$

This means that $|\Omega^{\vee \mathbf{x}}(X \wedge V(diag^* A, diag^* P)(\mathbf{x}))| \simeq |[q] \mapsto \Omega^{\vee \mathbf{x}}(X_q \wedge V(A_q, P_q))|$ (the loop may be performed degreewise, see A.1.5.0.5), and by Bökstedt's approximation lemma II.2.2.3 we get that

$$|THH(diag^* A, diag^* P; X)_n| \simeq |[q] \mapsto THH(A_q, P_q; X_q)_n|$$

which gives the result. ∎

1.3.2 Relation to Hochschild homology (over the integers)

{IV:132}

Since, à priori Hochschild homology is a simplicial Abelian group, whereas topological Hochschild homology is a Γ-space, we could consider HH to be a Γ-space by the Eilenberg-MacLane construction $H \colon \mathcal{A} = s\mathcal{A}b \to \Gamma\mathcal{S}_*$ in order to have maps between them.

We make a slight twist to make the comparison even more straight-forward. Recall the definitions of $\bar{H} \colon \mathcal{A} = s\mathcal{A}b \to \Gamma\mathcal{A}$ II.1, and the forgetful functor $U \colon \mathcal{A} \to \mathcal{S}_*$ which is adjoint to the free functor $\tilde{\mathbf{Z}} \colon \mathcal{S}_* \to \mathcal{A}$ of II.1.3.1. By definition $H = U\bar{H}$. An $\bar{H}\mathbf{Z}$-algebra A is a monoid in $(\Gamma\mathcal{A}, \otimes, \bar{H}\mathbf{Z})$ (see II.1.4.3), and is always equivalent to $\bar{H}$ of a simplicial ring (II.2.2.5). As noted in the proof of corollary II.2.2.5 the loops and hocolim used to stabilize could be exchanged for their counterpart in simplicial abelian groups if the input has values in simplicial abelian groups. This makes possible the following definition (loops and homotopy colimits are in $\mathcal{A}$):

{Def:IVHHZ}

Definition 1.3.3 Let A be an $\bar{H}\mathbf{Z}$-algebra, P an A-bimodule, and $X \in ob\Gamma^o$. Define the simplicial abelian group

$$HH^{\mathbf{Z}}(A,P;X)_q = \underset{\mathbf{x}\in\mathcal{I}^{q+1}}{\underrightarrow{\text{holim}}}\,\Omega^{\vee\mathbf{x}}\left(\tilde{\mathbf{Z}}[X]\otimes P(S^{x_0})\otimes\bigotimes_{1\le i\le q}A(S^{x_i})]\right)$$

with simplicial structure maps as for Hochschild homology. Varying X and q, this defines $\underline{HH}^{\mathbf{Z}}(A,P)\in ob\Gamma\mathcal{A}$.

{IV:onHHZ}

Remark 1.3.4 *Again (sigh), should A not take flat values, we replace it functorially by one that does. One instance where this is not necessary is when $A = \tilde{\mathbf{Z}}B$ for some $\mathbf{S}$-algebra B. Note that a $\tilde{\mathbf{Z}}B$-module is a special case of a B-module via the forgetful map $U\colon \Gamma\mathcal{A}\to\Gamma\mathcal{S}_*$ (it is a B-module "with values in $\mathcal{A}$").*

If A is a simplicial ring and P an A-bimodule, $HH^{\mathbf{Z}}(\bar{H}A,\bar{H}P)$ is clearly (pointwise) equivalent to

$$\bar{H}(HH(A,P)) = \{X\mapsto HH(A,P;X) = HH(A,P)\otimes\mathbf{Z}[X]\}$$

For A an $\bar{H}\mathbf{Z}$-algebra and P an A-bimodule, there is an obvious natural map

$$THH(UA,UP)(X)\to UHH^{\mathbf{Z}}(A,P)(X)$$

given by sending $X\to\tilde{\mathbf{Z}}[X]$ and smash of simplicial abelian groups to tensor product (again, if A should happen to be nonflat, we should take a functorial flat resolution, and in this case the "map" is really the one described preceeded by a homotopy equivalence pointing in the wrong direction). This is generally far from being an equivalence (it is for general reasons always two-connected). If $P = A$ it is a cyclic map.

However, we may factor $THH(UA,UP)\to UHH^{\mathbf{Z}}(A,P)$ through a useful equivalence:

{lemma:IV

Lemma 1.3.5 *Let A be an $\mathbf{S}$-algebra, P a $\tilde{\mathbf{Z}}A$-bimodule and $X\in ob\Gamma^o$. The inclusion*

$$X\wedge P(S^{x_0})\wedge\bigwedge_{1\le i\le q}A(S^{x_i})\to\tilde{\mathbf{Z}}[X]\otimes P(S^{x_0})\otimes\bigotimes_{1\le i\le q}\tilde{\mathbf{Z}}[A(S^{x_i})]$$

induces an equivalence

$$THH(A,UP)\xrightarrow{\sim}UHH^{\mathbf{Z}}(\tilde{\mathbf{Z}}A,P).$$

Proof: It is enough to prove it degreewise. If $M\in s\mathcal{A}b$ is m-connected, and $Y\in\mathcal{S}_*$ is y-connected, then $M\wedge Y\to M\otimes\tilde{\mathbf{Z}}[Y]$ is $2m+y+2$ connected (by induction on the cells of Y: assume $Y = S^{y+1}$, and consider $M\to\Omega^{y+1}(M\wedge S^{y+1})\to\Omega^{y+1}M\otimes\tilde{\mathbf{Z}}[S^{y+1}]$. The composite is an equivalence, and the first map is $2m+1$ connected by the Freudenthal suspension theorem A.1.10.0.9). Setting $M = P(S^{x_0})$ and $Y = X\wedge\bigwedge_{1\le i\le q}A(S^{x_i})$ we get that the map is $2x_0-2+\sum_{i=1}^q(x_i-1)+conn(X)+2$ connected, and so, after looping down the appropriate number of times, $x_0 - q + conn(X)$ connected, which goes to infinity with x_0. ∎

In the future we may not always be as pedantic as all this. We will often supress forgetful functors, and write this as $THH(A,P) \xrightarrow{\sim} HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}A,P)$.

If A is an $\bar{H}\mathbf{Z}$-algebra and P an A-bimodule, this gives a factorization

$$THH(UA,UP) \xrightarrow{\sim} UHH^{\mathbf{Z}}(\tilde{\mathbf{Z}}UA,P) \to UHH^{\mathbf{Z}}(A,P).$$

mark 1.3.4}

Remark 1.3.6 *Two words of caution:*

{ZHAneqHZA}

1. *Note, that even if $P = A$, $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}A,P)$ is not a cyclic device.*
2. *Note that if A is a simplicial ring, then $\tilde{\mathbf{Z}}HA$ is not equal to $H\tilde{\mathbf{Z}}A$. We will discover an interesting twist to this when we apply these lines of thought to additive categories instead of rings (see section 2.4).*
3. *In view of the equivalence $H\mathbf{Z}\wedge A \simeq \tilde{\mathbf{Z}}A$, this result should be interpreted as a change of ground ring equivalence*

$$HH^{\mathbf{S}}(A,UP) \simeq UHH^{H\mathbf{Z}}(H\mathbf{Z}\wedge A,P)$$

More generally, if $k \to K$ is a map of $\mathbf{S}$-algebras, A a cofibrant k-algebra and P a $K\wedge_k A$-bimodule, then

$$HH^k(A,f^*P) \simeq f^*HH^K(K\wedge_k A,P)$$

where $f\colon A \cong k\wedge_k A \to K\wedge_k A$.

For comparison the following lemmas are important (see [95, 4.2])

lemma{1PWSS}

Lemma 1.3.7 *If A is a ring and P an A-bimodule, then there is a spectral sequence*

$$E^2_{p,q} = HH_p(A, \pi_q THH(\mathbf{Z},P;X);Y) \Rightarrow \pi_{p+q}THH(A,P;X\wedge Y).$$

Proof: For a proof, see [95].On a higher level, it is just the change of ground ring spectral sequence: let $k \to K$ be a map of commutative $\mathbf{S}$-algebras, A a K-algebra and P a $K\wedge_k A$ bimodule, and assume A and K cofibrant as k-modules, then

$$HH^k(A,P) \simeq HH^K(K\wedge_k A,P) \simeq HH^K(A,HH^k(K,P))$$

where by abuse of notation P is regarded as a bimodule over the various algebras in question through the obvious maps. ∎

{1.3.6}

Lemma 1.3.8 *If A be a ring and P an A-bimodule, then the map $THH(A,P) \to HH^{\mathbf{Z}}(A,P)$ (and all the other variants) is a pointwise equivalence after rationalization, and also after profinite completion followed by rationalization.*

Proof: In the proof of the spectral sequence of the previous lemma, we see that the edge homomorphism is induced from the map $\pi_*THH(A,P) \to \pi_*HH(A,P)$. From the calculation of $\pi_*THH(\mathbf{Z},P)$ we get that all terms in the spectral sequence above the base line are torsion groups of bounded order. Thus $\pi_jTHH(A,P) = \pi_jTHH(A,P)$ and $\pi_jHH^{\mathbf{Z}}(A,P) = \pi_jHH(A,P)$ differs at most by groups of this sort, and the homotopy groups of the profinite completions $THH(A,P)\hat{}$ and $HH^{\mathbf{Z}}(A,P)\hat{}$ also differ by torsion groups of bounded order, and hence we have an equivalence $THH(A,P)\hat{}_{(0)} \to HH^{\mathbf{Z}}(A,P)\hat{}_{(0)}$.

If we the reader prefers not to use the calculation of $THH(\mathbf{Z})$, one can give a direct proof of the fact that the fiber of $THH(A,P) \to HH(A,P)$ has homotopy groups of bounded order directly from the definition.

Sketch:

1. Enough to do it in each dimension.
2. As A and P are flat as abelian groups we may resolve each by free abelian groups (mult. plays no role), and so it is enough to prove it for free abelian groups.
3. Must show that $\tilde{\mathbf{Z}}[X]\wedge\tilde{\mathbf{Z}}[Y]\wedge Z \to \tilde{\mathbf{Z}}[X\wedge Y]\wedge Z$ has fiber whose homotopy is of bounded order in a range depending on the connectivity of X, Y and Z, and this follows as $H_*(K(\mathbf{Z},n))$ are finite in a range.

∎

1.4 THH is determined by its values on simplicial rings

{IV:appro

In this section we prove the analogous statement of theorem III.3.2.1 for THH.

Let A be an **S**-algebra. Recall the definition of the functorial cube $\mathcal{A} = \{S \mapsto (A)_S\}$ of **S**-algebras from III.3.1.8 whose nodes $(A)_S$ were all equivalent to simplicial rings by proposition III.3.1.10. In particular, the S'th node was obtained by applying the free-forgetful pair $(\tilde{\mathbf{Z}}, U)$ as many times as there are elements in S. The functor $S \mapsto (-)^n_S$ can clearly be applied to A bimodules as well, and $S \mapsto (P)^n_S$ will be a cube of $S \mapsto (A)^n_S$ bimodules.

We will need the following result about the smashing of cubes

{1.4.1}

Lemma 1.4.1 *be $id + x_i$ cartesian cubes for $i = 1, \ldots, n$. Then*

$$\mathcal{X} = \{S \mapsto \bigwedge_{1\leq i\leq n} \mathcal{X}^i_S\}$$

is $id + \sum_i x_i$ cartesian.

Proof: Note that each d-subcube of $\mathcal{X}$ can be subdivided into d-cubes, each of whose maps are the identity on all the smash factors but one. Each of these d-cubes are by induction $2\cdot id + \sum_i x_i - 1$-cocartesian, and so the d-subcube we started with was $2\cdot id + \sum_i x_i - 1$-cocartesian. ∎

{1.4.2}

Proposition 1.4.2 *Let $\mathcal{A}$ be an id-cartesian cube of* **S***-algebras, and $\mathcal{P}$ an id-cartesian cube of $\mathcal{A}$ bimodules (i.e., each $S \to T$ induces a map of natural bimodules $(\mathcal{A}_S, \mathcal{P}_S) \to (\mathcal{A}_T, \mathcal{P}_T)$) and X a k-connected space. Then $THH(\mathcal{A}, \mathcal{P}; X)$ is $id + k + 1$ cartesian.*

Proof: Since realization commutes with homotopy colimits, this will follow if we can prove that for each $q \geq 0$, $S \mapsto THH(\mathcal{A}_S, \mathcal{P}_S; X)_q$ is $2 \cdot id + k$ cocartesian.

For any $q \geq 0$ the lemma above tells us that

$$S \mapsto X \wedge \mathcal{P}_S(S^{x_0}) \wedge \bigwedge_{1 \leq i \leq q} \mathcal{A}_S(S^{x_i})$$

is $id + k + 1 + \sum_{i=0}^{q} x_i$ cartesian. Looping down the appropriate number of times, this is $id + k + 1$ cartesian, and so

$$S \mapsto THH(\mathcal{A}_S, \mathcal{P}_S; X)_q$$

is $id + conn(X) + 1$ cartesian. ∎

{theo:THH} {1.4.3}

Theorem 1.4.3 (THH) *Let A be an* **S***-algebra and P an A-bimodule. Then the natural map*

$$THH(A, P) \to \underleftarrow{\operatorname{holim}}_{S \in \mathcal{P} - \emptyset} THH((A)_S, (P)_S)$$

is an equivalence.

Proof: This is a direct consequence of the above proposition applied to

$$\mathcal{A} = \{S \mapsto (A)_S\} \text{ and } \mathcal{P} = \{S \mapsto (P)_S\}$$

since the hypotheses are satisfied by theorem III.3.1.9. ∎

This means that we can reduce many questions about THH of **S**-algebras to questions about THH of (simplicial) rings, which again often may be reduced to questions about integral Hochschild homology by means of the spectral sequence 1.3.7.

As an example of this technique consider the following proposition.

{1.4.4}

Proposition 1.4.4 *Let A be an* **S***-algebra and P an A-bimodule, then Morita invariance holds for THH, i.e., the natural map*

$$THH(A, P) \xrightarrow{\sim} THH(Mat_n A, Mat_n P)$$

is a pointwise equivalence. If B is another **S***-algebra and Q a B-bimodules, then THH preserves products, i.e., the natural map*

$$THH(A \times B, P \times Q) \xrightarrow{\sim} THH(A, P) \times THH(B, Q)$$

is a pointwise equivalence.

Proof: Since

$$\tilde{\mathbf{Z}}[Mat_n A(X)] \cong \tilde{\mathbf{Z}}[\prod_n \bigvee_n A(X)] \longleftarrow \tilde{\mathbf{Z}}[\bigvee_n \bigvee_n A(X)] \cong \oplus_n \oplus_n \tilde{\mathbf{Z}}[A(X)] \longleftarrow Mat_n \tilde{\mathbf{Z}} A(X)$$

and

$$\tilde{\mathbf{Z}}[A(X) \times B(Y)] \longleftarrow \tilde{\mathbf{Z}}[A(X) \vee B(Y)] \cong \tilde{\mathbf{Z}}[A(X)] \oplus \tilde{\mathbf{Z}}[B(Y)] \longleftarrow \tilde{\mathbf{Z}}[A(X)] \vee \tilde{\mathbf{Z}}[B(X)]$$

are stable equivalences, it is in view of theorem 1.4.3, enough to prove the corresponding statements for rings. Appealing to the spectral sequence of lemma 1.3.7 together with the easy facts that

$$\pi_q THH(\mathbf{Z}, M_n P) \cong M_n(\pi_q THH(\mathbf{Z}, P))$$

and

$$\pi_q THH(\mathbf{Z}, P \oplus Q) \cong \pi_q THH(\mathbf{Z}, P) \oplus \pi_q THH(\mathbf{Z}, Q)$$

it follows from the corresponding statements in Hochschild homology, see e.g. [74, page 17] (use that matrices (resp. products) of flat resolutions of are flat resolutions of matrices (resp. products)). ∎

There are of course direct proofs of these statements, and they are essentially the same as in [74, page 17], except that one has to remember that sums are just equivalent to products (not isomorphic), see e.g. [27].

1.5 An aside: A definition of the trace from the K-theory space to topological Hochschild homology for S-algebras

{MSRI trac

In the next section we will give a natural construction of the (Bökstedt–Dennis) trace on the categorical level. However, for those not interested in this construction we give an outline of the trace map construction as it appeared in the unpublished MSRI notes [120], and later in [**?**, **?**, 6] and also, some of the elements showing up in the general definitions make an early appearance in the one we are going to give below.

This is only a weak transformation, in the sense that we will encounter weak equivalences going the wrong way, but this will cause no trouble in our context. Indeed, such arrows pointing the wrong way, can always be rectified by changing our models slightly. Furthermore, as we present it here, this only gives rise to a map of spaces, and not of spectra. We give a quick outline at the end, of how this can be extended to a map of spectra.

For any **S**-algebra A we will construct a weak map from $BA^* = B\widehat{GL}_1(A)$, the classifying space of the monoid of homotopy units of A, to $THH(A)(S^0)$. Applying this to the **S**-algebras $Mat_n A$, we get weak maps from $B\widehat{GL}_n(A)$ to $THH(Mat_n A)(S^0) \simeq THH(A)(S^0)$.

The map produced will respect stabilization, in the sense that

$$\begin{array}{ccc}
B\widehat{GL}_n(A) & \longrightarrow & THH(Mat_nA)(S^0) \\
\downarrow & & \downarrow \\
B\widehat{GL}_n(A)\times B\widehat{GL}_1(A) & \longrightarrow & THH(Mat_nA)(S^0)\times THH(Mat_1A)(S^0) \\
\simeq\uparrow & & \simeq\uparrow \\
B((Mat_nA\times Mat_1A)^*) & \longrightarrow & THH(Mat_nA\times Mat_1A)(S^0) \\
\downarrow & & \downarrow \\
B\widehat{GL}_{n+1}(A) & \longrightarrow & THH(Mat_{n+1}A)(S^0)
\end{array}$$

commutes where the upper vertical maps are induced by the indentity on the first factor, and the inclusion of $1\in\widehat{M}_1(A)=THH_0(Mat_1A)$ into the second factor. (Note that the horizontal maps are just weak maps, and that some of the intermediate stages may not have the property that the upwards pointing map is an equivalence, but this does not affect the argument.) Stabilizing this with respect to n and take the plus construction on both sides to get a weak transformation from $B\widehat{GL}(A)^+$ to $\lim_{n\to\infty}THH(M_nA)^+\simeq THH(A)$.

1.5.1 Construction

If M is a monoid, we may use the free forgetful adjoint pair to form a functorial free simplicial resolution $F(M)\xrightarrow{\simeq}M$. This extends to a functorial free resolution of any simplicial monoid, and in particular of $A^*=\widehat{GL}_1(A)$. The forgetful functor from groups to monoids has a left adjoint $M\mapsto M^{-1}M=\lim_{\leftarrow}G$ where the limit is over the category of groups under M. In the case M is free this is gotten by just adjoining formal inverses to all generators, and the adjunction $M\to M^{-1}M$ induces a weak equivalence $BM\to B(M^{-1}M)$ ($|BM|$ is just a wedge of circles, and the "inverses" are already included as going the opposite way around any circle. Alternatively, consider the "fiber" of $M\subset M^{-1}M$, that is the category C of objects elements in $M^{-1}M$, and a single morphism $m\colon g\cdot m\to g$ for every $m\in M$ and $g\in M^{-1}M$. Now, C is obviously connected, and between any two objects there is at most one morphism, and so C is contractible.)

In the case of the simplicial monoids $F(M)$ we get a transformation $F(M)\to G(M)=F(M)^{-1}F(M)$. If M is a group-like, then corollary A.1.5.0.12 tells us that the natural map $M\to\Omega BM$ is a weak equivalence. Furthermore, if M is group-like, then so is $F(M)$, and the diagram

$$\begin{array}{ccc}
F(M) & \longrightarrow & G(M) \\
\simeq\downarrow & & \simeq\downarrow \\
\Omega BF(M) & \xrightarrow{\simeq} & \Omega BG(M)
\end{array}$$

tells us that $F(M)\to G(M)$ is an equivalence.

Now, for any category $\mathcal{C}$, the nerve $\mathbf{N}\mathcal{C}$ may be considered as a simplicial category whose objects in $\mathbf{N}_q\mathcal{C}$ are the elements in the ordinary nerve $B_q\mathcal{C} = \{c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q\}$ (see A.1.1.5), and morphisms simply diagrams (in $\mathcal{C}$) like

$$\begin{array}{ccccccc} c_0 & \longleftarrow & c_1 & \longleftarrow & \dots & \longleftarrow & c_q \\ \downarrow & & \downarrow & & & & \downarrow \\ d_0 & \longleftarrow & d_1 & \longleftarrow & \dots & \longleftarrow & d_q \end{array}$$

If all morphisms in $\mathcal{C}$ are isomorphisms (i.e., $\mathcal{C}$ is a groupoid), then the face and degeneracies are all equivalences of categories. Hence, for any functor X from categories to simplicial sets sending equivalences to weak equivalences, $X(\mathcal{C}) = X(N_0\mathcal{C}) \xrightarrow{\sim} X(N\mathcal{C})$.

Also, just as we extended Hochschild homology from rings to (small) $\mathcal{A}b$-categories in I.3.2, the cyclic bar construction can be extended from monoids to categories: If $\mathcal{C}$ is a category and P is a $\mathcal{C}$-bimodule we define the *cyclic nerve* $B^{cy}(\mathcal{C}, P)$ to be the space whose q-simplices are given as

$$B^{cy}_q(\mathcal{C}, P) = \coprod_{c_0, c_1, \dots, c_q \in ob\mathcal{C}} P(c_0, c_q) \times \prod_{i=1}^{q} \mathcal{C}(c_i, c_{i-1}).$$

In particular, if G is a (simplicial) group regarded as a one point category in the ordinary sense, then we have a chain $BG = obNG \longrightarrow B^{cy}NG \xleftarrow{\sim} B^{cy}G$ where the first map sends $x \in BG$ to $x = x = \cdots = x \in B^{cy}_q NG$ and the last map is the weak equivalence induced by the equivalences $G \to N_qG$.

Assembling this information, we have a diagram

$$\begin{array}{ccccccc} BM & \xleftarrow{\sim} & BFM & & B^{cy}FM & \xrightarrow{\sim} & B^{cy}M \\ & & \simeq\downarrow & & \simeq\downarrow & & \\ & & BGM & \longrightarrow B^{cy}NGM \xleftarrow{\sim} & B^{cy}GM & & \end{array}$$

where the marked arrows are weak equivalences if M is group-like, giving a weak map $BM \to B^{cy}M$.

Recall the notation T_0 and R from chapter II (T_0 was like THH_0 used as a "fibrant replacement" for **S**-algebras, and R takes a Γ-space and evaluates at $1_+ = S^0$). For any **S**-algebra A, we have a map $B^{cy}RT_0A \to THH(A)(S^0)$ given by

$$B^{cy}_q RT_0(A) = \prod_{0 \le i \le q} \underset{\overrightarrow{x_i \in I}}{\operatorname{holim}} \Omega^{x_i} A(S^{x_i}) \to \underset{\overrightarrow{\mathbf{x} \in I^{q+1}}}{\operatorname{holim}} \Omega^{\vee \mathbf{x}} \bigwedge_{0 \le i \le q} A(S^{x_i})$$

where the map simply smashes functions together.

Composing weak map $BA^* \to B^{cy}A^*$ from the diagram above with the cyclic nerve of the monoid map $A^* = \widehat{GL}_1(A) \to \widehat{M}_1 A(S^0) = RT_0(A)$ and $B^{cy}RT_0(A) \to THH(A)(S^0)$ we have the desired "trace map" $BA^* \to THH(A)(S^0)$. **End construction.**

If we insist upon having a transformation on the spectrum level, we may choose a Γ space approach as in [6]. The action on the morphisms is far from obvious, and we refer the reader to [6] for the details.

2 Topological Hochschild homology of $\Gamma\mathcal{S}_*$-categories.

Recall the definition of $\Gamma\mathcal{S}_*$-categories. They were just like categories, except that instead of just morphism sets $\mathcal{C}(c,d)$ we have morphism Γ-spaces $\underline{\mathcal{C}}(c,d)$, the unit is a map $\mathbf{S} \to \underline{\mathcal{C}}(c,c)$ and the composition is a map

$$\underline{\mathcal{C}}(c,d) \wedge \underline{\mathcal{C}}(b,c) \to \underline{\mathcal{C}}(b,d)$$

Rings are $\mathcal{A}b$-categories with one object, and $\mathbf{S}$-algebras are $\Gamma\mathcal{S}_*$-categories with one object, so just like the extension in I.3.2 of Hochschild homology to cover the case of $\mathcal{A}b$-categories, we define topological Hochschild homology of general $\Gamma\mathcal{S}_*$-categories.

{2.1.1}

Definition 2.0.2 Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category, and P a $\mathcal{C}$-bimodule. Define for each tuple $\mathbf{x} = (x_0, \dots, x_q) \in ob\Gamma^{q+1}$

$$V(\mathcal{C},P)(\mathbf{x}) = \bigvee_{c_0,\dots,c_q \in ob\mathcal{C}} P(c_0,c_q)(S^{x_0}) \wedge \bigwedge_{1\le i\le q} \underline{\mathcal{C}}(c_i,c_{i-1})(S^{x_i})$$

and for each $X \in ob\Gamma$ and $q \ge 0$

$$THH(\mathcal{C},P;X)_q = \underset{\overrightarrow{\mathbf{x}\in \mathcal{I}^{q+1}}}{\text{holim}}\, \Omega^{\vee \mathbf{x}}(X \wedge V(\mathcal{C},P)(\mathbf{x})).$$

This is a simplicial space as before. It is functorial in X, and we write $THH(\mathcal{C},P)$ for the corresponding Γ-object, and $\underline{T}(\mathcal{C},P;X)$ for the corresponding Ω-spectrum.

2.1 Functoriality

We see that $THH(-,-)$ (as well any of the other versions) is a functor of $\Gamma\mathcal{S}_*$-natural bimodules $(\mathcal{C},P)$ B.NBNB.

Example 2.1.1 The example $(\mathcal{C}^\vee, P^\vee)$ of II.1.6.3 in the case where $\mathcal{C}$ an additive category is a slight generalization of the case considered in [27, part 2]. Here $\underline{\mathcal{C}}^\vee(c,d) = H(\mathcal{C}(c,d))$, but $P^\vee(c,d) = H(P(c,d))$ only if P is "bilinear". The restriction that P has to be additive (i.e., send sums in the first variable to products) is sometimes annoying. For instance, this is the reason that the comparison between THH and the homology of categories is often not done in the category of all "bifunctors" $\mathcal{C}^o \times \mathcal{C} \to \mathcal{A}b$ (see ref)NBNB.

Note 2.1.2 Since $\Gamma\mathcal{S}_*$-categories are examples of what was called ring functors in [27], it is worth noting that our current definition of THH agrees with the old one. In fact, a $\Gamma\mathcal{S}_*$-category is simply a ring functor restricted to Γ^o considered as the discrete finite pointed simplicial sets. The distinction between $\Gamma\mathcal{S}_*$-categories and ring functors is inessential in that topological Hochschild homology does not see the difference, and so all the general statements in [27, part 1] carries over to the new setting.

2.1.3 Cyclic structure and fixed points under the circle action

{II2.1.5}

Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category and X a space. Then, as before, $THH(\mathcal{C};X) = THH(\mathcal{C},\mathcal{C};X)$ is a cyclic space.

We promised in subsection 1.2.7 that we would take a closer look at the $\mathbf{S^1}$-fixed points. We consider $THH(\mathcal{C};X)$ as a simplicial cyclic set, and so if we apply $\sin|-|$ in the cyclic direction we get a simplicial $\mathbf{S^1}$-space which we write $\sin|THH(\mathcal{C};X)|$. As explained in A3?NBNB., if Z is a cyclic set, then the $\mathbf{S^1}$-fixed points of $\sin|Z|$ is nothing but $\lim_{\leftrightarrows\Lambda^o} Z$, or more concretely, the set of zero-simplices $z \in Z_0$ such that $ts_0 z = s_0 z \in Z_1$. So, we consider the simplices in the space

$$THH(\mathcal{C};X)_0 = \underset{\overrightarrow{x\in\mathcal{I}}}{\operatorname{holim}}\, \Omega^x(X \wedge \bigvee_{c\in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^x))$$

whose degeneracy is invariant under the cyclic action. In dimension q

$$\underset{\overrightarrow{x\in\mathcal{I}}}{\operatorname{holim}}\, \Omega^x(X\wedge \bigvee_{c\in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^x))_q = \coprod_{x_0\leftarrow\cdots\leftarrow x_q\in\mathcal{I}} \mathcal{S}_*(S^{x_q}\wedge\Delta[q]_+, \sin|(X\wedge \bigvee_{c\in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^{x_q})|)_0$$

The degeneracy sends $(x_0 \leftarrow \cdots \leftarrow x_q, f\colon S^{x_q}\wedge\Delta[q]_+ \to \sin|X\wedge \bigvee_{c\in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^{x_q})|)$ to

$$\left(\begin{matrix} x_0 \leftarrow \cdots \leftarrow x_q \\ 0_+ = \cdots = 0_+ \end{matrix}, S^{x_q}\wedge S^0\wedge\Delta[q]_+ \to \sin|(X\wedge \bigvee_{c_0,c_1\in ob\mathcal{C}} \underline{\mathcal{C}}(c_0,c_1)(S^{x_q})\wedge\underline{\mathcal{C}}(c_1,c_0)(S^0))|\right)$$

where the map is determined by f and the unit map $S^0 = \mathbf{S}(S^0) \to \underline{\mathcal{C}}(c,c)(S^0)$. For this to be invariant under the cyclic action, we first see that we must have $x_0 = \cdots = x_q = 0_+$. So f is a q-simplex in $\sin|X\wedge \bigvee_{c\in ob\mathcal{C}} \underline{\mathcal{C}}(c,c)(S^0)| \cong \sin(|X|\wedge \bigvee_{c\in ob\mathcal{C}} |\underline{\mathcal{C}}(c,c)(S^0)|)$ such that

$$|\Delta[q]|_+ \xrightarrow{f} |X|\wedge \textstyle\bigvee_{c\in ob\mathcal{C}} |\underline{\mathcal{C}}(c,c)(S^0)| \longrightarrow |X|\wedge \textstyle\bigvee_{c\in ob\mathcal{C}} |\underline{\mathcal{C}}(c,c)(S^0)|\wedge|\underline{\mathcal{C}}(c,c)(S^0)|$$

is invariant under permutation, where the last map is induced by the unit map $\underline{\mathcal{C}}(c,c)(S^0) \cong \underline{\mathcal{C}}(c,c)(S^0)\wedge S^0 \to \underline{\mathcal{C}}(c,c)(S^0)\wedge\underline{\mathcal{C}}(c,c)(S^0)$. Hence f only takes the value of the unit and factors through $|\Delta[q]| \to \bigvee_{c\in ob\mathcal{C}} |X| \cong \sin|X|\wedge(ob\mathcal{C})_+$, i.e.,

$$\lim_{\leftrightarrows\Lambda^o} THH(\mathcal{C};X) \cong \sin|THH(\mathcal{C};X)|^{\mathbf{S^1}} \cong \bigvee_{c\in ob\mathcal{C}} \sin|X|$$

We may be tempted to say that $\bigvee_{c\in ob\mathcal{C}} X$ is the "S^1-fixed point space" of $THH(\mathcal{C};X)$ because this is so after applying $\sin|-|$ to everything.

NB: If G is a topological group and X a G-space, then $\sin(X^G) \cong (\sin X)^{\sin G}$. Likewise for homotopy fixed points (up to homotopy).

2.2 The trace

{trace:Ct

{Def:Denn

There is a map, the "Dennis trace map"

$$ob\mathcal{C} \longrightarrow THH(\mathcal{C})(S^0)_0 \xrightarrow{\text{degeneracies}} THH(\mathcal{C})(S^0)$$

sending $d \in ob\mathcal{C}$ to the image of the non-basepoint of the unit map

$$S^0 = S(S^0) \to \mathcal{C}(d,d)(S^0)$$

composed with the obvious map

$$\mathcal{C}(d,d)(S^0) \subseteq \bigvee_{c \in ob\mathcal{C}} \mathcal{C}(c,c)(S^0) \to \underset{\overrightarrow{x \in ob\mathcal{I}}}{\text{holim}}\, \Omega^x \bigvee_{c \in ob\mathcal{C}} \mathcal{C}(c,c)(S^x) = THH(\mathcal{C})(S^0)_0$$

In other words, in view of the discussion in 2.1.3 the trace is (almost: just misses the base point) the inclusion of the $\mathbf{S^1}$ fixed points.

2.3 Comparisons with the $\mathcal{A}b$-cases

e ab-cases}

The statements which were made for $\bar{H}\mathbf{Z}$-algebras in section 1.3.2. have their analoges for $\Gamma\mathcal{A}$-categories:

{2.3.1}

Definition 2.3.1 Let $\mathcal{C}$ be a $\Gamma\mathcal{A}$-category, P a $\mathcal{C}$-bimodule and $X \in ob\Gamma^o$. Consider the simplicial Abelian group

$$HH^{\mathbf{Z}}(\mathcal{C}, P; X)_q = \underset{\overrightarrow{\mathbf{x} \in \mathcal{I}^{q+1}}}{\text{holim}}\, \Omega^{\vee \mathbf{x}} \bigoplus_{c_0, \dots, c_q \in ob\mathcal{C}} \left(P(c_0, c_q)(S^{x_0}) \otimes \bigotimes_{1 \le i \le q} \mathcal{C}(c_i, c_{i-1})(S^{x_i}) \right)$$

where loop and homotopy colimit is performed in simplicial abelian groups and with face and degeneracies as in Hochschild homology. Varying q and X, this defines $HH^{\mathbf{Z}}(\mathcal{C}, P) \in ob\Gamma\mathcal{A}$.

This is natural in $\Gamma\mathcal{A}$-natural pairs $(\mathcal{C}, P)$ (and is prone to all the irritating nonsense about nonflat values).

The prime example come from ordinary $\mathcal{A}b$-categories: by using the Eilenberg-MacLane construction on every morphism group, an $\mathcal{A}b$-category $\mathcal{E}$ cal be promoted to a $\Gamma\mathcal{A}$-category $\tilde{\mathcal{E}}$ (see II.1.6.2.2). Similarly we promote an $\mathcal{E}$-bimodule P to an $\tilde{\mathcal{E}}$ bimodule $\tilde{P}$. Since this construction is so frequent (and often in typographically challenging situations) we commit the small sin of writing $THH(\mathcal{E}, P)$ when we really ought to have written $THH(\tilde{\mathcal{E}}, \tilde{P})$.

Also, as in 1.3.4 it is clear that if $\mathcal{C}$ is an $\mathcal{A}b$-category and P a $\mathcal{C}$-bimodule, then $HH^{\mathbf{Z}}(\mathcal{C}, P)$ is pointwise equivalent to $\bar{H}\,(HH(\mathcal{C}, P))$

The proofs of the following statements are the same as the proofs for lemma **??** and 1.3.7

{2.3.2}

Lemma 2.3.2 *Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category, P a $\tilde{\mathbf{Z}}\mathcal{C}$-bimodule and $X \in ob\Gamma^o$. The map $THH(\mathcal{C}, UP) \to UHH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\mathcal{C}, P)$ is an equivalence.* ∎

{2.3.3}

Lemma 2.3.3 *Let $\mathfrak{C}$ be an $\mathcal{A}b$-category and P an $\mathfrak{C}$-bimodule. Then there is a first quadrant spectral sequences*

$$E^2_{p,q} = HH^{\mathbf{Z}}_p(\mathfrak{C}, \pi_q THH(H\mathbf{Z}, HP; X); Y) \Rightarrow \pi_{p+q} THH(H\mathfrak{C}, HP; X \wedge Y).$$

2.4 Topological Hochschild homology calculates the homology of additive categories

{THH of ac

There is another fact where the $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}-, -)$-construction is handy, but which has no analogy for $\mathbf{S}$-algebras.

Let $\mathcal{C}$ be an $\mathcal{A}b$-category, and let P be a $\mathcal{C}$-bimodule (i.e., an $\mathcal{A}b$-functor $\mathcal{C}^o \otimes \mathcal{C} \to \mathcal{A}b$). Then, by the results of section 2.3 you have that

$$THH(H\mathcal{C}, HP) \simeq U\, HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\bar{H}\mathcal{C}, \bar{H}P), \text{ and } HH^{\mathbf{Z}}(\bar{H}\tilde{\mathbf{Z}}\mathcal{C}, \bar{H}P) \simeq \bar{H}\, HH(\tilde{\mathbf{Z}}\mathcal{C}P),$$

but $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\bar{H}\mathcal{C}, \bar{H}P)$ is vastly different from $HH^{\mathbf{Z}}(\bar{H}\tilde{\mathbf{Z}}\mathcal{C}, \bar{H}P)$. As an example on may note that $THH(\mathbf{Z}, \mathbf{Z})$ is not equivalent to $HH(\tilde{\mathbf{Z}}\mathbf{Z}, \mathbf{Z}) = HH(\mathbf{Z}[t, t^{-1}], \mathbf{Z})$.

However, for additive categories ($\mathcal{A}b$-categories with sum) something interesting happens. Let $\mathfrak{C}$ be an additive category, and consider it as a $\Gamma\mathcal{A}$-category through the construction II.1.6.3: $\mathfrak{C}^{\oplus}(c, d)(k_+) = \mathfrak{C}(c, \overset{k}{\oplus} d)$. Since $\mathfrak{C}$ is additive we see that there is a canonical isomorphism $\tilde{\mathfrak{C}} \cong \mathfrak{C}^{\oplus}$, but this may not be so with the bimodules: if M is a $\tilde{\mathbf{Z}}\mathfrak{C}$-bimodule (which by adjointness is the same as a $U\mathfrak{C}$-bimodule) we define the $\mathfrak{C}^{\oplus}$-bimodule $M^{\oplus}$ by the formula $M^{\oplus}(c, d)(k_+) = M(c, \overset{k}{\oplus} d)$. If M is "linear" in the second factor (i.e., M is actually a $\tilde{\mathbf{Z}}\mathfrak{C}^o \otimes \mathfrak{C}$-module) the canonical map $\tilde{M} \to M^{\oplus}$ is an isomorphism, but for more general cases it won't even be a weak equivalence.

Theorem 2.4.1 *Let $\mathfrak{C}$ be an additive category and let M be a $\mathfrak{C}^o \otimes \tilde{\mathbf{Z}}\mathfrak{C}$-module. Then there is a canonical equivalence*

$$THH(U\mathfrak{C}^{\oplus}, UM^{\oplus}) \simeq H\,(HH(\tilde{\mathbf{Z}}\mathfrak{C}, M)).$$

Proof: In this proof we will use the model $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}(\mathcal{C}^{\oplus}), M^{\oplus})$ instead of $THH(U\mathfrak{C}^{\oplus}, UM^{\oplus})$ (see lemma 2.3.2), and since both expressions are very special it is enough to prove that the cononical stabilization map $HH((\tilde{\mathbf{Z}}\mathfrak{C}, M) \to HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}(\mathcal{C}^{\oplus}), M^{\oplus})(1_+)$ is an equivalence. Since the functors in the statement are homotopy functors in M, it is enough to prove the theorem for projective M. But all projectives are retracts of sums of projectives of the standard type

$$P_{x,y}(-, -) = \mathfrak{C}(-, y) \otimes \tilde{\mathbf{Z}}\mathfrak{C}(x, -)$$

and hence it is enough to show that the higher homotopy groups vanish, and the map induces an isomorphism on π_0 for these projectives. For $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}(\mathcal{C}^{\oplus}), M^{\oplus})$ and $HH(\tilde{\mathbf{Z}}\mathfrak{C}, M)$ this vanishing comes from the "extra degeneracy" defined by means of

$$\begin{array}{ccc} P_{x,y}(c,d)(k_+) & = & \mathfrak{C}(c,x)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(y,\overset{k}{\oplus}d) \\ & & \Big\downarrow{\scriptstyle f\otimes|\sum g_i|\mapsto f\otimes|1_y|\otimes|\sum g_i|} \\ P_{x,y}(c,y)(1_+)\otimes\tilde{\mathbf{Z}}\mathfrak{C}^{\oplus}(y,\oplus d)(k_+) & = & \mathfrak{C}(c,x)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(y,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(y,\overset{k}{\oplus}d) \end{array}$$

(the vertical lines are supposed to remind the reader that whatever is inside these are considered as generators in a free abelian group). This defines a contracting homotopy

$$s_{q+1}\colon HH(\tilde{\mathbf{Z}}\mathfrak{C}, P_{x,y})_q \to HH(\tilde{\mathbf{Z}}\mathfrak{C}, P_{x,y})_{q+1},$$

and likewise for $HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\mathfrak{C}^{\oplus}, P^{\oplus}_{x,y})$.

On π_0 we proceed as follows. Notice that $\pi_0(HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\mathfrak{C}^{\oplus}, P^{\oplus}_{x,y})_0) \cong \bigoplus_{c\in ob\mathfrak{C}}\mathfrak{C}(c,y)\otimes\mathfrak{C}(x,c)$ (essentially the Hurewicz theorem: if M is an abelian group $\pi_0\lim_{\overrightarrow{k}}\Omega^k\tilde{\mathbf{Z}}(M\otimes\tilde{\mathbf{Z}}[S^k])\cong M$) and $\pi_0(HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\mathfrak{C}^{\oplus}, P^{\oplus}_{x,y})_1)\cong\bigoplus_{c,d\in ob\mathfrak{C}}\mathfrak{C}(c,y)\otimes\mathfrak{C}(x,d)\otimes\mathfrak{C}(d,c)$. Hence the map $\pi_0 HH(\tilde{\mathbf{Z}}\mathfrak{C}, P_{x,y})\to\pi_0 HH^{\mathbf{Z}}(\tilde{\mathbf{Z}}\mathfrak{C}^{\oplus}, P_{x,y})$ is the map induced by the map of coequalizers

$$\begin{array}{ccc} \bigoplus_c\mathfrak{C}(c,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(x,c) & \leftleftarrows & \bigoplus_{c,d}\mathfrak{C}(c,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(x,d)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(d,c) \\ \downarrow & & \downarrow \\ \bigoplus_c\mathfrak{C}(c,y)\otimes\mathfrak{C}(x,c) & \leftleftarrows & \bigoplus_{c,d}\mathfrak{C}(c,y)\otimes\mathfrak{C}(x,d)\otimes\mathfrak{C}(d,c) \end{array}.$$

But both these coequalizers are isomorphic to $\mathfrak{C}(x,y)$, as can be seen by the unit map $\mathfrak{C}(x,y)\to\mathfrak{C}(x,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(y,y)$ and the composition $\mathfrak{C}(c,y)\otimes\mathfrak{C}(x,c)\to\mathfrak{C}(x,y)$ (here the linearity in the first factor is crucial: the class of $f\otimes|g|\in\mathfrak{C}(c,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(x,c)$ equals the class of $fg\otimes|1_x|\in\mathfrak{C}(x,y)\otimes\tilde{\mathbf{Z}}\mathfrak{C}(x,x)$). ∎

Remark 2.4.2 *The proof of this theorem is somewhat delicate in that it stears a middle course between various variants. We used the nonlinearity in the second factor of M to reduce to the projectives $P_{x,y}$ where this nonlinearity gave us the contracting homotopy. We then used the linearity in the first factor to identify the π_0 parts. A more general statement is that $THH(U\mathfrak{C}^{\oplus}, UM^{\oplus})$ is $HH(\tilde{\mathbf{Z}}\mathfrak{C}, LM)$ where L is linearization in the second factor. This first/second factor asymmetry is quite unnecessary and due to the fact that we stabilize in the second factor only. We could dualize and stabilize in the first factor only (the opposite of an additive category is an additive category), or we could do both at once. We leave the details to the interested reader.*

Corollary 2.4.3 (Pirashvili-Waldhausen [95]) *Let A be a discrete ring and M a bimodule. Then there is a natural chain of weak equivalences connecting $THH(HA, HM)$ and (the Eilenberg-MacLane spectrum associated to) $HH(\tilde{\mathbf{Z}}\mathcal{P}_A, M)$, where $\mathcal{P}_A$ is the category of finitely generated projective modules, and M is considered as a $\mathcal{P}_A$-bimodule by setting $M(c,d) = \mathcal{P}_A(c,d)\otimes M$.*

2.5 General results

Many results are most easily proven directly for $\Gamma\mathcal{S}_*$-categories, and not by referring to a reduction to special cases. We collect a few which will be of importance.

2.5.1 THH respect equivalences

This is the first thing that we should check, so that we need not worry too much about choosing this or that model for our categories.

{2.5.2}

Lemma 2.5.2 *Let $F_0, F_1 \colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ be maps of $\Gamma\mathcal{S}_*$-natural bimodules, and X a space. If there is a natural isomorphism $\eta \colon F_0 \to F_1$, then the two maps*

$$F_0, F_1 \colon THH(\mathcal{C}, P)(X) \to THH(\mathcal{D}, Q)(X)$$

are homotopic.

Proof: We construct a homotopy $H \colon THH(\mathcal{C}, P)(X) \wedge \Delta[1]_+ \to THH(\mathcal{D}, Q)(X)$ as follows. If $\phi \in \Delta([q], [1])$ and $\mathbf{x} \in \mathcal{I}^{q+1}$ we define the map $H_{\phi,\mathbf{x}} \colon V(\mathcal{C}, P)(\mathbf{x}) \to V(\mathcal{D}, Q)(\mathbf{x})$ by sending the $c_0, \dots, c_q \in \mathcal{C}^{q+1}$ summand into the $F_{\phi(0)}(c_0), \dots, F_{\phi(q)}(c_q) \in ob\mathcal{D}$ summand via the maps

$$\mathcal{C}(c, d) \xrightarrow{(\eta_d^j)_*(\eta_c^{-i})^* F_0} \mathcal{D}(F_i(c), F_j(d))$$

for $i, j \in \{0, 1\}$ (and $P(c, d) \longrightarrow Q(F_0(c), F_0(d)) \xrightarrow{(\eta_d^j)_*(\eta_c^{-i})^*} Q(F_i(c), F_j(d))$) ∎

{2.5.3}

Corollary 2.5.3 (THH respects $\Gamma\mathcal{S}_*$-equivalences) *Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be $\Gamma\mathcal{S}_*$-equivalence of $\Gamma\mathcal{S}_*$-categories, P a $\mathcal{D}$ bimodule and X a space. Then*

$$THH(\mathcal{C}, F^*P)(X) \xrightarrow{\simeq} THH(\mathcal{D}, P)(X).$$

Proof: Let G be an inverse, and $\eta \colon 1_{\mathcal{C}} \xrightarrow{\cong} GF$ and $\epsilon \colon 1_{\mathcal{D}} \xrightarrow{\cong} FG$ the natural isomorphisms. Consider the (non commutative) diagram

$$\begin{array}{ccc} THH(\mathcal{C}, F^*P)(X) & \xrightarrow{\eta} & THH(\mathcal{C}, (FGF)^*P)(X) \\ \downarrow{\scriptstyle F} & \nwarrow{\scriptstyle G} & \downarrow{\scriptstyle F} \\ THH(\mathcal{D}, P)(X) & \xrightarrow{\epsilon} & THH(\mathcal{D}, (FG)^*P)(X) \end{array}$$

The corollary above states that we get a map homotopic to the identity if we start with one of the horizontal isomorphism and go around a triangle. ∎

Recall the notion of stable equivalences of $\Gamma\mathcal{S}_*$-categories II.2.4.1.

{2.5.4}

Lemma 2.5.4 (THH respects stable equivalences of $\Gamma\mathcal{S}_*$-categories) *Let $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ be a map of $\Gamma\mathcal{S}_*$-natural bimodules, and assume F is a stable equivalence of $\Gamma\mathcal{S}_*$-categories inducing stable equivalences*

$$P(c, c') \to Q(F(c), F(c'))$$

for every $c, c' \in ob\mathcal{C}$. Then F induces a pointwise equivalence

$$THH(\mathcal{C}, P) \to THH(\mathcal{D}, Q).$$

Proof: According to lemma II.2.4.2 we may assume that F is either a $\Gamma\mathcal{S}_*$-equivalence, or a stable equivalence inducing an identity on the objects. If F is a $\Gamma\mathcal{S}_*$-equivalence we are done by corollary 2.5.3 once we notice that the conditions on P and Q imply that $THH(\mathcal{C}, P) \to THH(\mathcal{C}, F^*Q)$ is a pointwise equivalence.

If F is a stable equivalence inducing the identity on objects, then clearly F induces a pointwise equivalence

$$THH(\mathcal{C}, P)_q \to THH(\mathcal{C}, F^*Q)_q \to THH(\mathcal{D}, Q)_q$$

in every simplicial degree q. ∎

2.5.5 A collection of other results

The approximation in 1.4 of THH of arbitrary **S**-algebras by means of THH simplicial rings also works, mutatis mutandis, for $\Gamma\mathcal{S}_*$-categories to give an approximation of any $\Gamma\mathcal{S}_*$-category in terms of $s\mathcal{A}b$-categories.

The proof of the following lemma is just as for **S**-algebras (lemma 1.3.1)

Lemma 2.5.6 *Let $\mathcal{C}$ be a simplicial $\Gamma\mathcal{S}_*$-category and M a $\mathcal{C}$-bimodule (i.e., $\{[q] \mapsto (\mathcal{C}_q, M_q)\}$ is a natural bimodule). Then there is a natural pointwise equivalence*

$$THH(diag^*\mathcal{C}, diag^*M) \simeq diag^*\{[q] \mapsto THH(\mathcal{C}_q, M_q)\}.$$

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Definition 2.5.7 Let $\mathcal{A}$ and $\mathcal{B}$ be $\Gamma\mathcal{S}_*$-categories and M an $\mathcal{A}^o - \mathcal{B}$-bimodule. Then the *upper triangular matrix* $\Gamma\mathcal{S}_*$-category

$$\left[\begin{smallmatrix}\mathcal{A} & M \\ & \mathcal{B}\end{smallmatrix}\right]$$

is the $\Gamma\mathcal{S}_*$ category with objects $ob\mathcal{A} \times ob\mathcal{B}$ and with morphism object from (a, b) to (a', b') given by

$$\begin{bmatrix}\mathcal{A}(a, a') & M(a, b') \\ & \mathcal{B}(b, b')\end{bmatrix}$$

and with obvious matrix composition.

a:UT in gs}

Lemma 2.5.8 *With the notation as in the definition, the natural projection*

$$THH\left(\left[\begin{smallmatrix}\mathcal{A} & M\\ & \mathcal{B}\end{smallmatrix}\right]\right) \to THH(\mathcal{A}) \times THH(\mathcal{B})$$

is a pointwise equivalence.

Proof: Exchange some products with wedges and do an explicit homotopy as in [27, 1.6.20].

For concreteness and simplicity, let's do the analogous statement for Hochschild homology of k-algebras instead, where k is a commutative ring: let A_{11} and A_{22} be k-algebras, and let A_{12} be an $A_{11}^o \otimes_k A_{22}$-module. The group of q simplices in $HH\left(\left[\begin{smallmatrix}\mathcal{A}_{11} & A_{12}\\ & \mathcal{A}_{22}\end{smallmatrix}\right]\right)$ can be written as

$$\bigoplus \bigotimes_{i=0}^{q} A_{r_i,s_i}$$

where the sum is over the set of all functions $(r,s)\colon \{0,1\dots,q\} \to \{(11),(12),(22)\}$. The projection to $HH(A_{11}) \oplus HH(A_{22})$ is split by the inclusion onto the summands where $r_0 = \dots r_q = s_0 = \dots = s_q$. We make a simplicial homotopy showing that the non-identity composite is indeed homotopic to the identity. Let $\phi \in \Delta([q],[1])$ and y in the (r,s) summand of the Hochschild homology of the upper triangular matices. With the convention that $s_{q+1} = r_0$ we set

$$H(\phi, y) = y, \text{ if } r_k = s_{k+1} \text{ for all } k \in \phi^{-1}(0)$$

and zero otherwise. We check that for $j \in [q]$ we have equalitity $d_j H(\phi,y) = H(\phi d^j, d_j y)$, and so we have a simplicial homotopy. Note that $H(1,-)$ is the identity and $H(0,-)$ is the projection ($r_0 = s_1, \dots r_{q-1} = s_q, r_q = s_0$ implies that all indices are the same due to the upper triangularity).

The general result is proven by just the same method, exchanging products with wedges to use the distributivity of smash over wedge, and keeping track of the objects (this has the awkward effect that you have to talk about nonunital issues. If you want to avoid this you can obtain the general case from the $\mathcal{A}b$-case by approximating as in 1.4). Alternatively you can steal the result from I.3.6 via the equivalences

$$THH(\mathfrak{C}) \simeq H\,HH(\tilde{\mathbf{Z}}\mathfrak{C},\mathfrak{C}) \simeq H\,HH(\mathbf{Z}\mathfrak{C},\mathfrak{C}) = F(\mathfrak{C},\mathfrak{C})$$

to get an only slightly weaker result. ∎

Setting M to be the trivial module you get that THH preserves products (or again, you may construct an explicit homotopy as in [27, 1.6.15] (replacing products with wedges). There are noe added difficulties with the bimodule statement.

{2.4.4}

Corollary 2.5.9 *Let $\mathcal{C}$ and $\mathcal{D}$ be $\Gamma\mathcal{S}_*$-categories, P a $\mathcal{C}$-bimodule, Q a $\mathcal{D}$-bimodule. Then the canonical map is a pointwise equivalence*

$$THH(\mathcal{C}\times\mathcal{D}, P\times Q) \to THH(\mathcal{C},P;X)\times THH(\mathcal{D},Q;X).$$ ☺

2.5.10 Cofinality

Another feature which is important is the fact that topological Hochschild homology is insensitive to *cofinal inclusions* (see below). Note that this is very different from the K-theory case where there is a significant difference between the K-theories of the finitely generated free and projective modules: $K_0^f(A) \to K_0(A)$ is not always an equivalence.

{2.5.7}

Definition 2.5.11 Let $\mathcal{C} \subseteq \mathcal{D}$ be a $\Gamma\mathcal{S}_*$-full inclusion of $\Gamma\mathcal{S}_*$-categories. We say that $\mathcal{C}$ is cofinal in $\mathcal{D}$ if for every $d \in ob\mathcal{D}$ there exist maps

$$d \xrightarrow{\eta_d} c(d) \xrightarrow{\pi_d} d$$

such that $c(d) \in ob\mathcal{C}$ and $\pi_d \eta_d = 1_d$.

{2.5.8}

Lemma 2.5.12 *Let $j\colon \mathcal{C} \subset \mathcal{D}$ be an inclusion of a cofinal $\Gamma\mathcal{S}_*$-subcategory. Let P be a $\mathcal{D}$-bimodule. Then*

$$THH(\mathcal{C}, P) \to THH(\mathcal{D}, P)$$

is a pointwise equivalence.

Proof: For simplicity we prove it for $P = \mathcal{D}$. For each $d \in ob\mathcal{D}$ choose

$$d \xrightarrow{\eta_d} c(d) \xrightarrow{\pi_d} d,$$

such that η_c is the identity for all $c \in ob\mathcal{C}$. Then for every $\mathbf{x} \in \mathcal{I}^{q+1}$ we have a map $V(\mathcal{D})(\mathbf{x}) \to V(\mathcal{C})(\mathbf{x})$ sending the $d_0, \dots, d_q \in U\mathcal{D}^{q+1}$ summand to the $c(d_0), \dots, c(d_q) \in U\mathcal{C}^{q+1}$ summand via

$$\mathcal{D}(\pi_{d_0}, \eta_{d_q})(S^{x_0}) \wedge \dots \wedge \mathcal{D}(\pi_{d_q}, \eta_{d_{q-1}})(S^{x_q})$$

This map is compatible with the cyclic operations and hence defines a map

$$D(\pi, \eta)\colon THH(\mathcal{D}) \to THH(\mathcal{C})$$

Obviously $D(\pi, \eta) \circ THH(j)$ is the identity on $THH(\mathcal{C})$ and we will show that the other composite is homotopic to the identity. The desired homotopy can be expressed as follows. Let $\phi \in \Delta([q], [1])$ and let

$$d \xrightarrow{\eta_d^i} c^i(d) \xrightarrow{\pi_d^i} d \qquad \text{be} \qquad \begin{cases} d \xrightarrow{\eta_d} c(d) \xrightarrow{\pi_d} d & \text{if } i = 1 \\ d = d = d & \text{if } i = 0 \end{cases}$$

The homotopy $THH(\mathcal{D}) \wedge \Delta[1]_+ \to THH(\mathcal{D})$ is given by $H_{\phi,\mathbf{x}}\colon V(\mathcal{D})(\mathbf{x}) \to V(\mathcal{D})(\mathbf{x})$ sending the $d_0, \dots, d_q \in obU\mathcal{D}^{q+1}$ summand to the $c^{\phi(0)}(d_0), \dots, c^{\phi(q)}(d_q) \in obU\mathcal{D}^{q+1}$ summand via

$$\mathcal{D}(\pi_{d_0}^{\phi(0)}, \eta_{d_q}^{\phi(q)})(S^{x_0}) \wedge \dots \wedge \mathcal{D}(\pi_{d_q}^{\phi(q)}, \eta_{d_{q-1}}^{\phi(q-1)})(S^{x_q}).$$

∎

2.5.13 Application to the case of discrete rings

As an easy application, we will show how these theorems can be used to analyze the topological Hochschild homology of a discrete ring. The more general case of **S**-algebras will be treated later.

Let A be a discrete ring, and let P be an A bimodule, and by abuse of notation let P also denote the $\mathcal{P}_A$-bimodule $Hom_A(-, - \otimes_A P) \cong \mathcal{P}_A(-,-) \otimes_A P \colon \mathcal{P}_A \times \mathcal{P}_A^o \to \mathcal{A}b$.

{2.5.10}

Lemma 2.5.14 *Let A be a ring, $\mathcal{P}_A$ the category of finitely generated projective modules (I.2.1.3) and $\mathcal{F}_A$ the category of finitely generated free modules (I.2.1.4). Then the inclusion $\mathcal{F}_A \subseteq \mathcal{P}_A$ induces a pointwise equivalence*

$$THH(\mathcal{P}_A) \xrightarrow{\sim} THH(\mathcal{F}_A).$$ ☺

In the statement of the theorem we have again used the shorthand of writing $THH(A)$ when we really mean $THH(HA)$, and likewise for $THH(\mathcal{P}_A)$.

{2.5.11}

Theorem 2.5.15 *The inclusion of A in $\mathcal{P}_A$ as the rank 1 free module induces a pointwise equivalence*

$$THH(A,P) \xrightarrow{\sim} THH(\mathcal{P}_A,P).$$

Proof: Let $\mathcal{F}_A$ be the category of finitely generated free modules, and let $\mathcal{F}_A^k$ be the subcategory of free modules of rank less than or equal to k. We have a cofinal inclusion $M_kA \to \mathcal{F}_A^k$, given by regarding M_kA as the subcategory with only object: the rank k module. Consider the diagram where the limit is taken with respect to inclusion by zeros

$$\begin{array}{ccc}
THH(A,P) & \xrightarrow[\simeq]{\text{Morita}} & \lim_{k\to\infty} THH(M_kA, M_kP) \\
\downarrow & & \simeq\downarrow \text{cofinality} \\
THH(\mathcal{F}_A,P) & \xleftarrow[\simeq]{\text{filtered colimits}} & \lim_{k\to\infty} THH(\mathcal{F}_A^k,P) \\
\simeq\downarrow \text{cofinality} & & \\
THH(\mathcal{P}_A,P) & &
\end{array}$$

The leftward pointing map is a weak equivalence as loops respect filtered colimits (A.1.1.7.3) and $V(\mathcal{F}_A,P)(\mathbf{x}) = \lim_{k\to\infty} V(\mathcal{F}_A^k,P)(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{I}^{q+1}$. The other maps are weak equivalences for the given reasons and the result follows. ∎

2.5.16 Topological Hochschild homology of an finitely genrated free modules over an S-algebra

The category of finitely generated A-modules $\mathcal{F}_A$ is the $\Gamma\mathcal{S}_*$-category whose objects are the natural numbers (including zero), and where the morphisms are given by

$$\mathcal{F}_A(k_+,l_+) = \mathcal{M}_A(k_+ \wedge A, l_+ \wedge A) \cong \prod_k \bigvee_l A$$

An A-bimodule P is considered as an $\mathcal{F}_A$-bimodule in the obvious way. Except that the cofinality is not needed in the present situation, exactly the same proof as for the discrete case above give:

{2.6.1}

Lemma 2.5.17 *Let A be an $\mathbf{S}$-algebra and P an A-bimodule. Then the inclusion of the rank one module $A \to \mathcal{F}_A$ gives rise to an equivalence*

$$THH(A, P) \to THH(\mathcal{F}_A, P).$$

☺

Chapter V

The trace $K \to THH$

{IV2}

In this chapter we explain how the Dennis trace map IV.2.2 can be lifted to a trace map from algebraic K-theory to topological Hochschild homology. We first concentrate on the $\mathcal{A}b$-case since this is somewhat easier. This case is however sufficent to define the trace for discrete rings, and carries all the information we need in order to complete our proofs. The general construction is more complex, but this need not really concern us: the only thing we actually use it for is that it exists and is as functorial as anybody can wish.

The general construction occupies the second section, and tries to reconcile this construction with the others we have seen. In the third section we have another look at stable K-theory and verify that it agrees with topological Hochschild homology for **S**-algebras in general.

1 THH and K-theory: the $\mathcal{A}b$-case

In this section we define the trace map from algebraic K-theroy to the topological Hochschild homology of an additive or exact category much as was done in [27].

Before we do so, we have to prepare the ground a bit, and since these results will be used later we work in a wider generality for a short while.

Algebraic K-theory is preoccupied with the weak equivalences, topological Hochschild homology with the enrichment. The Dennis trace map 2.2 should seek to unite these opposite points of view.

Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category (see II.3.1.3), and recall the construction $\bar{H}\mathcal{C}$ from chapter II. This is a functor from Γ^o to symmetric monoidal $\Gamma\mathcal{S}_*$-categories such that for each $k_+ \in ob\Gamma^o$ the canonical map

$$\bar{H}\mathcal{C}(k_+) \to \mathcal{C}^{\times k}$$

is a $\Gamma\mathcal{S}_*$-equivalence. Hence

$$THH(\bar{H}\mathcal{C})$$

is a functor from Γ^o to $\Gamma\mathcal{S}_*$ or more symmetrically: a functor $\Gamma^o\times\Gamma^o \to \mathcal{S}_*$. For such functors we have again a notion of stable equivalences: if X and Y are functors $\Gamma^o \times \Gamma^o \to \mathcal{S}_*$, a map $X \to Y$ is a *stable equivalence* if

$$\lim_{\overrightarrow{k,l}} \Omega^{k+l} X(S^k, S^l) \to \lim_{\overrightarrow{k,l}} \Omega^{k+l} Y(S^k, S^l)$$

is a weak equivalence.

For each $k_+ \in ob\Gamma^o$ there is a map $k_+ \wedge THH(\mathcal{C}) \to THH(\bar{H}\mathcal{C}(k_+))$ (induced by the k functors $\mathcal{C} \to \bar{H}\mathcal{C}(k_+)$ given by the injections $1_+ \to k_+$) which assemble to a natural map $\Sigma^\infty THH(\mathcal{C}) \to THH(\bar{H}\mathcal{C})$ of functors $\Gamma^o \to \Gamma\mathcal{S}_*$.

additives}

Proposition 1.0.1 *Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category. Then for each $l_+ \in \Gamma^o$ the Γ-space*

$$k_+ \to THH(\bar{H}\mathcal{C}(k_+))(l_+)$$

is special, and the natural map

$$\Sigma^\infty THH(\mathcal{C}) \to THH(\bar{H}\mathcal{C})$$

is a stable equivalence.

Proof: For each $k_+, l_+ \in ob\Gamma^o$ the map

$$THH(\bar{H}\mathcal{C}(k_+))(l_+) \to THH(\mathcal{C}^{\times k})(l_+)$$

is a weak equivalence (since $\bar{H}\mathcal{C}$ is special and THH sends $\Gamma\mathcal{S}_*$-equivalences to pointwise equivalences 2.5.4), and so is

$$THH(\mathcal{C}^{\times k})(l_+) \to THH(\mathcal{C})(l_+)^{\times k}$$

(since THH respects products 2.5.9), and so the first part of the proposition is shown: $THH(\bar{H}\mathcal{C})(l_+)$ is special. For each k_+, the composite

$$k_+ \wedge THH(\mathcal{C}) \longrightarrow THH(\bar{H}\mathcal{C}(k_+)) \longrightarrow THH(\mathcal{C})^{\times k}$$

is a stable equivalence, and the last map is a pointwise equivalence, hence the first map is a weak equivalence, assembling to the stated result. ∎

This is a special case of a more general statement below which is proved similarly. A functor $(\mathcal{C}, P)$ from Γ^o to $\Gamma\mathcal{S}_*$-natural bimodules is nothing but a functor $\mathcal{C} \colon \Gamma^o \to \Gamma\mathcal{S}_*$-categories and for each $X \in ob\Gamma^o$ a $\mathcal{C}(X)$-bimodule $P(X)$, such that for every $f \colon X \to Y \in \Gamma^o$ there is a map of $\mathcal{C}(X)$-bimodules $\bar{f} \colon P(X) \to f^*P(Y)$ such that $\overline{gf} = f^*(\bar{g}) \circ \bar{f}$. (i.e., if in addition $g \colon Y \to Z$, then the diagram

$$\begin{array}{ccc} P(X) & \xrightarrow{\bar{f}} & f^*P(Y) \\ & \searrow^{\overline{gf}} & \downarrow{\scriptstyle f^*(\bar{g})} \\ & & (gf)^*P(Z) = f^*(g^*P(Z)) \end{array}$$

commutes). In particular $(\mathcal{C}, \mathcal{C})$ will serve as the easiest example.

{3.2.1}
Proposition 1.0.2 *Let $(\mathcal{C}, P)$ be a functor from Γ^o to $\Gamma\mathcal{S}_*$-natural bimodules. Assume that $\mathcal{C}$ is quite special (see II.3.2.1) and for all $X, Y \in ob\Gamma^o$ the map*

$$P(X \vee Y) \xrightarrow{(\overline{pr_X}, \overline{pr_Y})} pr_X^* P(X) \times pr_Y^* P(Y)$$

is a stable equivalence of $\mathcal{C}(X \vee Y)$-bimodules. Then

$$THH(\mathcal{C}, P) \xleftarrow{\sim} \Sigma^\infty THH(\mathcal{C}(1_+), P(1_+))$$

is a stable equivalence. ☺

Preparing for the way for the trace from the algebraic K-theory of exact categories, we make the following preliminary nerve construction (a more worked-out version will be needed later, see 2.1.4 below, but this will do for now). Note the connections to the nerve construction used in the proof of corollary I.2.3.2. Recall that THH preserves $\Gamma\mathcal{S}_*$-equivalences (2.5.4), and that if $\mathcal{C}$ is an $\mathcal{A}b$-category, then the degeneracy map $\mathcal{C} = N_0(\mathcal{C}, i) \to N_q i(\mathcal{C}, i)$ is an $\mathcal{A}b$-equivalence of categories.

Definition 1.0.3 Let $\mathcal{C}$ be a category. The nerve of $\mathcal{C}$ with respect to the isomorphisms is the simplicial category $N(\mathcal{C}, i)$ whose simplicial set of object is the classifying space $Bi\mathcal{C}$ of the subcategory of isomorphisms, and whose set of morphisms between $c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q$ and $c'_0 \leftarrow c'_1 \leftarrow \cdots \leftarrow c'_q$ is the set of all commuting diagrams {Def:nerve}

$$\begin{array}{ccccccc} c_0 & \xleftarrow{\simeq} & c_1 & \xleftarrow{\simeq} & \dots & \xleftarrow{\simeq} & c_q \\ \downarrow & & \downarrow & & & & \downarrow \\ c'_0 & \xleftarrow{\simeq} & c'_1 & \xleftarrow{\simeq} & \dots & \xleftarrow{\simeq} & c'_q \end{array}$$

in $\mathcal{C}$.

Note that the vertical maps need not be isomorphisms. Furthermore we have that

Lemma 1.0.4 *For all q the map $\mathcal{C} = N_0(\mathcal{C}, i) \to N_q(\mathcal{C}, i)$ induced by the degeneracies (i.e., sending c to $c = c = \cdots = c$) is an equivalence of categories.*

Lastly, if $\mathcal{C}$ is an $\mathcal{A}b$-category, $N(\mathcal{C}, i)$ will be a simplicial $\mathcal{A}b$-category.

If $\mathcal{C}$ is an $\mathcal{A}b$-category we will abuse notation by writing $THH(\mathcal{C})$ when we really should have written $THH(\tilde{\mathcal{C}})$ (where the functor $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ from $\mathcal{A}b$-categories to $\Gamma\mathcal{S}_*$-categories of II.1.6.2.2 allows us to consider all $\mathcal{A}b$-categories as $\Gamma\mathcal{S}_*$-categories.

A consequence of the lemma is that if $\mathcal{C}$ is an $\mathcal{A}b$-category the map

$$THH(\mathcal{C}) \to THH(N(\mathcal{C}, i))$$

induced by the degeneracies becomes a pointwise equivalence (since the functor $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ sends $\mathcal{A}b$-equivalences to $\Gamma\mathcal{S}_*$-equivalences and THH sends $\Gamma\mathcal{S}_*$-equivalences to pointwise equivalences).

This paves the way for our first definition of the trace from algebraic K-theory to topological Hochschild homology:

{Def:trac

Definition 1.0.5 (The trace for additive categories) Let $\mathcal{E}$ be an additive category. The trace map for $\mathcal{E}$ in the Segal formalism is the following chain of natural transformations where the leftward pointing arrows are all stable equivalences

$$\Sigma^\infty Bi\bar{H}\mathcal{E} = \Sigma^\infty obN(\bar{H}\mathcal{E}, i) \longrightarrow THH(N(\bar{H}\mathcal{E}, i)) \xleftarrow{\sim} THH(\bar{H}\mathcal{E}) \xleftarrow{\sim} \Sigma^\infty THH(\mathcal{E})$$

where the first map is the Dennis trace of 2.2, the second is the equivalence coming from the equivalences of categories $\mathcal{E} \to N_q(\mathcal{E}, i)$ and the third from from lemma 1.0.1.

1.1 Doing it with the S construction

We may also use the S construction of Waldhausen (see definition I.2.2.1). This has simplicial exact categories as output, and we may apply THH degreewise to these categories.

If $\mathfrak{C}$ is an exact category and X a space, there is a map $S^1 \wedge THH(S^{(k)}\mathfrak{C}; X) \to THH(S^{(k+1)}\mathfrak{C}; X)$ (since $S_0\mathfrak{C}$ is trivial), and so $THH(\underline{\mathbf{S}}\mathfrak{C}; X) = \{k \mapsto THH(S^{(k)}\mathfrak{C}; X)\}$ defines a spectrum. It is proven in [27] that the adjoint

$$THH(S^{(k)}\mathfrak{C}; X) \to \Omega THH(S^{(k+1)}\mathfrak{C}; X)$$

is an equivalence for $k > 0$. Furthermore if $\mathfrak{C}$ is split exact, that is all short exact sequences split, then it is an equivalence also for $k = 0$. Note that any additive category can be viewed as a split exact category by choosing exactly the split exact sequences as the admissible exact sequences. In fact, if we apply the S construction to an additive category with no mention of exact sequences, this is what we mean.

1.1.1 Split exact categories

Let $\mathfrak{C}$ be an additive category. We defined the $n \times n$ *upper triangular matrices*, $T_n\mathfrak{C}$, in I.2.2.4, to be the category with objects $ob\mathfrak{C}^{\times n}$, and morphisms

$$T_n\mathfrak{C}((c_1, \dots, c_n), (d_1, \dots, d_n)) = \bigoplus_{1 \le j \le i \le n} \mathfrak{C}(c_i, d_j)$$

and with composition given by matrix multiplication. Since $\mathfrak{C}$ is additive, so is $T_n\mathfrak{C}$. Consider the two functors

$$\mathfrak{C}^{\times n} \to T_n\mathfrak{C} \to \mathfrak{C}^{\times n}.$$

The first is the inclusion of $\mathfrak{C}^{\times n}$ as the diagonal subcategory of $T_n\mathfrak{C}$, the second forgets about off-diagonal entries, and the composite is the identity.

{3.4.2}

Proposition 1.1.2 *Let $\mathfrak{C}$ be an additive category. Then the inclusion of the diagonal $\mathfrak{C}^{\times n} \to T_n\mathfrak{C}$ induces a pointwise equivalence*

$$THH(\mathfrak{C}^{\times n}) \to THH(T_n\mathfrak{C}).$$

Proof: Using the stable equivalence of products and wedges, we see that the map of $\Gamma\mathcal{S}_*$-categories

$$\begin{bmatrix}\mathfrak{C}^{\oplus} & (\mathfrak{C}^{\times n-1})^{\oplus}\\ & (T_{n-1}\mathfrak{C})^{\oplus}\end{bmatrix} \to (T_n\mathfrak{C})^{\oplus},$$

where the left hand category is defined in 2.5.7, is a stable equivalence. Hence the statement follows by induction on n from lemma 2.5.4 and lemma 2.5.8.

Alternatively you can steal the result from I.3.6 via the equivalences

$$THH(\mathfrak{C}) \simeq HHH(\tilde{\mathbf{Z}}\mathfrak{C},\mathfrak{C}) \simeq HHH(\mathbf{Z}\mathfrak{C},\mathfrak{C}) = F(\mathfrak{C},\mathfrak{C}).$$

∎

Considering the additive category $\mathfrak{C}$ as a split exact category, the forgetful map $T_n\mathfrak{C} \to \mathfrak{C}^{\times n}$ factors through $S_n\mathfrak{C}$

$$T_n\mathfrak{C} \to S_n\mathfrak{C} \to \mathfrak{C}^{\times n}$$

The first map is given by sending $(c_1,\dots,c_n)$ to $i \le j \mapsto c_{i+1}\oplus\dots\oplus c_j$, and the second projects $i\le j \mapsto c_{ij}$ onto $i \mapsto c_{i-1,i}$.

{3.4.3}

Corollary 1.1.3 *Let $\mathfrak{C}$ be a additive category. Then*

$$THH(\mathfrak{C}^{\times n}) \to THH(S_n\mathfrak{C})$$

is a pointwise equivalence, and so for every $X \in \Gamma^o$ the natural map

$$THH(\mathfrak{C};X) \to \Omega THH(S\mathfrak{C};X)$$

is a weak equivalence.

Proof: This follows by proposition 1.1.2 since by I.2.2.5 $T_n\mathfrak{C}$ is equivalent to $S_n\mathfrak{C}$, and THH sends equivalences to pointwise equivalences. ∎

Hence we get

{cor:IV TH

Corollary 1.1.4 *Let $\mathfrak{C}$ be an additive category. Then for every $k \ge 0$ the natural map $\bar{H}\mathfrak{C}(S^k) \to S^{(k)}\mathfrak{C}$ induces a pointwise equivalence*

$$THH(\bar{H}\mathfrak{C}(S^k)) \xrightarrow{\sim} THH(S^{(k)}\mathfrak{C}).$$

Substituting $\mathfrak{C}$ with $S^{(k)}\mathfrak{C}$ in corollary 1.1.3 we get

{3.4.4}

Corollary 1.1.5 *Let $\mathfrak{C}$ be an additive category. Then the natural map*

$$THH(S^k\mathfrak{C}) \to \Omega THH(S^{k+1}\mathfrak{C})$$

is a pointwise equivalence for all $k \ge 0$. ∎

and exactly the same proof give the

{3.4.5}

Corollary 1.1.6 *Let $\mathfrak{C}$ be an additive category, and M a bilinear $\mathfrak{C}$ bimodule. Then the natural map*

$$THH(S^k\mathfrak{C}, S^kM) \to \Omega THH(S^{k+1}\mathfrak{C}, S^{k+1}M)$$

is a pointwise equivalence for all $k \geq 0$. ∎

This allows us to define the trace used in [27], competing with the one we gave in **??**.

{Def:nerv

Definition 1.1.7 (The nerveless trace for split exact categories) Let $\mathcal{E}$ be an additive category. The trace map for $\mathcal{E}$ in the Waldhausen formalism is the following chain of natural transformations where the leftward pointing arrows are all stable equivalences

$$\Sigma^\infty ob\underline{\mathbf{S}}\mathcal{E} \longrightarrow THH(\underline{\mathbf{S}}\mathcal{E}) \overset{\sim}{\longleftarrow} \Sigma^\infty THH(\mathcal{E})$$

where the first map is the Dennis trace of 2.2, the second is the equivalence coming from from corollary 1.1.3.

whose zeroth space is equivalent to the usual $THH(A, P; X)$.

1.1.8 Comparison of traces for the Waldhausen and Segal approaches

As a last step, we need to know that the two definitions for the trace for additive categories agree.

This information is collected in the following commutative diagram of bispectra (the Γ-spaces are tacitly evaluated on spheres)

$$\begin{array}{ccccccc}
\Sigma^\infty ob\underline{\mathbf{S}}\mathcal{E} & & & \searrow & & & \\
{\scriptstyle 2.3.2}\downarrow\wr & & & & & & \\
\Sigma^\infty Bi\underline{\mathbf{S}}\mathcal{E} & \longrightarrow & \underline{T}(N(\underline{\mathbf{S}}\mathcal{E}, i)) & \overset{\sim}{\longleftarrow} & \underline{T}(\underline{\mathbf{S}}\mathcal{E}) & & \\
{\scriptstyle 2.1.6}\uparrow\wr & & \wr\uparrow & & \wr\uparrow & \overset{1.1.3}{\underset{\sim}{\nwarrow}} & \\
\Sigma^\infty Bi\bar{H}\mathcal{E} & \longrightarrow & \underline{T}(N(\bar{H}\mathcal{E}, i)) & \overset{\sim}{\longleftarrow} & \underline{T}(\bar{H}\mathcal{E}) & \underset{\sim}{\overset{1.0.1}{\longleftarrow}} & \Sigma^\infty\underline{T}(\mathcal{E})
\end{array}$$

where each number refer to the result showing that the corresponding arrow is a weak equivalence.

1.2 Comparison with the homology of an additive category and the S-construction

nd H trace}

One thing that needs clarification is the relationship with the homology of a category which we used in I.3, and which we showed was equivalent to stable K-theory when applied to an additive category. We used the S-construction there, and we use it here, and in both places the outcome are Ω-spectra, and these coincide. We use as a comparison tool the model for topological Hochschild homology by means of abelian groups discussed in section 2 (where it was called something else??)

{3.5.1}
Remark 1.2.1 *If $\mathfrak{C}$ is an additive category, and M an additive bimodule, we have strict equivalences of spectra (indexed by m)*

$$F(S^{(m)}\mathfrak{C}, S^{(m)}M) \xrightarrow{\sim} HH^{\mathbf{Z}}(S^{(m)}\mathfrak{C}, S^{(m)}M) \xleftarrow{\sim} THH(S^{(m)}\mathfrak{C}, S^{(m)}M).$$

We have two independent proofs that these spectra are Ω spectra. Furthermore, the maps

$$F_0(S^{(m)}\mathfrak{C}, S^{(m)}M) \xrightarrow{\sim} HH^{\mathbf{Z}}(S^{(m)}\mathfrak{C}, S^{(m)}M)_0 \xleftarrow{\sim} THH(S^{(m)}\mathfrak{C}, S^{(m)}M)_0$$

are also strict equivalences, and so all maps in

$$\begin{array}{ccccc} F_0(S^{(m)}\mathfrak{C}, S^{(m)}M) & \xrightarrow{\sim} & HH^{\mathbf{Z}}(S^{(m)}\mathfrak{C}, S^{(m)}M)_0 & \xleftarrow{\sim} & THH(S^{(m)}\mathfrak{C}, S^{(m)}M)_0 \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ F(S^{(m)}\mathfrak{C}, S^{(m)}M) & \xrightarrow{\sim} & HH^{\mathbf{Z}}(S^{(m)}\mathfrak{C}, S^{(m)}M) & \xleftarrow{\sim} & THH(S^{(m)}\mathfrak{C}, S^{(m)}M) \end{array}$$

are (stable) equivalences of spectra. ∎

1.3 More on the trace map $K \to THH$ for rings

For comparison with earlier constructions, it is often fruitful to give a slightly different view of the trace map, where the cyclic nerve plays a more prominent rôle.

Furthermore, the comparison with the map defining the equivalence between stable K-theory and topological Hochschild homology has not yet been seen to relate to the trace. This will be discussed further in the next section.

In this section we let $\mathbf{T}(A, P; X)$ be the Ω-spectrum

$$\{k \mapsto THH(S^{(k)}\mathcal{P}_A, S^{(k)}\mathcal{M}_A(-, - \otimes_A P); X).$$

Consider

$$\begin{array}{ccc} ob\underline{\mathbf{S}}\mathcal{P}_A & \xrightarrow{c \mapsto c = c} & \operatorname{holim}_{\overrightarrow{x \in I}} \Omega^x \bigvee_{c \in ob\underline{\mathbf{S}}\mathcal{P}_A} \underline{\mathbf{S}}\mathcal{P}_A(c,c) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[S^x] \\ & & \| \\ \mathbf{T}(A, A; S^0) & \xleftarrow{\text{degeneracies}} & \mathbf{T}_0(A, A; S^0) \end{array}$$

This map agrees with the trace map given in the previous section, and displays the map as the composite $\mathbf{K}(A) = \mathbf{T}(A)^{S^1} \subset \mathbf{T}(A)$ and so tells you that the S^1 action on THH is important. You do not expect to be able to calculate fixed point sets in general, and so any approximation to the fixed points which are calculable should be explored.

If one want maps from $Ni\underline{\mathbf{S}}\mathcal{P}_A$ instead, one can either do as we did in section 1.2, or one may rewrite this slightly (and so destroying the circle action, but that does not concern us right now). As for groups, there is a map $Bi\mathfrak{C} \to N^{cy}i\mathfrak{C}$ for any category $\mathfrak{C}$, given by sending $c_0 \xleftarrow{\alpha_1} c_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_q} c_q \in B_q i\mathfrak{C}$ to

$$c_q \xleftarrow{(\prod \alpha_i)^{-1}} c_0 \xleftarrow{\alpha_1} c_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_q} c_q \in N^{cy}_q i\mathfrak{C}$$

This splits the natural map $N^{cy}i\mathfrak{C} \to Bi\mathfrak{C}$ given by forgetting (which is there regardless of maps being isomorphisms). If $\mathfrak{C}$ is a linear category we have a map $N^{cy}i\mathfrak{C} \to N^{cy}\mathfrak{C} \to THH(\mathfrak{C})$, where the first one is given by the inclusion of the isomorphism into all of $\mathfrak{C}$, and the second is stabilization. The diagram

$$\begin{array}{ccc} Bi\mathfrak{C} & \longleftarrow & B_0 i\mathfrak{C} = ob\mathfrak{C} \\ \downarrow & & \downarrow \\ N^{cy}i\mathfrak{C} & \longrightarrow & THH(\mathfrak{C}) \end{array}$$

commutes, where the rightmost map is defined as above. Setting $\mathfrak{C} = S^{(m)}\mathcal{P}_A$ we obtain the diagram

$$\begin{array}{ccc} Bi\underline{\mathbf{S}}\mathcal{P}_A & \longleftarrow & ob\underline{\mathbf{S}}\mathcal{P}_A \\ \downarrow & & \downarrow{\scriptstyle tr} \\ N^{cy}i\underline{\mathbf{S}}\mathcal{P}_A & \longrightarrow & \mathbf{T}(A) \end{array} .$$

There is no contradiciton to be had from the fact that the diagram

$$\begin{array}{ccc} Bi\underline{\mathbf{S}}\mathcal{P}_A & \longrightarrow & \underline{T}(N(\underline{\mathbf{S}}\mathcal{P}_A, i)) \\ \downarrow & & \uparrow\sim \\ N^{cy}i\underline{\mathbf{S}}\mathcal{P}_A & \longrightarrow & \underline{T}(\underline{\mathbf{S}}\mathcal{P}_A) \end{array} .$$

does not commute.

1.4 The trace, and the K-theory of endomorphisms

Let $\mathfrak{C}$ be an exact category and let $End(\mathfrak{C})$ be the category of endomorphisms in $\mathfrak{C}$. That is, it is the exact category with objects (c, f), with $f\colon c \to c \in \mathfrak{C}$, and with morphisms $(c, f) \to (d, g)$ commuting diagrams

$$\begin{array}{ccc} c & \longrightarrow & d \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ c & \longrightarrow & d \end{array}$$

A sequence $(c', f') \to (c, f) \to (c'', f'')$ is exact if the underlying sequence $c' \to c \to c''$ is exact. We note that

$$obSEnd(\mathfrak{C}) \cong \coprod_{c \in obS\mathfrak{C}} End(c)$$

There are two functors $\mathfrak{C} \to End(\mathfrak{C})$ given by $c \mapsto \{\xrightarrow{0} c\}$ and $c \mapsto \{c = c\}$ splitting the forgetful projection $End(\mathfrak{C}) \to \mathfrak{C}$ given by $(c, f) \mapsto c$. We let

$$\mathbf{End}(\mathfrak{C}) = \bigvee_{c \in ob\underline{\mathbf{S}}\mathfrak{C}} End(c) \simeq fiber\{ob\underline{\mathbf{S}}End(\mathfrak{C}) \to ob\underline{\mathbf{S}}\mathfrak{C}\}$$

(the $End(c)$s are here pointed at the zero maps) and note that the first step in the trace, $ob\underline{\mathbf{S}}\mathfrak{C} \to \mathbf{T}(\mathfrak{C})_0$ facors through $ob\underline{\mathbf{S}}\mathfrak{C} \to \mathbf{End}(\mathfrak{C})$ via the map $c \mapsto c = c$.

If $\mathfrak{C} \subseteq \mathfrak{D}$ is cofinal (ref) then $End(\mathfrak{C}) \subseteq End(\mathfrak{D})$ is also cofinal, and a quick calculation tells us that $K_0(End(\mathfrak{D}))/K_0(End(\mathfrak{C})) \cong K_0(\mathfrak{D})/K_0(\mathfrak{C})$, and hence by [113] we get that $\mathbf{End}(\mathfrak{C}) \to \mathbf{End}(\mathfrak{D})$ is an equivalence. This tells us that the "strong" cofinality of THH appears at a very early stage in the trace; indeed before we have started to stabilize.

2 The general construction of the trace

In order to state the nerve in the full generality, it is necessary to remove the dependence on the enrichment in $\mathcal{A}b$ we have used so far. This is replaced by an enrichment in $\Gamma\mathcal{S}_*$ which is always present for categories with sum. The second thing we have to relax is our previous preoccupation with isomorphisms. In general this involves a choice of weak equivalences, but in order to retain full functoriality of our trace construction we choose to restrict to the case where the weak equivalences come as a natural consequence of the $\Gamma\mathcal{S}_*$-category structure. This is sufficient for all current applications of the trace, and the modifications one would want in other (typically geometric) applications are readily custom built from this.

2.1 The category of pairs $\mathfrak{P}$, nerves and localization

For applications to K-theory, one needs to consider categories with a given choice of weak equivalences. The weak equivalences are at the outset unaware of the enrichment of our categories: they only form a $\mathcal{S}$-category (just like the units in a ring only form a group, disregarding the additive structure). This is the reason we have to introduce pairs of categories.

Let the category of *free pairs* $\mathfrak{P}^{\text{free}}$ be the category whose objects are pairs $(\mathcal{C}, w)$ where $\mathcal{C}$ is a small $\Gamma\mathcal{S}_*$-category and $w\colon \mathcal{W} \to R\mathcal{C}$ an $\mathcal{S}$-functor of small $\mathcal{S}$-categories.

A morphism

$$(\mathcal{C}, w) \longrightarrow (\mathcal{C}', w')$$

in $\mathfrak{P}^{\text{free}}$ is a pair

$$F\colon \mathcal{C} \longrightarrow \mathcal{C}', \qquad G\colon \mathcal{W} \longrightarrow \mathcal{W}'$$

where F is a $\Gamma\mathcal{S}_*$-functor and G an $\mathcal{S}$-functor such that

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{w} & R\mathcal{C} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle RF} \\ \mathcal{W}' & \xrightarrow{w'} & R\mathcal{C}' \end{array}$$

commutes.

A *weak equivalence* in $\mathfrak{P}^{\text{free}}$ is a map

$$(\mathcal{C}, w) \xrightarrow{(F,G)} (\mathcal{C}', w')$$

such that F is a stable equivalence of $\Gamma\mathcal{S}_*$-categories and G is a weak equivalence of $\mathcal{S}$-categories.

2.1.1 The *object functor*

from the category of free pairs to sets is the functor ob which sends $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C})$ to the set of objects $ob\mathcal{W}$. If for some reason the objects in question are naturally pointed (as they will be in the applications to algebraic K-theory), we use the same letters for the functor into pointed sets.

2.1.2 The subcategories $\mathfrak{P}^{\text{fix}} \subseteq \mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$

Consider the full subcategory $\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$ whose objects are the pairs $(\mathcal{C}, w)$ that have the property that $w\colon \mathcal{W} \to R\mathcal{C}$ is the identity on objects. The subcategory of *fixed pairs* $\mathfrak{P}^{\text{fix}} \subseteq \mathfrak{P}$ contains all objects, but a morphism of fixed pairs is a morphism of pairs

$$(\mathcal{C}, w) \xrightarrow{(F,G)} (\mathcal{C}', w')$$

where F (and hence G) is the identity on objects.

These subcategories inherit the notion of weak equivalences from $\mathfrak{P}^{\text{free}}$ by requiring that the relevant maps lie in the appropriate category.

2.1.3 The right adjoint $\phi\colon \mathfrak{P}^{\text{free}} \to \mathfrak{P}$

The inclusion functor $\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$ has a right adjoint $\phi\colon \mathfrak{P}^{\text{free}} \to \mathfrak{P}$ given by sending $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C}) \in ob\mathfrak{P}^{\text{free}}$ to $\phi(\mathcal{C}, w) = (\phi_w\mathcal{C}, \phi w)$ each of whose factors are defined below.

The set of objects of $\phi(\mathcal{C}, w)$ is $ob\mathcal{W}$, and given two objects c and d the Γ-space of morphism is given by

$$\phi_w\mathcal{C}(c, d) = \mathcal{C}(wc, wd)$$

and ϕw is

$$\mathcal{W}(c, d) \xrightarrow{w} R\mathcal{C}(wc, wd) = R\phi_w\mathcal{C}(c, d).$$

Note that the composite

$$\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}} \xrightarrow{\phi} \mathfrak{P}$$

is the identity. When considered as an endofunctor on $\mathfrak{P}^{\text{free}}$ ϕ is idempotent ($\phi^2 = \phi$) and there is a natural transformation $\phi \to id_{\mathfrak{P}^{\text{free}}}$ given by the obvious map $\phi_w\mathcal{C} \to \mathcal{C}$ which is the identity on morphisms. Using this we get that ϕ is right adjoint to the inclusion as promised.

ichednerve}

2.1.4 The nerve

For each nonnegative integer q, let $[q] = \{0 < 1 < \cdots < q\}$, and consider it as the category $\{0 \leftarrow 1 \leftarrow \cdots \leftarrow q\}$. If $\mathcal{D}$ is a category, the nerve of $\mathcal{D}$ is the simplicial category which in dimension q is the category of functors $N_q\mathcal{D} = [[q], \mathcal{D}]$.

In contrast, if $\mathcal{W}$ is a $\mathcal{S}$-category, we get a bisimplicial category $N\mathcal{W}$ which in bidegree p, q is the functor category $(N_q\mathcal{W})_p = [[q], \mathcal{W}_p]$. Note that $N\mathcal{W}$ is **not** a simplicial $\mathcal{S}$-category since the set of objects may vary in both the p and q direction. However, it is convenient to consider $N\mathcal{W}$ as a bisimplicial $\mathcal{S}$-category with discrete morphism spaces. Likewise, if $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category, $N\mathcal{C}$ is the bisimplicial $\Gamma\mathcal{S}_*$-category $[p], [q] \mapsto [[q], \mathcal{C}_p]$. Both constructions are appropriately functorial.

Let the *free nerve*

$$N^{\mathrm{free}}\colon \mathfrak{P}^{\mathrm{free}} \to [\Delta^o \times \Delta^o, \mathfrak{P}^{\mathrm{free}}]$$

be the functor which sends $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C})$ to the bisimplicial pair $N^{\mathrm{free}}(\mathcal{C}, w)$ which in bidegree q, p is given by

$$N_q(\mathcal{C}, w)_p = (N_q\mathcal{C}_p, N_q\mathcal{W}_p \xrightarrow{N_q w} N_qR\mathcal{C}_p = RN_q\mathcal{C}_p)\cdot$$

More interestingly, we have the *nerve* N which is defined by the following diagram

$$\begin{array}{ccc} \mathfrak{P}^{\mathrm{free}} & \xrightarrow{N^{\mathrm{free}}} & [\Delta^o \times \Delta^o, \mathfrak{P}^{\mathrm{free}}] \\ \subseteq\big\uparrow & & \phi\big\downarrow \\ \mathfrak{P} & \xrightarrow{N} & [\Delta^o \times \Delta^o, \mathfrak{P}] \end{array} .$$

We will often write $(N^w\mathcal{C}, Nw)$ for the bisimplicial $\Gamma\mathcal{S}_*$-category $N(\mathcal{C}, w)$. Note that $obN(\mathcal{C}, w)$ is (the simplicial set of objects of) the usual degreewise nerve of $\mathcal{W}$.

2.1.5 The localization functor

In a pair $(\mathcal{C}, w)$ one may think of the map $w\colon \mathcal{W} \to R\mathcal{C}$ as an inclusion of a subcategory of "weak equivalences", and the purpose of the localization of [25] is to invert the weak equivalences. More precisely, the localization consists of two functors $L, B\colon \mathfrak{P} \to \mathfrak{P}$ connected by natural transformations

$$(\mathcal{C}, w) \longleftarrow B(\mathcal{C}, w) \longrightarrow L(\mathcal{C}, w)$$

where $L(\mathcal{C}, w)$ is the *localization* of $(\mathcal{C}, w)$ ("with respect to the weak equivalences"). This construction enjoys various properties listed in 2.4.2 below. First we recall the relevant definitions concerning groupoids.

2.1.6 Groupoids and groupoid-like pairs

Let $\mathcal{B}$ be a $\mathcal{S}$-category. The category $\pi_0\mathcal{B}$ is the category with the same objects as $\mathcal{B}$, but with morphism sets from a to b the path components $\pi_0\mathcal{B}(a, b)$.

We say that a $\mathcal{S}$-category $\mathcal{B}$ is *groupoid-like* if $\pi_0\mathcal{B}$ is a groupoid (i.e., all its morphisms are isomorphisms). We say that $\mathcal{B}$ is a *groupoid* if for every q the category $\mathcal{B}_q$ is a groupoid. A pair $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C}) \in \mathfrak{P}$ is called a *groupoid(-like) pair* (resp. a *groupoid-like pair*) if $\mathcal{W}$ is a groupoid (resp. groupoid-like). We say that a functor

$$(\mathcal{C}, w\colon \mathcal{W} \longrightarrow R\mathcal{C})\colon \Gamma^o \longrightarrow \mathfrak{P}$$

is *groupoid-like* if $\mathcal{W}(X)$ is groupoid-like for all $X \in \Gamma^o$.

2.1.7 Properties of the localization.

calization}

See [25] for further information. Given a pair $(\mathcal{C}, w) \in \mathfrak{P}$ the following is true.

1. The maps giving the natural transformations

$$(\mathcal{C}, w) \longleftarrow B(\mathcal{C}, w) \longrightarrow L(\mathcal{C}, w)$$

alization1} are in $\mathfrak{P}^{\mathrm{fix}}$.

2. alization2} $L(\mathcal{C}, w)$ is a groupoid pair.

3. alization3} $B(\mathcal{C}, w) \to (\mathcal{C}, w)$ is a weak equivalence.

4. If $(\mathcal{C}, w) \to (\mathcal{C}', w') \in \mathfrak{P}^{\mathrm{fix}}$ is a weak equivalence, then $L(\mathcal{C}, w) \to L(\mathcal{C}', w')$ is a weak equivalence. alization4}

5. alization5} If $(\mathcal{C}, w)$ is a groupoid-like pair, then $B(\mathcal{C}, w) \to L(\mathcal{C}, w)$ is a weak equivalences.

6. On the subcategory of $\mathfrak{P}^{\mathrm{fix}}$ of groupoid pairs $(\mathcal{C}, w)$ there is a natural weak equivalence

$$L(\mathcal{C}, w) \xrightarrow{\sim} (\mathcal{C}, w)$$

such that

$$\begin{array}{ccc} L(\mathcal{C}, w) & \longleftarrow & B(\mathcal{C}, w) \\ & \searrow & \downarrow \\ & & (\mathcal{C}, w) \end{array}$$

alization6} commutes.

2.1.8 A definition giving the K-theory of a symmetric monoidal $\Gamma\mathcal{S}_*$-category using the canonical choice of weak equivalences

We are ready for yet another definition of algebraic K-theory to be used in this book. This formulation uses the uniform choice of weak equivalences we made in section II.3.3. This is convenient, and covers most known examples, however we must be open to relax this if a given application is not of this sort.

Let

$$\mathfrak{k}\colon \text{symmetric monoidal } \Gamma\mathcal{S}_*\text{-categories} \longrightarrow [\Gamma^o, \mathfrak{P}]_*$$

(the $_*$ subscript means that the functors are pointed) be the composite

$$\begin{array}{c} \text{symmetric monoidal } \Gamma\mathcal{S}_*\text{-categories} \\ \bar{H}\big\downarrow \\ [\Gamma^o, \Gamma\mathcal{S}_*\text{-categories}]_* \\ T_0\big\downarrow \\ [\Gamma^o, \Gamma\mathcal{S}_*\text{-categories}]_* \\ \mathcal{C}\mapsto W(\mathcal{C})=(\mathcal{C},w_{\mathcal{C}})\big\downarrow \\ [\Gamma^o, \mathfrak{P}]_* \end{array}.$$

and let

$$\mathcal{K}\colon \text{symmetric monoidal } \Gamma\mathcal{S}_*\text{-categories} \longrightarrow [\Delta^o, [\Gamma^o, \mathfrak{P}]_*]$$

be the composite

$$\begin{array}{c} \text{symmetric monoidal } \Gamma\mathcal{S}_*\text{-categories} \\ \mathfrak{k}\big\downarrow \\ [\Gamma^o, \mathfrak{P}]_* \\ L\big\downarrow \\ [\Gamma^o, \mathfrak{P}]_* \\ N\big\downarrow \\ [\Delta^o, [\Gamma^o, \mathfrak{P}^{\text{free}}]_*] \\ \phi\big\downarrow \\ [\Delta^o, [\Gamma^o, \mathfrak{P}]_*] \end{array}$$

If $\mathcal{D}$ is a symmetric monoidal category, then we call $\mathcal{K}(\mathcal{D})$ the algebraic K-theory category of $\mathcal{D}$, whereas its objects $ob\mathcal{K}(\mathcal{D})$ is the algebraic K-theory spectrum of $\mathcal{D}$.

2.2 Redundancy in the definition of $\mathcal{K}$ from the point of view of algebraic K-theory

From $\mathcal{K}$ we get algebraic K-theory by applying the object functor, and then the extra stuff which is put there to make the morphisms right, can be peeled away.

Lemma 2.2.1 *Let $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C}) \in \mathfrak{P}$ and assume $\mathcal{W}$ is groupoid like. Then the natural maps*

$$ob\phi NL(\mathcal{C}, w) \longleftarrow ob\phi NB(\mathcal{C}, w) \longrightarrow ob\phi N(\mathcal{C}, w)$$

are weak equivalences.

Proof: This follows from the properties 2.4.2.2.3.5, 2.4.2.2.3.5 and 2.4.2.2.4.2 of the localization since by [29, 9.5] the nerve preserve weak equivalences of $\mathcal{S}(ob\mathcal{W})$-categories, and so the claim follows. ∎

Together with lemma II.3.3.2 this gives the following theorem justifying our claim that $\mathcal{K}$ measures algebraic K-theory. Recall that $Bi\bar{H}\mathcal{C}$ (the nerve of the isomorphisms of the Segal construction which we call $\bar{H}$) is one of the formulae for the algebraic K-theory, which we in III.2 compared with Waldhausen's construction and in III.3 compared with the plus construction.

Theorem 2.2.2 *Let $\mathcal{D}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category. Then $ob\mathcal{K}(\mathcal{D})$ is connected to $obN\mathfrak{k}(\mathcal{D}) = obN\mathcal{W}_{T_0\bar{H}\mathcal{D}}$ by a chain of natural weak equivalences. If $\mathcal{D}$ has stably fibrant morphism spaces, then this is naturally equivalent to $obN\mathcal{W}_{\bar{H}\mathcal{D}}$, and if it has discrete morphism spaces it is naturally equivalent to $obNi\bar{H}\mathcal{D}$.*

2.3 The cyclotomic trace

Due to the fact that our nerves give simplicial discrete $\Gamma\mathcal{S}_*$-categories as output, it is convenient to consider a $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ as a simplicial category enriched in Γ-sets $\{[q] \mapsto \mathcal{C}_q\}$ before applying THH, and we define

$$thh(\mathcal{C}) = \{[q] \mapsto THH(\mathcal{C}_q)\}.$$

By an argument just like the proof of lemma IV.1.3.1 regarding THH of simplicial objects we get that the canonical map

$$thh(\mathcal{C}) \xrightarrow{\sim} THH(\mathcal{C})$$

is a pointwise equivalence, and so it does not really matter.

We need these constructions to be functors on the level of pairs, and for $(\mathcal{C}, w) \in \mathfrak{P}$ we simply set

$$thh(\mathcal{C}, w) = thh(\mathcal{C}).$$

Consider the transformation

$$thh(N(\mathcal{C}, w)) \longleftarrow thh(N_0(\mathcal{C}, w)) =\!=\!= thh(\mathcal{C}, w)$$

induced by the degeneracies (note that the equality $thh(\mathcal{C}, w) = thh(N_0(\mathcal{C}, w)$ gets all messed up if we use THH instead of our equivalent thh).

{lem:IV3.3}

Lemma 2.3.1 *If $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C}) \in \mathfrak{P}$ is a groupoid pair, then the natural map*

$$thh(\mathcal{C}, w) \longrightarrow thh(N(\mathcal{C}, w))$$

is a pointwise equivalence.

Proof: Fix a simplicial dimension p. Note that since all maps in $\mathcal{W}_p$ are isomorphisms, the map induced by the degeneracy maps

$$(\mathcal{C}, w)_p = N_0(\mathcal{C}, w)_p \to N_q(\mathcal{C}, w)_p$$

gives an equivalence $\mathcal{C}_p = N_0^w\mathcal{C}_p \to N_q^w\mathcal{C}_p$ of (Γ-set)-categories for each $q \geq 0$. The statement follows immediately. ∎

Because of naturality, the same goes through, even though our applications will not be to individual pairs, but rather to functors

$$(\mathcal{C}, w)\colon \Gamma^o \to \mathfrak{P}$$

2.3.2 The Dennis trace and cyclotomic trace maps of $\Gamma\mathcal{S}_*$-categories

Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category. There is a map, the "Dennis trace map"

$$ob\mathcal{C} \longrightarrow thh(\mathcal{C}_0)(S^0)_0 \xrightarrow{\text{degeneracies}} thh(\mathcal{C})(S^0).$$

Here the first map is defined by sending $d \in ob\mathcal{C}$ to the image of the non-base point under the unit map

$$S^0 = 1_+ = \mathbf{S}(1_+) \longrightarrow \mathcal{C}_0(d, d)(1_+)$$

composed with the obvious map

$$\mathcal{C}_0(d, d)(S^0) \subseteq \bigvee_{c \in ob\mathcal{C}} \mathcal{C}_0(c, c)(S^0) \longrightarrow \left(\underset{\overrightarrow{x \in ob\mathcal{I}}}{\operatorname{holim}} \Omega^x \bigvee_{c \in ob\mathcal{C}} \mathcal{C}_0(c, c)(S^x)\right)_0 = thh(\mathcal{C}_0)(S^0)_0.$$

If $\mathcal{C}$ has an initial object, then the Dennis trace is a pointed map, and we get a map of Γ-spaces $\Sigma^\infty ob\mathcal{C} \to thh(\mathcal{C})$ given by the "assembly"

$$X \wedge ob\mathcal{C} \longrightarrow X \wedge thh(\mathcal{C})(S^0) \longrightarrow thh(\mathcal{C})(X).$$

Definition 2.3.3 The *THH-trace* of symmetric monoidal $\Gamma\mathcal{S}_*$-categories is the natural transformation of bispectra which to a symmetric monoidal $\Gamma\mathcal{S}_*$-category $\mathcal{D}$ gives the map

$$\Sigma^\infty ob\mathcal{K}(\mathcal{D}) \longrightarrow THH(\mathcal{K}(\mathcal{D}))$$

induced by the Dennis trace.

{theo:IV3

Theorem 2.3.4 *Let $\mathcal{D}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category. Then $THH(\mathcal{K}(\mathcal{D}))$ is naturally equivalent to $\Sigma^\infty THH(\mathcal{D})$.*

Proof: This is corollary 2.3.8 below. ∎

Before we prove theorem 2.3.4, we must make some preparations which will also be useful later in the paper. Let

$$(\mathcal{C}, w)\colon \Gamma^o \longrightarrow \mathfrak{P}$$

and consider the commutative diagram

2.3.5 Main Diagram

$$\begin{array}{ccccccc}
\Sigma^\infty obN(\mathcal{C},w) & \longrightarrow & THH(N(\mathcal{C},w)) & \longleftarrow & THH(\mathcal{C},w) & \xleftarrow{j} & \Sigma^\infty THH(\mathcal{C}(1_+)) \\
\simeq\uparrow & & \uparrow & & \simeq\uparrow & & \\
\Sigma^\infty obN(B(\mathcal{C},w)) & \longrightarrow & THH(N(B(\mathcal{C},w))) & \longleftarrow & THH(B(\mathcal{C},w)) & & \\
i_K\downarrow & & \downarrow & & i_{THH}\downarrow & & \\
\Sigma^\infty obN(L(\mathcal{C},w)) & \longrightarrow & THH(N(L(\mathcal{C},w))) & \xleftarrow{\simeq} & THH(L(\mathcal{C},w)) & &
\end{array}$$

aindiagram} We note that if $(\mathcal{C},w)=\mathfrak{k}(\mathcal{D})$ where $\mathcal{D}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category NBNBrefon$\mathfrak{k}$, the lower left hand horizontal map is exactly the cyclotomic trace $\Sigma^\infty ob\mathcal{K}(\mathcal{D})\to THH(\mathcal{K}\mathcal{D})$, and the top rightmost spectrum is $THH(T_0\mathcal{D})\simeq THH(\mathcal{D})$.

$$\begin{array}{ccccccc}
\Sigma^\infty obN\mathfrak{k}\mathcal{D} & \longrightarrow & THH(N\mathfrak{k}\mathcal{D}) & \longleftarrow & THH(\mathfrak{k}\mathcal{D}) & \xleftarrow{j} & \Sigma^\infty THH(T_0\mathcal{D}) \\
\simeq\uparrow & & \uparrow & & \simeq\uparrow & & \simeq\uparrow \\
\Sigma^\infty obN(B(\mathfrak{k}\mathcal{D})) & \longrightarrow & THH(N(B(\mathfrak{k}\mathcal{D}))) & \longleftarrow & THH(B(\mathfrak{k}\mathcal{D})) & & \Sigma^\infty THH(\mathcal{D}) \\
i_K\downarrow & & \downarrow & & i_{THH}\downarrow & & \\
\Sigma^\infty ob\mathcal{K}(\mathcal{D}) & \longrightarrow & THH(\mathcal{K}(\mathcal{D})) & \xleftarrow{\simeq} & THH(L(\mathfrak{k}\mathcal{D})) & &
\end{array}$$

Lemma 2.3.6 *The arrows marked with $\simeq$ in the main diagram 2.3.5 are stable equivalences.*

Proof: It is enough to consider the simpler case where $(\mathcal{C}, w\colon \mathcal{W}\to R\mathcal{C})$ is a single object in $\mathfrak{P}$ (and not a functor $\Gamma^o\to\mathfrak{P}$).

alization8} First consider the two maps induced by $B(\mathcal{C},w)\to(\mathcal{C},w)\in\mathfrak{P}^{\text{fix}}$. By . $B(\mathcal{C},w)\to(\mathcal{C},w)$ is a weak equivalence, giving the result since both $\Sigma^\infty obN$ and THH send weak equivalences to stable equivalences.

Lastly the map

$$THH(N(L(\mathcal{C},w)))\xleftarrow{\sim} THH(L(\mathcal{C},w))$$

is a stable equivalence by 2.3.1 since $L(\mathcal{C},w)$ is a groupoid pair. ∎

Theorem 2.3.7 *Let $(\mathcal{C},w)\colon\Gamma^o\to\mathfrak{P}$.*

1. *If $(\mathcal{C},w)$ is groupoid-like then the arrows marked i_K and i_{THH} in the main diagram 2.3.5 are stable equivalences of spectra.*
2. *If $\mathcal{C}$ is quite special, then the arrow marked j in the main diagram 2.3.5 is a stable equivalence.*

Proof: Assume that $(\mathcal{C},w)$ is groupoid-like. That i_K is a stable equivalence follows from alization5} lemma NBNBrefThat i_{THH} is a stable equivalence follows by . since THH preserves stable equivalences.

If $\mathcal{C}$ is quite special, j is a stable equivalence since THH preserves products and stable equivalences. ∎

cor:IV3.10}

Corollary 2.3.8 *If $\mathcal{D}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category, then the main diagram gives a natural chain of stable equivalences*

$$\begin{array}{ccccc}
 & & THH((\mathfrak{k}\mathcal{D}) & \xleftarrow[j]{\sim} \Sigma^\infty THH(T_0\mathcal{D}) \xleftarrow{\sim} & \Sigma^\infty THH(\mathcal{D}) \\
 & & \simeq\big\uparrow & & \\
 & & THH(B(\mathfrak{k}\mathcal{D})) & & \\
 & & \simeq\big\downarrow i_{THH} & & \\
THH(\mathcal{K}(\mathcal{D})) & \xleftarrow{\sim} & THH(L(\mathfrak{k}\mathcal{D})) & &
\end{array}$$

2.4 Weak cyclotomic trace

The price we have to pay for having a single map representing our cyclotomic trace is that the models of either side are more involved than their classical counterparts. At the cost of having to talk about weak transformations (some weak equivalences point in the "wrong" direction) this can be remedied by just exchanging the complicated models with their simpler, but equivalent cousins.

This is useful when we want to compare our definition to previous definitions of cyclotomic traces which were all examples of quite special groupoid-like pairs.

Definition 2.4.1

1. Let
$$(\mathcal{C}, w)\colon \Gamma^o \to \mathfrak{P}$$
be quite special and groupoid-like. Then the *weak cyclotomic trace* is the functorial weak composite $\Sigma^\infty obN(\mathcal{C}, w) \to \Sigma^\infty THH(\mathcal{C}(1_+))$ along the lower outer edge of the diagram 2.3.5 above.

2. If $\mathcal{D}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category, then the weak trace of $\mathcal{D}$ is the composite weak map
$$\Sigma^\infty obN\mathfrak{k}(\mathcal{D}) \longrightarrow \Sigma^\infty THH(\mathfrak{k}(\mathcal{D})(1_+)) = \Sigma^\infty THH(T_0\mathcal{D}) \xleftarrow{\sim} \Sigma^\infty THH(\mathcal{D})$$
where the leftmost weak map is the weak trace of $\mathfrak{k}(\mathcal{D})$ (which is quite special and groupoid-like).

Note that the only map in this weak trace of $\mathcal{D}$ that is not a weak equivalence is the cyclotomic trace $\Sigma^\infty ob\mathcal{K}(\mathcal{D}) \to THH(\mathcal{K}(D))$ of definition 3.5(recall that by definition $\mathcal{K}(\mathcal{D}) = NL\mathfrak{k}(\mathcal{D})$).

2.4.2 The quite special groupoid case

If $(\mathcal{C}, w) \in \mathfrak{P}$ is a groupoid pair, then lemma 2.3.1 says that

$$THH(N(\mathcal{C}, w)) \longleftarrow THH(\mathcal{C}, w)$$

is an equivalence, and we are free to consider the weak map

$$\Sigma^\infty obN(\mathcal{C}, w) \longrightarrow THH(N(\mathcal{C}, w)) \xleftarrow{\sim} THH(\mathcal{C}, w)$$

from the upper line of the main diagram 2.3.5. This gives rise to the simpler definition of the cyclotomic trace which was used in refNBNB. Compare Wald and Segal In our context we have to keep the nerves in place, and in view of the commutativity of

$$\begin{array}{ccc} obS\mathcal{E} & \longrightarrow & THH(S\tilde{\mathcal{E}})(S^0) \\ \simeq\uparrow & & \simeq\uparrow \\ obNiS\mathcal{E} & \longrightarrow & THH(N(S\tilde{\mathcal{E}}, iS\mathcal{E} \subseteq S\mathcal{E}))(S^0) \end{array}$$

the relevant translation is the following:

Definition 2.4.3 Let $\mathcal{E}$ be a symmetric monoidal $\mathcal{A}b$-category. The weak cyclotomic trace of $\mathcal{E}$ is the weak map

$$\Sigma^\infty obNi\bar{H}\mathcal{E} \longrightarrow THH(N(\bar{H}\tilde{\mathcal{E}}, i\bar{H}\mathcal{E} \subseteq \bar{H}\mathcal{E})) \xleftarrow{\sim} THH(\bar{H}\tilde{\mathcal{E}}) \xleftarrow{\sim} \Sigma^\infty THH(\tilde{\mathcal{E}})$$

obtained from the top row of the main diagram 2.3.5 with $(\mathcal{C}, w) = (\bar{H}\tilde{\mathcal{E}}, i\bar{H}\mathcal{E} \subseteq \bar{H}\mathcal{E})$.

The following gives the comparison between the weak traces. That two weak transformations "agree up to homotopy" can have many interpretations. We use the term for the equivalence relation generated by the relation gotten by saying that if

$$\begin{array}{ccccccccccc} A_0 & \longrightarrow & B_0 & \xleftarrow{\sim} & C_0 & \longrightarrow & \dots & \xleftarrow{\sim} & Y_0 & \longrightarrow & Z_0 \\ \| & & \simeq\downarrow & & \simeq\downarrow & & & & \simeq\downarrow & & \| \\ A_1 & \longrightarrow & B_1 & \xleftarrow{\sim} & C_1 & \longrightarrow & \dots & \xleftarrow{\sim} & Y_1 & \longrightarrow & Z_1 \end{array}$$

is a commutative diagram of natural transformations where the marked arrows are weak equivalences, then the top and the bottom row define weak transformations that agree up to homotopy.

prop:IV4.4}

Proposition 2.4.4 *Let $\mathcal{E}$ be a symmetric monoidal $\mathcal{A}b$-category. Then the weak cyclotomic trace of $\tilde{\mathcal{E}}$ precomposed with the map $\Sigma^\infty obNi\bar{H}\mathcal{E} \xrightarrow{\sim} \Sigma^\infty obN\mathfrak{t}\tilde{\mathcal{E}}$ agrees up to homotopy with the weak cyclotomic trace of $\mathcal{E}$.*

alization6} *Proof:* If we let $(\mathcal{C}, w) = (\bar{H}\tilde{\mathcal{E}}, i\bar{H}\mathcal{E} \subseteq \bar{H}\mathcal{E})$, in the main diagram 2.3.5, we have by . that there are natural vertical equivalences from the bottom to the top rows making everything commute

$$\begin{array}{ccccccc} \Sigma^\infty obN(\mathcal{C}, w) & \longrightarrow & THH(N(\mathcal{C}, w)) & \xleftarrow{\sim} & THH(\mathcal{C}, w) & \xleftarrow{\sim} & \Sigma^\infty THH(\mathcal{C}(1_+)) \\ \simeq\uparrow & & \uparrow & & \simeq\uparrow & & \\ \Sigma^\infty obN(L(\mathcal{C}, w)) & \longrightarrow & THH(N(L(\mathcal{C}, w))) & \xleftarrow{\simeq} & THH(L(\mathcal{C}, w)) & & \end{array}$$

The top row is the weak cyclotomic trace of $\mathcal{E}$ whereas going around the lower edge agrees up to homotopy with the weak cyclotomic trace of $(\mathcal{C}, w)$. But all nodes of the weak trace are homotopy invariants, and so the weak equivalence $(\mathcal{C}, w) \to \mathfrak{k}\tilde{\mathcal{E}}$ shows that the weak cyclotomic trace of $(\mathcal{C}, w)$ agrees up to homotopy with the weak cyclotomic trace of $\tilde{\mathcal{E}}$ precomposed with the map $\Sigma^\infty obNi\bar{H}\mathcal{E} \xrightarrow{\sim} \Sigma^\infty obN\mathfrak{k}\tilde{\mathcal{E}}$. ∎

2.5 The category of finitely generated A-modules

Let A be an $\mathbf{S}$-algebra, and consider the $\Gamma\mathcal{S}_*$-full subcategory of category of A-modules with objects $k_+ \wedge A$ for $k \geq 0$. More precisely we could equally characterize it as the $\Gamma\mathcal{S}_*$-category whose objects are the natural numbers (including zero), and where the morphisms are given by

$$\mathcal{F}_A(k_+, l_+) = \mathcal{M}_A(k_+ \wedge A, l_+ \wedge A) \cong \prod_k \bigvee_l A$$

This forms a symmetric monoidal $\Gamma\mathcal{S}_*$-category via the sum. Let

$$\mathfrak{k}(\mathcal{F}_A) = (\mathcal{C}_A, w_A) \colon \Gamma^o \to \mathfrak{P}$$

be the functor produced by the machinery of section 2: $\mathcal{C}_A = T_0\bar{H}\mathcal{F}_A$ and $w_A \colon \mathcal{W}_A \to R\mathcal{C}_A$ the pullback of $i\pi_0\mathcal{C}_A \to \pi_0\mathcal{C}_A \leftarrow R\mathcal{C}_A$.

By Morita equivalence NBNBref, the map $THH(\mathcal{F}_A) \leftarrow THH(A)$ induced by the inclusion of the rank one module is also a stable equivalence.

Definition 2.5.1 The algebraic K-theory of an $\mathbf{S}$-algebra A is the $\Gamma\mathcal{S}_*$

$$K(A) = obN\mathfrak{k}(\mathcal{F}_A)$$

and the *trace* for $\mathbf{S}$-algebras is the weak natural transformation

$$\Sigma^\infty K(A) \longrightarrow \Sigma^\infty THH(A)$$

given by the weak cyclotomic trace for $\mathcal{F}_A$ followed by the equivalence induced by the inclusion of the rank one module

$$THH(\mathcal{F}_A) \xleftarrow{\sim} THH(A).$$

Hence the only thing left to claim is that

1. this definition of K-theory agrees with the one in NBNBref, and
2. this definition of the cyclotomic trace agrees with the one in NBNBref.

Recall the definition of the group-like simplicial monoid $\widehat{GL}_k(A)$ where A is an $\mathbf{S}$-algebra as the pullback of $GL_k(\pi_0 A) \to M_k(\pi_0 A) \leftarrow RT_0 M_k A$.

Theorem 2.5.2 *There is a natural chain of weak equivalences*

$$\Omega^\infty ob\mathcal{K}(\mathcal{F}_A) \simeq \Omega^\infty K(A) \simeq K_0^f(\pi_0 A) \times B\widehat{GL}(A)^+.$$

Proof: The first weak equivalence follows from NBNBref. To simplify notation, let $W = \mathcal{W}_{T_0\bar{H}\mathcal{F}_A}$. Note that $K(A) \simeq obNW$. Since the associated spectrum is special

$$\Omega^\infty K(A) \simeq \Omega obNW(S^1),$$

and for each $n_+ \in \Gamma^o$ we have that $obNW(n_+) \simeq (obNW(1_+))^{\times n}$. For each $k \geq 0$, let W^k be the full subcategory of $W(1_+)$ whose only object is $k_+ \wedge A$. Note that by definition, this is nothing but $\widehat{GL}_k(A)$ considered as a simplicial category with only one object. Hence we are done, for by Segal [?] there is a chain of weak equivalences

$$\Omega obNW(S^1) \simeq K_0^f(\pi_0 A) \times \lim_{\overrightarrow{k}} (obNW^k)^+ = K_0^f(\pi_0 A) \times B\widehat{GL}(A)^+.$$

∎

Theorem 2.5.3 *The current definition of the weak cyclotomic trace agrees up to homotopy with the one given in NBNBref for rings.*

Proof: There are two steps to this. The first is to note that if A is a ring, then the definition we have given of the category of finitely generated HA-modules, agrees with the more down-to-earth definition of the category of finitely generated A-modules. More precisely, let $\mathcal{M}_A$ be the symmetric monoidal $\mathcal{A}b$-category whose objects are the natural numbers (including zero) , and where a morphism from m to n is an $n \times m$-matrix. Consider this as a $\Gamma\mathcal{S}_*$-category $\tilde{\mathcal{M}}_A$ as in NBNBref We see that there is a $\Gamma\mathcal{S}_*$-weak equivalence $\mathcal{F}_A \to \tilde{\mathcal{M}}_A$ given by sending $\vee$ to $\oplus$, and so also an equivalence

$$RT_0\mathcal{F}_A \xrightarrow{\sim} RT_0\tilde{\mathcal{M}}_A \xleftarrow{\sim} \mathcal{M}_A.$$

Hence the K theory and THH as given in this paper are naturally equivalent to the usual ones when we choose the weak equivalences to be the isomorphisms, since $\widehat{GL}_k(HA) \simeq GL_k(A)$.

The second thing we have to see is that the two definitions of the trace agree. Since the model we have for finitely generated free HA-modules is equivalent to $\tilde{\mathcal{M}}_A$, this follows from proposition 2.4.4. ∎

3 Stable K-theory and topological Hochschild homology.

In this section we are going to give a proof of Goodwillies conjecture $K^S \simeq THH$ for **S**-algebras. For rings, this is almost clear already, but for **S**-algebras we need to know that some of the maps used in the ring case have their analog in the **S**-algebra world. These considerations runs parallel with a need which will be apparent in chapter IV, namely: we need to know what consequences the equivalence $K^S \simeq THH$ has for the trace map.

3.1 Stable K-theory

Recall the discussion of stable K-theory in (ref?). The discussion in the previous chapter give that

{4.1.1}

Theorem 3.1.1 *Let A be a simplicial ring and P a simplicial A bimodule. Then*

$$\mathbf{K}^S(A,P)\simeq \mathbf{T}(A,P)$$

and the equivalence is induced by

$$\mathbf{K}^S(A,P)\simeq D_1\mathsf{C}_A(P)\cong \mathbf{T}(A,P)_0\xrightarrow{\simeq}\mathbf{T}(A,P)$$

and this is compatible with the definition for the $\mathbf{F}$ *construction.*

Proof: As both K-theory (of radical extensions) and THH may be computed degreewise we may assume that A and P are discrete. Then the only thing which need verification is the compatibility. Recall that the equivalence $\mathbf{K}^S(A,P)\simeq\mathbf{F}(A,P)$ was given by

$$D_1\mathsf{C}_A(-)\xrightarrow{\simeq}D_1\mathbf{F}_0(A,-)\xleftarrow{\simeq}\mathbf{F}_0(A,-)\xrightarrow{\simeq}\mathbf{F}(A,-)$$

Now considering the diagram

$$\begin{array}{ccccc}
D_1\mathsf{C}_A(P) & \xrightarrow{\cong} & \mathbf{T}(A,P)_0 & \longrightarrow & \mathbf{T}(A,P)\\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow\\
D_1\mathbf{F}_0(A,-)(P) & \xrightarrow{\cong} & \mathbf{R}(A,P)_0 & \longrightarrow & \mathbf{R}(A,P)\\
 & & \simeq\uparrow & & \simeq\uparrow\\
 & & \mathbf{F}_0(A,P) & \xrightarrow{\simeq} & \mathbf{F}(A,P)
\end{array}$$

where $\mathbf{R}(A,P)$ represents the spectrum $R(\underline{\mathbf{S}}\mathcal{P}_A,\underline{\mathbf{S}}\mathcal{M}_A(-,-\otimes_A P))$ and R is as in (ref?) we see that the equivalences are compatible. ∎

3.2 THH of split square zero extensions

Let A be an $\mathbf{S}$-algebra and P an A bimodule. We want to study $THH(A\vee P)$ closer. In the ring case, we see that $A\vee P\to A\ltimes P$ is a stable equivalence of $\mathbf{S}$-algebras, and so $A\vee P$ will cover all the considerations for split square zero extensions of rings.

The first thing one notices, is that the natural distributivity of smash and wedge give us a decomposition of $THH(A\vee P;X)$, or more precisely a decomposition of $V(A\vee P)(\mathbf{x})$ for every $\mathbf{x}\in\mathcal{I}$, as follows. Let

$$V^{(j)}(A,P)(\mathbf{x})=\bigvee_{\phi\in\Delta_m([j-1],[q])}\ \bigwedge_{0\leq i\leq q}F_{i\phi}(x_i)$$

where

$$F_{i,\phi}(x) = \begin{cases} A(S^x) & \text{if } i \notin im\phi \\ P(S^x) & \text{if } i \in im\phi \end{cases}$$

Then

$$V(A \vee P)(\mathbf{x}) \cong \bigvee_{j \geq 0} V^{(j)}(A,P)(\mathbf{x})$$

(note that $V^{(j)}(A,P)(\mathbf{x}) = *$ for $j > q+1$). Set

$$THH^{(j)}(A,P;X)_q = \underrightarrow{\text{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \Omega^{\vee\mathbf{x}}(X \wedge V^{(j)}(A,P)(\mathbf{x}))$$

and

$$\underline{T}^{(j)}(A,P;X) = \{k \mapsto THH^{(j)}(A,P;S^k \wedge X)\}$$

We see that this define cyclic objects (the transformations used to define THH respect the number of occurrences of the bimodule), when varying q. The inclusions and projections

$$V^{(j)}(A,P)(\mathbf{x}) \subseteq V(A \vee P)(\mathbf{x}) \to V^{(i)}(A,P)(\mathbf{x})$$

define cyclic maps

$$\bigvee_{j\geq 0} THH^{(j)}(A,P;X) \to THH(A \vee P;X) \to \prod_{j \geq 0} THH^{(j)}(A,P;X)$$

The approximation lemma assures us that

$$\underrightarrow{\text{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \prod_{j\geq 0} \Omega^{\vee\mathbf{x}}(X \wedge V^{(j)}(A,P)(\mathbf{x})) \to \prod_{j\geq 0} THH^{(j)}(A,P;X)_q$$

is an equivalence. In effect, the cyclic map

$$THH(A \vee P;X) \xrightarrow{\sim} \prod_{0 \leq j} THH^{(j)}(A,P;X)$$

is a weak equivalence (more is true, see...). As $THH^{(j)}(A,P;X)$ is $j-1$ reduced, the product is equivalent to the weak product, and so both maps in

$$\bigvee_{j\geq 0} \underline{T}^{(j)}(A,P;X) \to \underline{T}(A \vee P;X) \to \prod_{j\geq 0} \underline{T}^{(j)}(A,P;X)$$

are equivalences.

If P is $k-1$ connected and X is $m-1$ connected, we see that $THH^{(j)}(A,P;X)$ is $jk+m-1$ connected, and so

$$THH(A \vee P;X) \to THH(A;X) \times THH^{(1)}(A,P;X)$$

is $2k+m$ connected. This means that the space $THH^{(1)}(A,P;X)$ merits special attention as a first approximation to the difference between $THH(A \vee P;X)$ and $THH(A;X)$.

3.3 Free cyclic objects

Let $\mathcal{C}$ be a category with finite sums. Then the forgetful functor from cyclic $\mathcal{C}$ objects to simplicial $\mathcal{C}$ objects have a left adjoint, the free cyclic functor N^{cy} defined as follows (see [129]: it is just yet another example of the fact that Hochschild homology can be defined in any monoidal category: this time in sets). If $\phi \in \Lambda$ we can write $\tau^{-s}\phi\tau^{s} = \psi\tau^{r}$ in a unique fashion with $\psi \in \Delta$. If X is a simplicial object, $N^{cy}X$ is given in dimension q by $\coprod_{C_{q+1}} X_q$, and with ϕ^* sending x in the $s \in C_{q+1}$ summand to $\psi^* x$ in the $r+s$th summand.

Example: if X is a constant simplicial object, then this is $S^1_+ \otimes X$.

Lemma 3.3.1 Lemma 3.3.2 *4.2.2 The map adjoint to the inclusion*

$$N^{cy}\underline{T}(A,P;X) \to \underline{T}^{(1)}(A,P;X)$$

is an equivalence. More precisely, if P is $k-1$ connected and X is $m-1$ connected, then

$$N^{cy}THH(A,P;X) \to THH^{(1)}(A,P;X)$$

is a $2k+2m$ connected cyclic map.

Proof: Note that, $V(A,P)(\mathbf{x}) \subseteq V^{(1)}(A,P)(\mathbf{x})$ defines the summand in which the P appears in the zeroth place. There are q other possibilities for placing P, and we may encode this by defining the map

$$C_{q+1_+} \wedge THH(A,P;X)_q \to THH^{(1)}(A,P;X)_q$$

taking $t^i \in C_{q+1}$, $\mathbf{x} \in \mathcal{I}^{q+1}$ and $f\colon S^{\vee\mathbf{x}} \to X \wedge V(A,P)(\mathbf{x})$ and sending it to

$$t^i\mathbf{x}, \qquad \begin{array}{ccccc} S^{\vee t^i\mathbf{x}} & & & & X\wedge V^{(1)}(A,P)(t^i\mathbf{x}) \\ \cong\uparrow & & & & \uparrow \\ S^{\vee\mathbf{x}} & \xrightarrow{f} & X\wedge V(A,P)(\mathbf{x}) & \xrightarrow{\subseteq} & X\wedge V^{(1)}(A,P)(\mathbf{x}) \end{array}$$

Varying q, this is the cyclic map

$$N^{cy}THH(A,P;X) \to THH^{(1)}(A,P;X)$$

Let $V^{(1,i)}(A,P)(\mathbf{x}) \subset V^{(1)}(A,P)(\mathbf{x})$ be the summand with the P at the ith place. The map may be factored as

$$\begin{array}{ccc} \bigvee_{t^i\in C_{q+1}} \underrightarrow{\operatorname{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \Omega^{\vee\mathbf{x}}(X\wedge V(A,P)(\mathbf{x})) & \xrightarrow{\cong} & \underrightarrow{\operatorname{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \bigvee_{t^i\in C_{q+1}} \Omega^{\vee\mathbf{x}}(X\wedge V^{(1,i)}(A,P)(\mathbf{x})) \\ & & \downarrow \\ & & \underrightarrow{\operatorname{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \Omega^{\vee\mathbf{x}}(X\wedge V^{(1)}(A,P)(\mathbf{x})) \end{array}$$

where the first map is given by the same formula with $V^{(1,i)}$ instead of $V^{(1)}$, and where the latter is induced by the inclusions

$$V^{(1,j)}(A,P)(\mathbf{x}) \subseteq \bigvee_{t^i \in C_{q+1}} V^{(1,i)}(A,P)(\mathbf{x}) \cong V^{(1)}(A,P)(\mathbf{x})$$

We may exchange the wedges by products

$$\begin{array}{ccc}
\underset{\mathbf{x}\in\mathcal{I}^{q+1}}{\underrightarrow{\operatorname{holim}}} \bigvee_{t^i\in C_{q+1}} \Omega^{\vee\mathbf{x}}(X\wedge V^{(1,i)}(A,P)(\mathbf{x})) & \longrightarrow & \underset{\mathbf{x}\in\mathcal{I}^{q+1}}{\underrightarrow{\operatorname{holim}}} \Omega^{\vee\mathbf{x}}(X\wedge V^{(1)}(A,P)(\mathbf{x})) \\
\downarrow & & \simeq\downarrow \\
\underset{\mathbf{x}\in\mathcal{I}^{q+1}}{\underrightarrow{\operatorname{holim}}} \prod_{t^i\in C_{q+1}} \Omega^{\vee\mathbf{x}}(X\wedge V^{(1,i)}(A,P)(\mathbf{x})) & \xrightarrow{\cong} & \underset{\mathbf{x}\in\mathcal{I}^{q+1}}{\underrightarrow{\operatorname{holim}}} \Omega^{\vee\mathbf{x}}(X\wedge \prod_{t^i\in C_{q+1}} V^{(1,i)}(A,P)(\mathbf{x}))
\end{array}$$

and the left vertical arrow is $2(k+m)$ connected and the right vertical arrow is an equivalence by Blakers–Massey. ∎

These considerations carry over to the $\mathbf{T}(A \ltimes P)$ spectra. As we saw in (ref), $\mathcal{D}_A^{(m)}P \subseteq S^{(m)}\mathcal{P}_{A\ltimes P}$ were degreewise isomorphisms of categories, so $THH(\mathcal{D}_AP) \xrightarrow{\sim} \mathbf{T}(A \ltimes P)$. Furthermore, recall that the objects of $\mathcal{D}_AP$ were $ob\underline{\mathbf{S}}\mathcal{P}_A$, and $\mathcal{D}_AP(c,d) = \underline{\mathbf{S}}\mathcal{P}_A(c,d) \oplus \underline{\mathbf{S}}\mathcal{M}_A(c, d \otimes_A P)$. Substituting $X \mapsto \mathcal{D}_AP(c,d) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X]$ with the stably equivalent $X \mapsto \underline{\mathbf{S}}\mathcal{P}_A(c,d) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X] \vee \underline{\mathbf{S}}\mathcal{M}_A(c, d\otimes_A P) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X]$ we may define $\mathbf{T}^{(j)}(A,P)$ as before, and we get that the cyclic map

$$\bigvee_{j\geq 0} \mathbf{T}^{(j)}(A,P;X) \to THH(\mathcal{D}_AP) \to \mathbf{T}(A \ltimes P)$$

is an equivalence, and that if P is $k-1$ connected then

$$\mathbf{T}(A;X) \vee \mathbf{T}^{(1)}(A,P;X) \to \mathbf{T}(A \ltimes P; X)$$

is $jk-1$ connected. Furthermore, as N^{cy} preserves equivalences, have that the composite

$$S^1_+ \wedge \mathbf{T}(A,P;X)_0 = N^{cy}(\mathbf{T}(A,P;X)_0) \to N^{cy}\mathbf{T}(A,P;X) \to \mathbf{T}^{(1)}(A \ltimes P)$$

is an equivalence, and so $N^{cy}\mathbf{T}(A,P;X) \to S^1_+\wedge\mathbf{T}(A,P;X)$ is an equivalence (this is a shadow of a more general fact about N^{cy} which we won't need, see [?]).

3.4 Relations to the trace $\tilde{\mathbf{K}}(A \ltimes P) \to \tilde{\mathbf{T}}(A \ltimes P)$

Our definition of the trace $\tilde{K}(A \ltimes P) \to \widetilde{THH}(A \ltimes P)$ is the map

$$\tilde{\mathbf{K}}(A \ltimes P) = \widetilde{ob}\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P} \xrightarrow{tr} \widetilde{THH}(\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P}) = \tilde{\mathbf{T}}(A \ltimes P)$$

(ref). Another definition could be via

$$\mathsf{C}_A(BP) \longrightarrow \mathsf{C}_A(N^{cy}P) \cong \widetilde{N^{cy}}t\mathcal{D}_AP \longrightarrow \widetilde{THH}(\mathcal{D}_AP) \xrightarrow{\simeq} \widetilde{THH}(\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P})$$

The two are related by the diagram

{4.3.1}
$$\begin{array}{ccccc} \mathsf{C}_A(BP) & \xrightarrow{\sim} & \tilde{N}i\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P} & \xleftarrow{\sim} & \tilde{ob}\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P} \\ \downarrow = & \searrow & & \searrow & \downarrow tr \\ \mathsf{C}_A(BP) & \longleftarrow & \mathsf{C}_A(N^{cy}P) & \longrightarrow & T\tilde{H}H(\underline{\mathbf{S}}\mathcal{P}_{A\ltimes P}) \end{array} \tag{3.4.0}$$

{4.3.2}

Lemma 3.4.1 *If P is $k-1$ connected, and X a finite pointed simplicial set, then*

$$X\wedge \mathsf{C}_A(P) \to \mathsf{C}_A(P\otimes_{\mathbf{Z}}\tilde{\mathbf{Z}}[X])$$

is $2k$ connected.

Proof: It is enough to prove it for a finite set X. The smash moves past the wedges in the definition of C_A, and the map is simply $\bigvee_{c\in obS_q^{(m)}\mathcal{P}_A}$ of the inclusion

$$\begin{array}{ccc} X\wedge S_q^{(m)}\mathcal{M}(c,c\otimes_A P) & \xrightarrow{\cong} & \bigvee_{X-*} S_q^{(m)}\mathcal{M}(c,c\otimes_A P) \\ & & \subseteq\downarrow \\ \tilde{\mathbf{Z}}[X]\otimes_{\mathbf{Z}} S_q^{(m)}\mathcal{M}(c,c\otimes_A P) & \xleftarrow{\cong} & \prod_{X-*} S_q^{(m)}\mathcal{M}(c,c\otimes_A P) \end{array}$$

which is $2k$ connected by BM. The usual considerations about m reducedness in the q direction(s), give the lemma ∎

{4.3.3}

Lemma 3.4.2 *If P is $k-1$ connected, then the composite*

$$\mathsf{C}_A(BP) \longrightarrow \mathsf{C}_A(N^{cy}P) \longleftarrow S^1_+\wedge\mathsf{C}_A(P) \longrightarrow S^1\wedge\mathsf{C}_A(P)$$

is $2k$ connected (i.e., induce isomorphism on homotopy groups in the expected range).

Proof: Follows from lemma 3.4.1, and the diagram 3.4.0 preceding it ∎

Consider the diagram (of bispectra)

$$\begin{array}{ccccc} \tilde{\mathbf{T}}(A\ltimes P) & \xrightarrow{\sim} & \tilde{\underline{\mathbf{T}}}(A\ltimes P) & \xleftarrow{\sim} & \tilde{\underline{T}}(A\ltimes P) \\ \uparrow & & \uparrow & & \uparrow \\ N^{cy}\mathbf{T}(A,P) & \xrightarrow{\sim} & N^{cy}\underline{\mathbf{T}}(A,P) & \xleftarrow{\sim} & N^{cy}\underline{T}(A,P) \\ \sim\downarrow & & \downarrow & & \downarrow \\ S^1_+\wedge\mathbf{T}(A,P) & \xrightarrow{\sim} & S^1_+\wedge\underline{\mathbf{T}}(A,P) & \xleftarrow{\sim} & S^1_+\wedge\underline{T}(A,P) \\ \downarrow & & \downarrow & & \downarrow \\ S^1\wedge\mathbf{T}(A,P) & \xrightarrow{\sim} & S^1\wedge\underline{\mathbf{T}}(A,P) & \xleftarrow{\sim} & S^1\wedge\underline{T}(A,P) \end{array}$$

The upwards pointing arrows are induced by the inclusion $V(A,P)(\mathbf{x})\subseteq V(A\ltimes P)(\mathbf{x})$ (likewise with $V(\underline{\mathbf{S}}\mathcal{P}_A,P)$ instead of $V(A,P)$). The rightmost upper vertical map is $2k$ connected by (ref) and so all up going arrows are $2k$ connected.

Proposition 3.4.3 *If P is $k-1$ connected, then the composites* {4.3.4}

$$\tilde{\mathbf{K}}(A \ltimes P) \longrightarrow \tilde{\mathbf{T}}(A \ltimes P) \longleftarrow N^{cy}\mathbf{T}(A,P) \xrightarrow{\sim} S^1_+ \wedge \mathbf{T}(A,P) \longrightarrow S^1 \wedge \mathbf{T}(A,P)$$

and

$$\tilde{\mathbf{K}}(A \ltimes P) \longrightarrow \tilde{\underline{\mathbf{T}}}(A \ltimes P) \xleftarrow{\sim} \tilde{\underline{T}}(A \ltimes P) \longleftarrow N^{cy}\underline{T}(A,P) \longrightarrow S^1 \wedge \underline{T}(A,P)$$

are $2k$ connected (i.e., induce isomorphism on homotopy groups in the expected range).

Proof: The second statement follows from the first. As $\mathsf{C}_A(P) \to (D_1\mathsf{C}_A)(P) \simeq \mathbf{T}(A,P)_0$ is $2k$ connected (ref), the lemma gives that all composites from top left to bottom right in

$$\begin{array}{ccccccc}
\mathsf{C}_A(BP) & \longrightarrow & \mathsf{C}_A(N^{cy}P) & \longleftarrow & S^1_+ \wedge \mathsf{C}_A(P) & \longrightarrow & S^1 \wedge \mathsf{C}_A(P) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \tilde{\mathbf{T}}(A \ltimes P) & \longleftarrow & S^1_+ \wedge \mathbf{T}(A,P)_0 & \longrightarrow & S^1 \wedge \mathbf{T}(A,P)_0 \\
 & & \uparrow & & \simeq\downarrow & & \simeq\downarrow \\
 & & N^{cy}\mathbf{T}(A,P) & \xrightarrow{\simeq} & S^1_+ \wedge \mathbf{T}(A,P) & \longrightarrow & S^1 \wedge \mathbf{T}(A,P)
\end{array}$$

are $2k$ connected. Diagram (?) then imply the proposition. ∎

3.5 Stable K-theory and THH for S-algebras

The functor $S \mapsto A^n_S$ displayed in section III.3.1.8, can clearly be applied to A bimodules as well, and $S \mapsto P^n_S$ will be a cube of $S \mapsto A^n_S$ bimodules, which ultimately gives us a cube $S \mapsto A^n_S \vee P^n_S$ of **S**-algebras. If P is an A bimodule, so is $X \mapsto \Sigma^m(X) = P(S^m \wedge X)$. We defined

$$K^S(A,P) = \operatorname{holim}_{\overrightarrow{k}} \Omega^k \text{fiber}\{K(A \vee \Sigma^{k-1}P) \to K(A)\}$$

The trace map induces a map to

$$\operatorname{holim}_{\overrightarrow{k}} \Omega^k \text{fiber}\{THH(A \vee \Sigma^{k-1}P) \to THH(A)\}$$

and we may compose with the weak map to

$$\operatorname{holim}_{\overrightarrow{k}} \Omega^k (S^1 \wedge THH(A, \Sigma^{k-1}P)$$

given by the discussion of the previous section. We know that this is an equivalence for A a simplicial ring and P a simplicial A bimodule.

{4.4.1} **Theorem 3.5.1** *Let A be an* **S***-algebra and P an A bimodule. Then $K^S(A,P) \simeq THH(A,P)$.*

Proof: There is a stable equivalence $A^n_S \vee P^n_S \to (A \vee P)^n_S$, consisting of repeated applications of the $2k$ connected map $\tilde{\mathbf{Z}}[A(S^k)] \vee \tilde{\mathbf{Z}}[P(S^k)] \to \tilde{\mathbf{Z}}[A(S^k)] \oplus \tilde{\mathbf{Z}}[P(S^k)] \cong \tilde{\mathbf{Z}}[A(S^k) \vee P(S^k)]$. The noninitial nodes in these cubes are all equivalent to a simplicial ring case, and is hence taken care of by theorem 3.1.1 (or rather proposition 3.4.3 since the identification of the equivalence in theorem 3.1.1 with the trace map is crucial in order to have functoriality for **S**-algebras), and all we need to know is that

$$K(A \vee P) \to \underset{\overleftarrow{S \neq \emptyset}}{\operatorname{holim}} K(A^n_S \vee P^n_S)$$

in $n+1$ connected, that

$$THH(A \vee P) \to \underset{\overleftarrow{S \neq \emptyset}}{\operatorname{holim}} THH(A^n_S \vee P^n_S)$$

and

$$THH(A, P) \to \underset{\overleftarrow{S \neq \emptyset}}{\operatorname{holim}} THH(A^n_S, P^n_S)$$

are n connected. This follow from the theorems III.3.2.2 and 1.4.3. ∎

Chapter VI

Topological Cyclic homology, and the trace map

A motivation for the definitions to come can be found by looking at the example of a $\Gamma\mathcal{S}_*$-category $\mathcal{C}$. Consider the trace map {V}

$$ob\mathcal{C} \to THH(\mathcal{C})$$

Topological Hochschild homology is a cyclic space, $ob\mathcal{C}$ is merely a set. However, the trace IV.2.2 is universal in the sense that $ob\mathcal{C} = \lim_{\overleftarrow{\Lambda^o}} THH(\mathcal{C})$. A more usual way of putting this, is to say that $ob\mathcal{C} \to |THH(\mathcal{C})|$ is **the inclusion of the $\mathbb{T}$-fixed points**, which also makes sense since the realization of a cyclic space is a topological spaces with a circle action (see 1.1 below).

In particular, the trace from K-theory has this property. The same is true for the other definition of the trace (IV.1.5), but this follows more by construction than by fate. In fact, any reasonable definition of the trace map should factor through the $\mathbb{T}$-fixed point space, and so, if one wants to approximate K-theory one should try to mimic $\mathbb{T}$-fixed point space by any reasonable means. The awkward thing is that forming the $\mathbb{T}$-fixed point space as such is really not a reasonable thing to do, in the sense that it does not preserve weak equivalences. Homotopy fixed point spaces are nice approximations which are well behaved, and strangely enough it turns out that so are the actual fixed point spaces with respect to finite subgroups of the circle. The aim is now to assemble as much information from these nice construction as possible.

0.6 Connes' Cyclic homology

The first time the circle comes into action for trace maps, is when Alain Connes defines his cyclic cohomology [20]. We are mostly concerned with homology theories, and in one of its many guises, *cyclic homology* is just the $\mathbb{T}$ homotopy orbits of the Hochschild homology spectrum. This is relevant to K-theory for several reasons, and one of the more striking reasons is the fact of Loday and Quillen [75] and Tsygan [124]: just as the K-theory is rationally the primitive part of the group homology of $GL(A)$, cyclic homology is rationally the primitive part of the Lie-algebra homology of $\mathfrak{gl}(A)$.

However, in the result above there is a revealing dimension shift, and, for the purposes of comparison with K-theory via trace maps, it is not the homotopy orbits, but the homotopy fixed points which play the central rôle. The homotopy fixed points of Hochschild homology give rise to T. Goodwillie and J. D. S. Jones' *negative cyclic homology* $HC^-(A)$. In [39] Goodwillie proves that if $A \to B$ is a map of simplicial $\mathbf{Q}$-algebras inducing a surjection $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel, then the relative K-theory $K(A \to B)$ was equivalent to the relative negative cyclic homology $HC^-(A \to B)$.

All told, the cyclic theories associated with Hochschild homology seem to be right rationally, but just as for the comparison with stable K-theory, we must replace Hochschild homology by topological Hochschild homology to obtain integral results.

0.7 [6] and $TC^{\widehat{}}_p$

Topological cyclic homology appears in Bökstedt, Hsiang and Madsen's proof on the algebraic K-theory analog of the Novikov conjecture [6], and is somewhat of a surprise. The obvious generalization of negative cyclic homology would be the $\mathbb{T}$ homotopy fixed point space of topological Hochschild homology, but this turns out not to have all the desired properties. Instead, they consider actual fixed points under the actions of the finite subgroups of $\mathbb{T}$.

After completing at a prime, looking only at the action of the finite subgroups is not an unreasonable thing to do, since you can calculate the $\mathbb{T}$ homotopy fixed points by looking at a tower of homotopy fixed points with respect to cyclic groups of prime power order (see example A.1.9.8.5). The equivariant nature of Bökstedt's formulation of THH is such that the actual fixed point spaces under the finite groups are nicely behaved 1.3.14, and in one respect they are highly superior to the homotopy fixed point spaces: The fixed point spaces with respect to the finite subgroups of $\mathbb{T}$ are connected by more maps than you would think of by considering the homotopy fixed points or the linear analogs, and the interplay between these maps can be summarized in topological cyclic homology, TC, to give an amazingly good approximation of K-theory.

Topological cyclic homology, as we define it, is a a non-connective spectrum, but its completions $\underline{TC}(-)^{\widehat{}}_p$ are all -2-connected. As opposed to topological Hochschild homology, the topological cyclic homology of a ring is generally not an Eilenberg-MacLane spectrum.

In [6] the problem at hand is reduced to studying topological cyclic homology and trace maps of $\mathbf{S}$-algebras of the form $\mathbf{S}[G]$ where $\mathbf{S}$ is the sphere spectrum (see chapter II) and G is some simplicial group, i.e., the $\mathbf{S}$-algebras associated to Waldhausen's A theory of spaces (see II.2.4.6). In this case, TC is particularly easy to describe: for each prime p, there is a cartesian square

$$\begin{array}{ccc} \underline{TC}(\mathbf{S}[G])^{\widehat{}}_p & \longrightarrow & (\Sigma\underline{T}(\mathbf{S}[G])_{h\mathbb{T}})^{\widehat{}}_p \\ \downarrow & & \downarrow \\ \underline{T}(\mathbf{S}[G])^{\widehat{}}_p & \longrightarrow & \underline{T}(\mathbf{S}[G])^{\widehat{}}_p \end{array}$$

(in the homotopy category) where the right vertical map is the circle transfer, and the lower horizontal map is analogous to something like the difference between the identity and a pth power map.

0.8 TC of the integers

Topological cyclic homology is much harder to calculate than topological Hochschild homology, but according to Goodwillie's ICM'90 conjecture, it is worth while pursuing anyhow. The first calculation to appear is in fact one of the hardest ones produced to date, but also the most prestigious: in [7] Bökstedt and Madsen set forth to calculate $TC(\mathbf{Z})^\wedge_p$ for $p > 2$, and found that they could describe $TC(\mathbf{Z})^\wedge_p$ in terms of objects known to homotopy theorists:

$$TC(\mathbf{Z})^\wedge_p \simeq imJ^\wedge_p \times BimJ^\wedge_p \times SU^\wedge_p$$

provided a certain spectral sequence behaved as they suspected it did. In his thesis "The equivariant structure of topological Hochschild homology and the topological cyclic homology of the integers", [Ph.D. Thesis, Brown Univ., Providence, RI, 1994] Stavros Tsalidis proved that the spectral sequence was as Bökstedt and Madsen had supposed, by adapting an argument in G. Carlsson's proof of the Segal conjecture [17] to suit the present situation. Using this Bökstedt and Madsen calculates in [8] $TC(A)^\wedge_p$ for A the Witt vectors of finite fields of odd characteristic, and in particular get the above formula for $TC(\mathbf{Z})^\wedge_p \simeq TC(\mathbf{Z}^\wedge_p)^\wedge_p$. See also Tsalidis' papers [122] and [123]. Soon after J. Rognes showed in [101] that an analogous formula hold for $p = 2$ (you do not have the splitting, and the image of J should be substituted with the complex image of J) in a series of papers ending with .

0.9 Other calculations of TC

All the calculations below are due to the impressive effort of Hesselholt and Madsen. As the calculations below were made after Goodwillie's conjecture was known for rings, they were stated for K-theory whenever possible, even though they were actually calculations of TC. For a ring A, let $W(A)$ be the p-typical Witt vectors, see [110] for the commutative case and [48] for the general case. Let $W(A)_F$ be the coinvariants under the Frobenius action, i.e., the cokernel of $1 - F\colon W(A) \to W(A)$. Note that $W(\mathbf{F}_p) = W(\mathbf{F}_p)_F = \mathbf{Z}^\wedge_p$.

1. Hesselholt [48] $\pi_{-1}\underline{TC}(A)^\wedge_p \cong W(A)_F$.

2. Hesselholt and Madsen (cf. [49] and [80]) Let k be a perfect field of characteristic $p > 0$. Then $\underline{TC}(A)$ is an Eilenberg-MacLane spectrum for any k-algebra A, and

$$\pi_i\underline{TC}(k)^\wedge_p = \begin{cases} W(k)_F & \text{if } i = -1 \\ \mathbf{Z}^\wedge_p & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_i \underline{TC}(k[t]/t^n)\hat{_p} = \begin{cases} \pi_i \underline{TC}(k)\hat{_p} & \text{if } i = -1 \text{ or } i = 0 \\ \mathbf{W}_{nm-1}/V_n \mathbf{W}_{m-1} & \text{if } i = 2m-1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{W}_j = (1 + tk[[t]])^\times/(1 + t^{j+1}k[[t]])^\times$ is the truncated Witt vectors, and $V_n \colon \mathbf{W}_{m-1} \to \mathbf{W}_{nm-1}$ is the Vershibung map sending $f(t) = 1 + t\sum_{i=0}^{\infty} a_i t^1$ to $f(t^n)$. Let C be the cyclic group of order p^N. Then

$$\pi_i \underline{TC}(k[C])\hat{_p} = \begin{cases} \pi_i \underline{TC}(k)\hat{_p} & \text{if } i = -1 \text{ or } i = 0 \\ K_1^{\oplus n} & \text{if } i = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

where K_1 is the p part of the units $k[C]^*$.

3. Hesselholt ([48]). Let A be a free associative $\mathbf{F}_p$ algebra. Then

$$\pi_i \underline{TC}(A)\hat{_p} = \begin{cases} W(A)_F & \text{if } i = -1 \\ \mathbf{Z}\hat{_p} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand,

$$\pi_i \underline{TC}(\mathbf{F}_p[t_1, \dots t_n])\hat{_p} = \begin{cases} (\bigoplus_{g \in G_m} \mathbf{Z}\hat{_p})\hat{_p} & \text{for } -1 \le i \le n-2 \\ 0 & \text{otherwise} \end{cases}$$

where G_m is some explicit (non-empty) set (see [**?**, page 140])

4. Hesselholt and Madsen [52]. Let K be a complete discrete valuation field of characteristic zero with perfect residue field k of characteristic $p > 2$. Let A be the valuation ring of K. Then Hesselholt and Madsen analyze $TC(A)\hat{_p}$ and in particular they give very interesting algebraic interpretations of the relative term of the transfer map $TC(k)\hat{_p} \to TC(A)\hat{_p}$ (gotten by inclusion of the k-vector spaces into the torsion modules of A). See [52].

5. Rognes and Ausoni [3]. As a first step towards calculating the algebraic k-theory of connective complex K-theory ku, Ausoni and Rognes calculate topological cyclic homology of the Adams summand ℓ_p.

0.10 Where to read

The literature on TC is naturally even more limited than on THH. Böksted, Hsiang and Madsen's original paper [6] is still very readable. The first chapters of Hesselholt and Madsen's [51] can serve as a streamlined introduction for those familiar with equivariant G-spectra. For more naïve readers, the unpublished lecture notes of Goodwillie can be of great help. Again, the survey article of Madsen [80] is recommendable.

1 The fixed point spectra of THH.

We will define TC by means of a homotopy cartesian square of the type (i.e., it will be the homotopy limit of the rest of the diagram)

$$\begin{array}{ccc} TC(-) & \longrightarrow & THH(-)^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \prod_{p \text{ prime}} TC(-;p)^{\widehat{}}_p & \longrightarrow & (\prod_{p \text{ prime}} THH(-)^{\widehat{}}_p)^{h\mathbb{T}} \end{array}$$

(as it stands, this strictly does not make sense: there are some technical adjustments we shall return to) The $\mathbb{T}$ homotopy fixed points are formed with respect to the cyclic structure.

In this section we will mainly be occupied with preparing the ground for the lower left hand corner of this diagram. Let $C_n \subseteq \mathbb{T}$ be the n-th roots of unity, and we choose our generator of the cyclic group C_n to be $t_{n-1} = t = e^{2\pi i/n}$. For each prime number p, the functor $TC(-;p)$ is defined as the homotopy limit of a diagram of fixed point spaces $|THH(-)|^{C_{p^n}}$. The maps in the diagrams are partially inclusion of fixed points $|THH(-)|^{C_{p^{n+1}}} \subseteq |THH(-)|^{C_{p^n}}$, and partially some more exotic maps - the "restriction maps" - which we will describe below. The contents of this section is mostly fetched from the very readable, but unpublished, MSRI notes of Goodwillie [120]. If desired, the reader can consult appendix C for some facts on group actions.

1.1 Cyclic spaces and the edgewise subdivision

{cyclic s

First we need to revisit Connes' category Λ. For once we record a formal definition.

Definition 1.1.1 Let Λ be the category with the same objects as Δ, but with morphism sets given by

$$\Lambda([n],[q]) = \Delta([n],[q]) \times C_{n+1}$$

where a pair (σ, t^a) is considered as a composite

$$[n] \xrightarrow{t^a} [n] \xrightarrow{\sigma} [q]$$

(where $t = t_n$ is the generator of C_{n+1}, so that $t_n^{n+1} = 1_{[n]}$). Composition is subject to the extra relations

$$\begin{aligned} t_n d^i &= d^{i-1} t_{n-1} && 1 \le i \le n \\ t_n d^0 &= d^n && \\ t_n s^i &= s^{i-1} t_{n+1} && 1 \le i \le n \\ t_n s^0 &= s^n t_{n+1}^2 && \end{aligned}$$

A *cyclic object* in some category $\mathcal{C}$ is a functor $\Lambda^o \to \mathcal{C}$ and a *cyclic map* is a natural transformation between cyclic objects.

Notice that the description above gives that any map in Λ can be written as a composite ϕt^a where $\phi \in \Delta$. Furthermore, this factorization is unique.

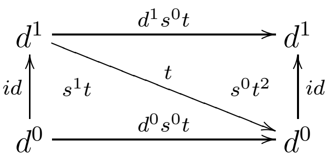
Due to the inclusion $j\colon \Delta \subset \Lambda$, any cyclic object X gives rise to a simplicial object j^*X.

As noted by Connes [19], cyclic objects are intimately related to objects with a circle action (see also [59], [28] and [6]). In analogy with the standard n-simplices $\Delta[n] = \{[q] \mapsto \Delta([q],[n])\}$, we define the cyclic sets

$$\Lambda[n] = \Lambda(-,[n])\colon \Lambda^o \to \mathcal{E}ns$$

Lemma 1.1.2 *For all n, $|j^*\Lambda[n]|$ is an $\mathbb{T}$-space, naturally (in $[n] \in ob\Lambda^o$) homeomorphic to $\mathbb{T} \times |\Delta[n]|$.*

Proof: For a proof, see e.g., [28, 2.7]. For a more visual idea, you can stare at the homeomorphism e.g., between $|j^*\Lambda[1]|$ and $\mathbb{T} \times |\Delta[1]|$ directly:



where the left edge is identified with the right edge (notice that s^1t and s^0t^2 are the only non-degenerate two-simplices). ∎

We note two adjoint pairs that are nice to have. We have already seen half of the first pair in IV.1.1.1, namely the cyclic bar construction. If $\mathcal{C}$ is a category with finite coproducts we get an adjoint pair

$$\mathcal{C}^{\Lambda^o} \underset{j^*}{\overset{B^{cy}}{\leftrightarrows}} \mathcal{C}^{\Delta^o}$$

where j^* is induced by $j\colon \Lambda \subset \Delta$, and the cyclic bar construction (with respect to the coproduct $\vee$) $B^{cy}_{\vee} = B^{cy}$ is the left adjoint to j^* given in degree q by $B^{cy}X([q]) = \bigvee_{C_{q+1}} X_q$, but with a twist in the simplicial structure (in fact, $\Lambda[n] \cong B^{cy}\Delta[n]$). To be precise, consider the bijection

$$\Lambda([m],[n]) \xrightarrow{f \mapsto \psi(f) = (\psi_\Delta(f), \psi_C(f))} \Delta([m],[n]) \times C_{m+1} \cong B^{cy}_m\Delta[n]$$

given by the unique factorization of maps in Λ, with inverse given by composition $\psi^{-1}(\sigma, t^a) = \sigma t^a$. Hence we can identify $\Lambda[n]$ with $B^{cy}\Delta[n]$ if we give the latter the cyclic structure $\phi^*((\sigma, t^a)) = \psi(\sigma t^a \phi)$. In general, for $y \in B^{cy}_m X$ in the t^a-summand this reads $\phi^*(y)) = (\phi_\Delta(t^a\phi)^* y$ in the $\Delta_C(t^a\phi)$-summand.

The other adjoint pair is given by the realization/singular functors connecting (pointed) cyclic sets with $\mathbb{T}$-spaces

$$\mathbb{T} - Top_* \underset{\sin_\Lambda}{\overset{|-|_\Lambda}{\leftrightarrows}} \mathcal{E}ns_*{}^{\Lambda^o}$$

given by

$$|X|_\Lambda = \int^{[q]\in\Lambda^o} |\Lambda[q]|_\Lambda {}_+\wedge X_q \cong \coprod_{[q]\in\Lambda^o} |\Lambda[q]|_\Lambda {}_+\wedge X_q/\sim$$

where X is a cyclic set and $|\Lambda[q]|_\Lambda$ is $|j^*\Lambda[q]| \cong \mathbb{T}\times|\Delta[n]|$ considered as a $\mathbb{T}$-space, and for a $\mathbb{T}$-space Z

$$\sin_\Lambda Z = \{[q]\mapsto (\mathbb{T}-Top)(|\Lambda[q]|_\Lambda, Z)\}.$$

We record some formal isomorphisms. Let U be the forgetful functor from $\mathbb{T}$-spaces to (topological pointed) spaces (right adjoint to $\mathbb{T}_+\wedge -$).

Lemma 1.1.3 *There are natural isomorphisms*

$$j^*\sin_\Lambda Z \cong \sin(UZ), \qquad U|X|_\Lambda \cong |j^*X| \qquad \textit{and} \qquad |N^{cy}Y|_\Lambda \cong \mathbb{T}_+\wedge|Y|$$

where X is a cyclic set, Y a simplicial set and Z an $\mathbb{T}$-space (the statements are true both in the pointed and unpointed case).

Proof: The first follows by the isomorphism $|\Lambda[q]|_\Lambda \cong \mathbb{T}\times|\Delta[n]|$, and the adjointness of U with $\mathbb{T}_+\wedge -$; and the last two follow by formal nonsense. ∎

These isomorphisms will mean that we won't be fanatic about remembering to put the subscript Λ on sin and $|-|$.

Lemma 1.1.4 *Let X be a pointed cyclic set. Then*

$$\varprojlim_{\Lambda^o} X \cong \{x\in X_0|s_0x = ts_0x\} \cong |X|_\Lambda^{\mathbb{T}}$$

Proof: The first equation is a direct calculation, and the second from the adjunction isomorphism $|X|^{\mathbb{T}}_\Lambda = (\mathbb{T}-Top_*)(S^0,|X|_\Lambda) \cong \mathcal{E}ns_*^{\Lambda^o}(S^0,X) = \lim_{\overleftarrow{\Lambda^o}} X$. ∎

Note in particular that if we consider a cyclic space as a simplicial cyclic set, then the formula always holds true if applied degreewise. For those who worry about the difference between spaces (simplicial sets) and topological spaces, we note that if G is a group and X a simplicial G-set, then the two fixed-point constructions $|X^G|$ and $|X|^G$ are naturally homeomorphic.

1.2 The edgewise subdivision

Let $a\in\mathbf{N}$. The subdivision functor $sd^a\colon\Delta\to\Delta$ is the composite of the diagonal $\Delta\to\Delta^{\times a}$ composed with the concatenation $\Delta^{\times a}\to\Delta$ which sends $(S_1,\dots S_a)$ to the concatenation $\S_1\sqcup\dots\sqcup S_a$ which as a set is the disjoint union, but with ordering such that $s\in S_i$ is less that $t\in S_j$ if either $i<j$ or $i=j$ and $s<t\in S_i$. Note that $sd^a[j-1]=[ja-1]$. This construction extends to the cyclic world as follows

$$\begin{array}{ccc} \Delta & \xrightarrow{sd^a} & \Delta \\ \big\downarrow{\scriptstyle\text{inclusion}} & & \big\downarrow{\scriptstyle\text{inclusion}} \\ \Lambda\times C_a & \longrightarrow & \Lambda \end{array}$$

where the cyclic group C_a is considered as a category with one object, and where the lower map sends $(t^m, T^n) \in (\Lambda \times C_a)([j-1],[j-1])$ to $t^{am+jn} in \Lambda([ja-1],[ja-1])$, and T is our chosen generator for C_a.

Precomposing any simplicial object X with sd^a gives $sd_a X = X \circ sd^a$, the *a-fold edgewise subdivision of* X. We note that $(sd_a X)_{j-1} = X_{aj-1}$.

Furthermore, given a cyclic object X, we see that $sd_a X$ becomes a new cyclic object, with a C_a-action.

Lemma 1.2.1 ([6]) *Let X be a cyclic space. There is a natural C_a-equivariant homeomorphism $|sd_a X| \cong |X|$, where the action on $|sd_a X|$ comes from the C_a-action on $sd_a X$, and the action on $|X|$ comes from the cyclic structure on X. The resulting homeomorphism $|sd_a X^{C_a}| \cong |X|^{C_a}$ is $\mathbb{T}$-equivariant if we let $\mathbb{T}$ act on $|sd_a X^{C_a}|$ via the cyclic structure, and on $|X|^{C_a}$ through the isomorphism $\mathbb{T} \cong \mathbb{T}/C_q$.*

Proof: Must write ∎

1.3 The restriction map

Let A be an $\mathbf{S}$-algebra and X a space. We will now define an important cyclic map

$$R \colon sd_q THH(A;X)^{C_q} \to THH(A;X)$$

called the *restriction map*. This map is modeled on the fact that if C is a group and $f \colon Z \to Y$ is a C-map, then f sends the C-fixed points to C-fixed points; and hence we get a map

$$Map_*(Z,Y)^C \to Map_*(Z^C Y^C)$$

by restricting to fixed points. Notice that the $j-1$ simplices of $sd_a THH(A;X)$ are given by

$$THH(A,X)_{aj-1} = \underset{\overrightarrow{x_{k,l}\in\mathcal{I},1\leq k\leq a,1\leq l\leq j}}{\text{holim}} Map_*(\bigwedge_{k,l} S^{x_{k,l}}, X \wedge V(A)((x_{k,l})))$$

The C_a fixed points under the action on $\mathcal{I}^{aj}$ are exactly the image of the diagonal $\mathcal{I}^j \to \mathcal{I}^{aj}$ sending $\mathbf{x}$ to $\mathbf{x}^a = (\mathbf{x},\dots,\mathbf{x})$, and the C_a fixed points are given by

$$THH(A,X)^{C_a}_{aj-1} \cong \underset{\overrightarrow{x_1,\dots,x_j\in\mathcal{I}^j}}{\text{holim}} Map_* \left((\bigwedge_{1\leq i\leq j} S^{x_i})^{\wedge a}, X \wedge V(A)((x_1,\dots x_j)^a) \right)^{C_a}$$

Note that $V(A)((x_1,\dots x_j)^a) \cong V(A)(x_1,\dots,x_j)^{\wedge a} = (\bigwedge_{1\leq i\leq j} A(S^{x_i}))^{\wedge a}$. In the mapping space, both the domain and target are a-fold smash products with C_a-action given by permutation (except for the C_a-fixed space X which just stays on for the ride) and so we get a restriction map to the mapping space of the fixed points:

$$Map_*(\bigwedge_{1\leq i\leq j} S^{x_i}, X \wedge V(A)(x_1,\dots,x_j))$$

Taking the homotopy limit we get a map $sd_aTHH(A;X)^{C_a}_{j-1} \to THH(A;X)_{j-1}$ which assembles to a cyclic map

$$R\colon sd_aTHH(A,X)^{C_a} \to THH(A;X)$$

giving the pair $(THH(A;X), R)$ the structure of an *epicyclic space* in the sense of Goodwillie's MSRI notes:

Definition 1.3.1 An epicyclic space (Y, ϕ) is a cyclic space Y equipped with maps

$$\phi_q\colon Y^{C_q}_{qj-1} \to Y_{j-1}$$

for all $q, j \geq 1$ satisfying

1. $\phi_q\colon (sd_qY)^{C_q} \to Y$ is simplicial
2. $\phi_q t = t\phi_q$ (which implies that $\phi_q(Y^{C_{aq}}_{qaj-1}) \subseteq Y^{C_a}_{aj-1}$)
3. $\phi_a\phi_q = \phi_{aq}\colon Y^{C_{aq}}_{aqj-1} \to Y_{j-1}$
4. $\phi_1 = 1$

Note that ϕ_q can be regarded as a cyclic map $(sd_qY)^{C_q} \to Y$, and also as a C_a-equivariant simplicial map $(sd_{aq}Y)^{C_q} \to sd_aY$ for any a. Let for $a \geq 1$

$$Y\langle a\rangle = |(sd_aY)^{C_a}|$$

In addition to the map $\phi_q\colon Y\langle aq\rangle \to Y\langle a\rangle$ we have a map – the "inclusion of fixed points" – given as $i_q\colon Y\langle aq\rangle \cong |Y|^{C_{qa}} \subseteq |Y|^{C_a} \cong Y\langle a\rangle$. By the definition of an epicyclic space we get that these maps obey the following relations

$$\begin{array}{ll} \phi_q\phi_r = \phi_{qr} & \phi_1 = i_1 = id \\ i_qi_r = i_{qr} & i_q\phi_r = \phi_r i_q \end{array}$$

In other words, $a \mapsto Y\langle a\rangle$ is a functor to topological spaces from the category $\mathcal{RF}$:

Definition 1.3.2 Let $\mathcal{RF}$ be the category whose objects are the natural numbers, and where

$$\mathcal{RF}(a,b) = \{f_{r,s} | a = rsb\}$$

with composition $f_{r,s} \circ f_{p,q} = f_{rp,sq}$. An epicyclic space (Y, ϕ) give rise to a functor from $\mathcal{RF}$ to spaces by sending a to $Y\langle a\rangle$, $f_{q,1}$ to ϕ_q and $f_{1,q}$ to i_q. Sloppily, we write $R = f_{r,1}$ and $F = f_{1,r}$ for any unspecified r (and range), hence the name of the category. For any given prime p, the full subcategory of $\mathcal{RF}$ containing only the powers of p is denoted $\mathcal{RF}_p$.

Example 1.3.3 We have seen that THH is defines an epicyclic space, and a map of **S**-algebras give rise to a map respecting the epicyclic structure.

Another example is the cyclic nerve. Let $\mathcal{C}$ be any category, and consider the *cyclic nerve* $N^{cy}\mathcal{C}$. This is a straight-forward generalization of the cyclic bar of a monoid:

$$N_q^{cy}\mathcal{C} = \{c_q \leftarrow c_0 \leftarrow c_1 \leftarrow \dots \leftarrow c_{q-1} \leftarrow c_q \in \mathcal{C}\}$$

with face and degeneracies given by composition and insertion of identities, and with cyclic structure given by cyclic permutation. This is a cyclic set, and $|N^{cy}\mathcal{C}|^{\mathbb{T}} \cong \lim_{\overleftarrow{\Lambda^o}} N^{cy}\mathcal{C} = ob\mathcal{C}$ where an object is identified with its identity morphism in $N_0^{cy}\mathcal{C}$. The fixed point sets under the finite subgroups of the circle are more interesting as $sd_rN^{cy}\mathcal{C}^{C_r} \cong N^{cy}\mathcal{C}$. In fact, an element $x \in (sd_rN^{cy}\mathcal{C})_{q-1} = N_{rq-1}^{cy}\mathcal{C}$ which is fixed by the C_r action must be of the form

$$c_q \overset{f_1}{\longleftarrow} c_1 \overset{f_2}{\longleftarrow} \dots \overset{f_q}{\longleftarrow} c_q \overset{f_1}{\longleftarrow} c_1 \overset{f_2}{\longleftarrow} \dots \overset{f_q}{\longleftarrow} c_q \overset{f_1}{\longleftarrow} c_1 \overset{f_2}{\longleftarrow} \dots \overset{f_q}{\longleftarrow} c_q$$

and we get an isomorphism $\phi_r\colon sd_rN^{cy}\mathcal{C}^{C_r} \cong N^{cy}\mathcal{C}$ by forgetting the repetitions. This equips the cyclic nerve with an epicyclic structure, and obviously a functor of categories give rise to a map of cyclic nerves respecting the epicyclic structure.

An interesting example is the case where $\mathcal{C}$ is replaced by the simplicial monoid $M = THH_0(A)$ connected to any **S**-algebra A. We have a map $N^{cy}M \to THH(A)$ given by smashing together functions $S^{x_i} \to A(S^{x_i})$

$$\prod_{0\leq i\leq q} \underset{\overrightarrow{x_i\in\mathcal{I}}}{\operatorname{holim}} \Omega^{x_i}A(S^{x_i}) \to \underset{\overrightarrow{\mathbf{x}\in\mathcal{I}^{q+1}}}{\operatorname{holim}} \Omega^{\vee\mathbf{x}} \bigwedge_{0\leq i\leq q} A(S^{x_i})$$

This map preserves the epicyclic structure.

1.3.4 Remark

Our notion of an epicyclic space is **not** the same as the one earlier Goodwillie proposed in a letter to Waldhausen [119], and which later was used by Burghelea, Fiedorowicz, and Gajda in [16] to compare Adams operators. This older definition generalized the so-called power maps $P_q = \phi_q^{-1}\colon N^{cy}\mathcal{C} \to (sd_qN^{cy}\mathcal{C})^{C_q}$ instead. Cyclic nerves are epicyclic spaces under either definition.

Remark 1.3.5 *An epicyclic space (Y, ϕ) is more than a functor from $\mathcal{RF}$ to spaces. In fact, as each $(sd_aY)^{C_a}$ is again a cyclic space, each $Y\langle a\rangle = |(sd_aY)^{C_a}|$ comes equipped with an $\mathbb{T}$-action. However, $Y\langle a\rangle$ is not a functor to $\mathbb{T}$-spaces: the inclusion of fixed point spaces under the finite subgroups of $\mathbb{T}$ are not $\mathbb{T}$-equivariant, but speed up the action. We may encode this as a continuous functor sending $\theta \in \mathbf{R}/\mathbf{Z} \cong \mathbb{T}$ to $\rho_\theta\colon Y\langle a\rangle \to Y\langle a\rangle$ we get the additional relations*

$$\phi_q\rho_\theta = \rho_\theta\phi_q \qquad i_q\rho_\theta = \rho_{q\theta}i_q \qquad \rho_\theta\rho_\tau = \rho_{\theta+\tau}$$

This can again be encoded in a topological category $S\mathcal{RF}$ with objects the natural numbers and morphisms $S\mathcal{RF}(a,b) = \mathbb{T} \times \mathcal{RF}(a,b)$. Composition is given by

$$(\theta, f_{r,s})(\tau, f_{p,q}) = (\theta + s\tau, f_{rp,sq})$$

Sending θ to ρ_θ we see that any epicyclic space give rise to a continuous functor $a \mapsto Y\langle a\rangle$ from $S\mathcal{RF}$ to topological spaces. In the MSRI notes Goodwillie defines

$$\underline{TC}(A;X) = \{k \mapsto \operatorname*{holim}_{\overleftarrow{a \in S\mathcal{RF}}} |sd_a THH(A; S^k \wedge X)^{C_a}|\}$$

(the homotopy limit remembers the topology in $\mathbb{T}$), and gives a proof that this elegant definition agrees with the one we are going to give. The only reasons we have chosen to refrain from giving this as our definition is that our definition is custom built for our application (and for computations), and the proof that they agree would lengthen the discussion further.

1.3.6 Properties of the fixed point spaces

We now make a closer study of the C_q-fixed point spaces of THH when q is a prime power. The most important result is proposition 1.4.1, often referred to as "the fundamental cofibration sequence" which guarantees that the actual fixed point spaces will have good homotopical properties.

Definition 1.3.7 Let

$$T\langle\ \rangle(A;X)\colon \mathcal{RF} \to \mathcal{S}_*$$

with $T\langle a\rangle(A,X) = \sin|sd_a THH(A,X)^{C_a}|$, be the functor associated with the epicyclic space $(THH(A;X), R)$. We set $R = T\langle f_{r,1}\rangle$ (for "Restriction", which it is) and $F = T\langle f_{1,r}\rangle$ (for "Frobenius", see 1.18, which here is the inclusion of fixed points

$$T\langle rq\rangle \cong \sin|THH(A,X)|^{C_{rq}} \subseteq \sin|THH(A,X)|^{C_q} \cong T\langle\rangle q)$$

This construction is functorial in A and X, and we set

$$\underline{T}\langle a\rangle(A;X) = \{k \mapsto T\langle a\rangle(A; S^k \wedge X)\}$$

Remember that each $\underline{T}\langle a\rangle$ can be considered as functors to cyclic spaces (but they do not assemble when varying a). We will not distinguish notationally whether we think of $\underline{T}[a](A;X)$ as a simplicial or cyclic space, and we offer the same ambiguity to $\underline{T}(A;X) \cong \underline{T}\langle 1\rangle(A;X)$.

The $\underline{T}\langle a\rangle(A;X)$ are Ω-spectra for any a, but we will just now we prove this only for a a prime power (which is all we will really need due to the form of our definition of integral TC. For the general result see remark 1.3.17 below.). It follows as a corollary of:

{prop:Vfur

Proposition 1.3.8 *Let p be a prime. Then there is a chain of natural equivalences from the fiber of*

$$\underline{T}\langle p^n\rangle(A;X) \xrightarrow{R} \underline{T}\langle p^{n-1}\rangle(A;X)$$

to $sd_{p^n}\underline{T}(A;X)_{hC_{p^n}}$. *Indeed, for each* j*, the fiber of*

$$(sd_{p^n}THH(A,X)^{C_{p^n}})_{j-1} \xrightarrow{R} (sd_{p^{n-1}}THH(A,X)^{C_{p^{n-1}}})_{j-1}$$

is naturally weakly equivalent to $\operatorname{holim}_{\overrightarrow{k}} \Omega^k((sd_{p^n}THH(A,S^k\wedge X)_{j-1})_{hC_{p^n}})$.

Proof: The first statement follows from the second. Let $q=p^n$, $G=C_q$ and $H=C_p$. For $\mathbf{x}\in\mathcal{I}^j$, let

$$Z(\mathbf{x})=\left(\bigwedge_{1\leq i\leq j} S^{x_i}\right)^{\wedge q}, \text{ and } W(\mathbf{x})=X\wedge\left(\bigwedge_{1\leq i\leq j} A(S^{x_i})\right)^{\wedge q}$$

By the approximation lemma, the fiber of R is naturally equivalent to

$$\underset{\overrightarrow{\mathbf{x}\in\mathcal{I}^j}}{\operatorname{holim}}\, fiber\{Map_*(Z(\mathbf{x}),W(\mathbf{x}))^G\to Map_*(Z(\mathbf{x})^H,W(\mathbf{x})^H)^{G/H}\}$$

which, by the isomorphism

$$Map_*(Z(\mathbf{x})^H,W(\mathbf{x})^H)^{G/H}\cong Map_*(Z(\mathbf{x})^H,W(\mathbf{x}))^G$$

is isomorphic to

$$\underset{\overrightarrow{\mathbf{x}\in\mathcal{I}}}{\operatorname{holim}}\, Map_*(U(\mathbf{x}),W(\mathbf{x}))^G$$

where $U(\mathbf{x})=Z(\mathbf{x})/(Z(\mathbf{x})^H)$. As $U(\mathbf{x})$ is a free finite based G complex, corollary C.C.2.1.3 tells us that there is a natural chain

$$\begin{aligned} Map_*(U(\mathbf{x}),W(\mathbf{x}))^G &\longrightarrow Map(U(\mathbf{x}),\lim_{\overrightarrow{k}}\Omega^k(S^k\wedge W(\mathbf{x})))^G\\ &\xleftarrow{\sim} \lim_{\overrightarrow{k}}\Omega^k Map_*(U(\mathbf{x}),S^k\wedge W(\mathbf{x}))_{hG})\end{aligned}.$$

and that the first map is $\vee\mathbf{x}-1$ connected. Furthermore, the cofibration sequence $Z(\mathbf{x})^H\subseteq Z(\mathbf{x})\to U(\mathbf{x})=Z(\mathbf{x})/(Z(\mathbf{x})^H)$ induces a fibration sequence

$$\begin{array}{ccc}\Omega^k Map_*(U(\mathbf{x}),S^k\wedge W(\mathbf{x}))_{hG}) & \longrightarrow & \Omega^k Map_*(Z(\mathbf{x}),S^k\wedge W(\mathbf{x}))_{hG})\\ & & \downarrow\\ & & \Omega^k Map_*(Z(\mathbf{x})^H,S^k\wedge W(\mathbf{x}))_{hG})\end{array}.$$

Since $Z(\mathbf{x})^H$ is $\mathbf{x}q/p$-dimensional and $S^k\wedge W(\mathbf{x}))_{hG}$ is $\mathbf{x}q+k-1$-connected, the first map in the fiber sequence is $\mathbf{x}(q-q/p)-1$-connected.

Taking the homotopy colimit over $\mathcal{I}^j$, this gives the proposition. ∎

A variant of this proposition was proven by Madsen in a letter to Hsiang around 1988. It does not play a major role in [6] (perhaps too obvious to mention in the equivaraint context in which that paper was written), but it is vital for all calculations of TC. In [120] Goodwillie shows how it can be used to simplify many of the arguments in [6]. This is how we will use it. For instance, the following important results are immediate corollaries. The proofs are by induction on n, noting that homotopy orbits preserve equivalences.

Corollary 1.3.9 *Any map $A \to B$ of* **S**-*algebras inducing an equivalence $THH(A) \to THH(B)$ induces an equivalence*

$$\underline{T}\langle p^n\rangle(A;X) \to \underline{T}\langle p^n\rangle(B;X)$$

Corollary 1.3.10 *Let A be an* **S**-*algebra and X a space. Then for any prime power p^n*

1. *$\underline{T}\langle p^n\rangle(A;X)$ is a connective Ω-spectrum.*
2. *$\underline{T}\langle p^n\rangle(-;X)$ takes stable equivalences of* **S**-*algebras to equivalences.*
3. *$\underline{T}\langle p^n\rangle(-;X)$ is Morita invariant.*
4. *$\underline{T}\langle p^n\rangle(-;X)$ preserves products up to equivalence.*

Proof: Follows by 1.3.8 and the corresponding properties of THH, plus the fact that homotopy orbits preserve loops and products of spectra. ∎

The corollary is true if we exchange p^n for an arbitrary number a, but that does not follow from the proposition, but rather from a more complicated argument running over all the prime factors of a, see [120, 11].

1.3.11 $\Gamma\mathcal{S}_*$-categories

Essentially just the same construction can be applied to the case of $\Gamma\mathcal{S}_*$-categories.

If $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category $THH(\mathcal{C};X)$ also has its restriction map R, and $(THH(\mathcal{C};X),R)$ is an epicyclic space: If $\mathbf{x} \in \mathcal{I}^q$, then we have a restriction map

$$(\Omega^{\vee\mathbf{x}^a}(X \wedge V(\mathcal{C})(\mathbf{x}^a)))^{C_a} \to \Omega^{\vee\mathbf{x}}(X \wedge V(\mathcal{C})((\mathbf{x}^a))^{C_a})$$

as before, and note that $V(\mathcal{C})(\mathbf{x}^a)^{C_a} = V(\mathcal{C})(\mathbf{x})$. Proceeding just as for **S**-algebras we see that

$$a \mapsto T\langle a\rangle(\mathcal{C};X) = \sin|sd_a THH(\mathcal{C};X)^{C_a}|$$

defines a functor from $\mathcal{RF}$ (or better: from $S\mathcal{RF}$) to spaces.

Lemma 1.3.12 *Let p be a prime, $\mathcal{C}$ a $\Gamma\mathcal{S}_*$-category and X a space. The R map fits into a fiber sequence*

$$|\underline{T}(\mathcal{C};X)|_{hC_{p^n}} \longrightarrow \underline{T}\langle p^n\rangle(\mathcal{C};X) \xrightarrow{R} \underline{T}\langle p^{n-1}\rangle(\mathcal{C};X)$$

Indeed, for each j, the fiber of

$$(sd_{p^n}THH(\mathcal{C},X)^{C_{p^n}})_{j-1} \xrightarrow{R} (sd_{p^{n-1}}THH(\mathcal{C},X)^{C_{p^{n-1}}})_{j-1}$$

is naturally weakly equivalent to $\operatorname{holim}_{\overrightarrow{k}} \Omega^k((sd_{p^n}THH(\mathcal{C},S^k \wedge X)_{j-1})_{hC_{p^n}})$.

Proof: Exactly the same as the **S**-algebra case. ∎

As before, this gives a series of corollaries.

{lem:VTHH

Corollary 1.3.13 *Any $\Gamma\mathcal{S}_*$-functor $F\colon \mathcal{C}\to\mathcal{D}$ inducing an equivalence $THH(\mathcal{C})\to THH(\mathcal{D})$ induces an equivalence*

$$\underline{T}\langle p^n\rangle(\mathcal{C};X)\to\underline{T}\langle p^n\rangle(\mathcal{D};X)$$ ☺

{cor:V148

{cor:V1481} **Corollary 1.3.14** *Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category and X a space. Then for any prime power p^n*

{cor:V1482} 1. *$\underline{T}\langle p^n\rangle(\mathcal{C};X)$ is a connective Ω-spectrum.*

{cor:V1483} 2. *The functor $\underline{T}\langle p^n\rangle(-;X)$ takes $\Gamma\mathcal{S}_*$-equivalences of categories to equivalences.*

3. *If A is a ring, then the inclusion $A\subseteq\mathcal{P}_A$ as a rank one module induces an equivalence $T\langle p^n\rangle(A;X)\xrightarrow{\sim}T\langle p^n\rangle(\mathcal{P}_A;X)$*

4. *$\underline{T}\langle p^n\rangle(-;X)$ preserves products up to equivalence.* ☺

Corollary 1.3.15 *If $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category, then $T\langle p^n\rangle(\bar{H}\mathcal{C};X)=\{k\mapsto T\langle p^n\rangle(\bar{H}\mathcal{C}(S^k);X)\}$ is an Ω-spectrum, equivalent to $\underline{T}\langle p^n\rangle(\mathcal{C};X)$.*

Proof: That $T\langle p^n\rangle(\bar{H}\mathcal{C};X)$ is an Ω-spectrum follows for instance from 1.3.14.2 and 1.3.14.**??** since $\bar{H}\mathcal{C}(k_+)$ is $\Gamma\mathcal{S}_*$-equivalent to $\mathcal{C}^{\times k}$. That the two Ω-spectra are equivalent follows by comparing both to the bispectrum $\underline{T}\langle p^n\rangle(\bar{H}\mathcal{C};X)$. ∎

Corollary 1.3.16 *If $\mathcal{C}$ is an exact category, then*

$$T\langle p^n\rangle(\bar{H}\mathcal{C};X)\to\{k\mapsto T\langle p^n\rangle(S^{(k)}\mathcal{C};X)\}$$

is an equivalence of Ω-spectra.

Proof: Follows by corollary 1.3.13 since $THH(\bar{H}\mathcal{C}(S^k);X)\to THH(S^{(k)}\mathcal{C};X)$ is an equivalence NBNB(ref). ∎

ark:Rfiber}

Remark 1.3.17 *An analog of lemma 1.3.8 holds for integers q that are not prime powers as well. The statement is that if a is a positive integer the homotopy fiber of the map*

$$\underline{T}(A)^{C_q}\to\underset{\overleftarrow{r|q}}{\operatorname{holim}}\,\underline{T}(A)^{C_{q/r}}$$

induced by the restriction map and where the homotopy limit is over the positive numbers dividing q is naturally equivalent to $\underline{T}(A)_{hC_q}$. The proof is gotten by entering the proof of lemma 1.3.8, and letting $G=C_q$, $Z(\mathbf{x})$ and $W(\mathbf{x})$ be as before, but forgetting that q was a prime power. Assume by induction that the statement has been proven for all groups of cardinality less than G, and so that all for these groups the fixed point spectra of THH are homotopy functors and Bökstedt's approximation lemma applies.

This means that the canonical map

$$\underset{\overrightarrow{x\in\mathcal{I}^j}}{\operatorname{holim}}\ \underset{\overleftarrow{0\neq H\subset G}}{\operatorname{holim}}Map_*(Z(\mathbf{x})^H,W(\mathbf{x})^H)^{G/H}\longrightarrow\underset{\overleftarrow{0\neq H\subset G}}{\operatorname{holim}}\ \underset{\overrightarrow{x\in\mathcal{I}^j}}{\operatorname{holim}}Map_*(Z(\mathbf{x})^H,W(\mathbf{x})^H)^{G/H}$$

is an equivalence. The right hand side is isomorphic to

$$\underset{0\neq H\subset G}{\underleftarrow{\operatorname{holim}}}\left(\underset{x\in\mathcal{I}^{j\cdot|G/H|}}{\underrightarrow{\operatorname{holim}}} Map_*(Z(\mathbf{x})^H, W(\mathbf{x})^H)\right)^{G/H} = \underset{0\neq H\subset G}{\underleftarrow{\operatorname{holim}}} sd^{|G/H|}THH(A,X)^{G/H},$$

and the left hand side is isomorphic to

$$\underset{x\in\mathcal{I}^j}{\underrightarrow{\operatorname{holim}}}\,\underset{0\neq H\subset G}{\underleftarrow{\operatorname{holim}}} Map_*(Z(\mathbf{x})^H, W(\mathbf{x}))^G$$

which is equivalent to

$$\underset{x\in\mathcal{I}^j}{\underrightarrow{\operatorname{holim}}} Map_*(\cup_{0\neq H\subset G} Z(\mathbf{x})^H, W(\mathbf{x}))^G$$

(the union can be replaced by the corresponding homotopy colimit). Via this equivalence the homotopy fiber of

$$sd^{|G|}THH(A,X)^G_{j-1} \longrightarrow \underset{0\neq H\subset G}{\underleftarrow{\operatorname{holim}}} sd^{|G/H|}THH(A,X)^{G/H}_{j-1}$$

is equivalent to

$$\underset{x\in\mathcal{I}^j}{\underrightarrow{\operatorname{holim}}} Map_*(U(\mathbf{x}), W(\mathbf{x}))^G$$

where $U(\mathbf{x}) = Z(\mathbf{x})/\cup_{0\neq H\subset G} Z(\mathbf{x})^H$. Then the same argument leads us to our conclusion, using that $U(\mathbf{x})$ is a free finite based G-space.

1.4 Spherical group rings

In the special case of spherical group rings the restriction maps split, making it possible to give explicit models for the C_{p^n} fixed point spectra of topological Hochschild homology.

{lem:Rspl

Lemma 1.4.1 *The restriction maps split for spherical group rings.*

Proof: Let G be a simplicial group. We will prove that the restriction map $sd_{ab}THH(\mathbf{S}[G],X)^{C_{ab}} \to sd_aTHH(\mathbf{S}[G],X)^{C_a}$ splits. We fix an object $\mathbf{x}\in\mathcal{I}^j$, and consider the restriction map

$$\begin{array}{c} Map_*\left(\left(\bigwedge_{i=1}^j S^{x_i}\right)^{\wedge ab}, X\wedge\left(\bigwedge_{i=1}^j (S^{x_i}\wedge G_+)\right)^{\wedge ab}\right)^{C_{ab}} \\ \downarrow \\ Map_*\left(\left(\bigwedge_{i=1}^j S^{x_i}\right)^{\wedge a}, X\wedge\left(\bigwedge_{i=1}^j (S^{x_i}\wedge G_+)\right)^{\wedge a}\right)^{C_a} \end{array}.$$

Let $S = \left(\bigwedge_{i=1}^j S^{x_i}\right)^{\wedge a}$, and consider the isomorphism

$$|S^{\wedge b}| \cong |S|\wedge S^{\perp}$$

coming from the one-point compactification of

$$\mathbf{R}^n\otimes\mathbf{R}^b\cong\mathbf{R}^n\otimes(\mathrm{diag}\oplus\mathrm{diag}^\perp)\cong\mathbf{R}^n\oplus(\mathbf{R}^n\otimes\mathrm{diag}^\perp)$$

where diag $\subseteq R^b$ is the diagonal line and $n=a\cdot\vee\mathbf{x}$. The desired splitting

$$Map_*(S,X\wedge S\wedge G_+^{\times ja})^{C_a}\to Map_*(S^{\wedge b},X\wedge S^{\wedge b}\wedge G_+^{\times jab})^{C_{ab}}$$

is gotten by sending $f\colon|S|\to|X\wedge S\wedge G_+^{\times ja}|$ to

$$\begin{array}{ccccc}|S^{\wedge b}|\cong|S|\wedge S^\perp & \xrightarrow{f\wedge id} & |X\wedge S\wedge G_+^{\times ja}|\wedge S^\perp & & \\ \cong & & |X\wedge S|\wedge S^\perp\wedge|G_+^{\times ja}| & & . \\ \cong & & |X\wedge S^{\wedge b}\wedge G_+^{\times ja}| & \xrightarrow{id\wedge\mathrm{diag}} & |X\wedge S^{\wedge b}\wedge G_+^{\times jab}|\end{array}$$

∎

Example 1.4.2 To see how the isomorphism $|S|^{\wedge b}\cong|S|\wedge S^\perp$ of the above proof works, consider the following example.

$S=S^1$, $b=2$, $|S|^2\cong|S|\wedge S^\perp$ is gotten from

$$\mathbf{R}^2\cong\mathbf{R}\oplus\mathbf{R},\qquad \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\mapsto((a+b)/2,(a-b)/2)$$

and the $\mathbf{Z}/2$ action is trivial in the first factor and mult by -1 in the other. Notice that if $f\colon|S|\to|X\wedge S\wedge G_+|$ sends $a\in\mathbf{R}^*=|S|$ to $x_a\wedge s_a\wedge g_a$, then the composite

$$|S^2|\cong|S|\wedge S^\perp\xrightarrow{f\wedge id}|X\wedge S\wedge G_+|\wedge S^\perp\cong|X\wedge S^2\wedge G_+|\to|X\wedge S^2\wedge G_+^{\times 2}|$$

sends $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ to $((a+b)/2,(a-b)/2)$ to

$$x_{(a+b)/2}\wedge s_{(a+b)/2}\wedge g_{(a+b)/2}\wedge(a-b)/2$$

to

$$x_{(a+b)/2}\wedge\left[\begin{smallmatrix}s_{(a+b)/2+(a-b)/2}\\ s_{(a+b)/2-(a-b)/2}\end{smallmatrix}\right]\wedge g_{(a+b)/2}$$

to

$$x_{(a+b)/2}\wedge\left[\begin{smallmatrix}s_{(a+b)/2+(a-b)/2}\\ s_{(a+b)/2-(a-b)/2}\end{smallmatrix}\right]\wedge g_{(a+b)/2}\wedge g_{(a+b)/2}.$$

Exchanging a and b in this formula transforms it to

$$x_{(a+b)/2}\wedge\left[\begin{smallmatrix}s_{(a+b)/2-(a-b)/2}\\ s_{(a+b)/2+(a-b)/2}\end{smallmatrix}\right]\wedge g_{(a+b)/2}\wedge g_{(a+b)/2}.$$

sphgprings}

Corollary 1.4.3 *The map*

$$\bigvee_{j=0}^{n}|THH(\mathbf{S}[G])|_{hC_{p^j}}\to|THH(\mathbf{S}[G])|^{C_{p^n}}$$

induced by the splitting is a stable equivalence, and the restriction map correspond to the projection.

2 Topological cyclic homology.

{sec:VTC}

In this section we finally will give a definition of topological cyclic homology. We first will define the pieces $TC(-,p)$ which are relevant to the p-complete part of TC, and later merge this information with the rational information coming from the homotopy fixed points of the whole circle action.

2.1 The definition and properties of $TC(-,p)$

as an intermediate stage, we define the functors $TC(-,p)$ which captures the information of topological cyclic homology when we complete at the prime p. We still list the case of an $\mathbf{S}$-algebra separately, in case the reader feels uncomfortable with $\Gamma\mathcal{S}_*$-categories.

Definition 2.1.1 Let p be a prime, A an $\mathbf{S}$-algebra and X a space. Recall that $\mathcal{RF}_p \subset \mathcal{RF}$ is the full subcategory of powers of p. We define

$$TC(A;X,p) = \underset{\overleftarrow{p^n\in\mathcal{RF}_p}}{\text{holim}}\, T\langle p^n\rangle(A;X)$$

This gives rise to the spectrum

$$\underline{TC}(A,X;p) = \underset{\overleftarrow{p^n\in\mathcal{RF}_p}}{\text{holim}}\, \underline{T}\langle p^n\rangle(A;X) = \{k \mapsto TC(A;S^k\wedge X,p)\}$$

If $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$-category we define

$$TC(\mathcal{C};X,p) = \underset{\overleftarrow{p^n\in\mathcal{RF}_p}}{\text{holim}}\, T\langle p^n\rangle(\mathcal{C};X)$$

with associated spectrum

$$\underline{TC}(\mathcal{C};X,p) = \underset{\overleftarrow{p^n\in\mathcal{RF}_p}}{\text{holim}}\, \underline{T}\langle p^n\rangle(\mathcal{C};X) = \{k \mapsto TC(\mathcal{C};S^k\wedge X,p)\}$$

If $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category we have a spectrum

$$TC(\bar{H}\mathcal{C};X,p) = \{k \mapsto TC(\bar{H}\mathcal{C}(S^k);X,p)\}$$

We get the analogs of the results in the previous chapter directly:

Lemma 2.1.2 *Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category, X a space and p a prime. Then*

1. *$\underline{TC}(\mathcal{C};X,p)$ is an Ω-spectrum.*
2. *The functor $\underline{TC}(-;X,p)$ takes $\Gamma\mathcal{S}_*$-equivalences of categories to equivalences.*
3. *If A is a ring, then the inclusion $A \subseteq \mathcal{P}_A$ as a rank one module induces an equivalence $\underline{TC}(A;X,p) \xrightarrow{\sim} \underline{TC}(\mathcal{P}_A;X,p)$*

4. *$\underline{TC}(-;X,p)$ preserves products up to pointwise equivalence.*

5. *If $\mathcal{C} \to \mathcal{D}$ is a $\Gamma\mathcal{S}_*$-functor inducing an equivalence $THH(\mathcal{C}) \to THH(\mathcal{D})$, then it induces an equivalence*
$$\underline{TC}(\mathcal{C};X,p) \to \underline{TC}(\mathcal{D};X,p)$$

6. *If $\mathcal{C}$ is a symmetric monoidal $\Gamma\mathcal{S}_*$-category, then $TC(\bar{H}\mathcal{C};X,p)$ is an Ω-spectrum equivalent to $\underline{TC}(\mathcal{C};X,p)$.*

Proof: This follows from the corresponding properties for $\underline{T}\langle p^n\rangle$ from section 1.4, and the properties of homotopy limits. ∎

Here we see that it made a difference that we considered $T\langle a\rangle(\bar{H}\mathcal{C};X)$ as the spectrum $\{n \mapsto T\langle a\rangle(\bar{H}\mathcal{C}(S^n);X)\}$, and not as a Γ-space $\{k_+ \mapsto T\langle a\rangle(\bar{H}\mathcal{C}(k_+);X)\}$: the spectrum associated to the (pointwise) homotopy limit of a Γ-space is not the same as the (pointwise) homotopy limit of the spectrum, since the homotopy limits can destroy connectivity. We will shortly see that this is not a real problem, since $\underline{TC}(\mathcal{C};X,p)$ is always -2-connected, and so $TC(\bar{H}\mathcal{C};X,p)$ and the spectrum associated with $\{k_+ \mapsto TC(\bar{H}\mathcal{C}(k_+);X,p)\}$ will be equivalent once X is connected. In any case, it may be that the correct way of thinking of this, is to view TC of symmetric monoidal $\Gamma\mathcal{S}_*$-categories as Γ-spectra:

$$\{k_+ \mapsto \underline{TC}(\bar{H}\mathcal{C}(k_+);X,p)\}$$

This point will become even more acute when we consider the homotopy fixed point spectra for the entire circle actions since these are not connective.

revise!! If $\mathcal{C}$ is exact we have an equivalent Ω-spectrum

$$\mathbf{TC}(\mathcal{C};X,p) = \underset{\overleftarrow{p^n\in\mathcal{RF}_p}}{\text{holim}}\, T\langle p^n\rangle(\S\mathcal{C};X) = \{k \mapsto TC(S^{(k)}\mathcal{C};X,p)\}$$

If A is a ring we let
$$\mathbf{TC}(A,X;p) = \mathbf{TC}(\mathcal{P}_A,X)\}$$
and we see that $\mathbf{TC}(A;X,p)$ is equivalent to $\underline{TC}(A;X,p)$.

2.2 Some structural properties of $TC(-,p)$

A priori, the category $\mathcal{RF}_p$ can seem slightly too big for comfort, but it turns out to be quite friendly, especially if we consider the F and R maps separately. This separation gives us good control over the homotopy limit defining $TC(-,p)$. For instance, we shall see that it implies that $\underline{TC}$ is connective, can be computed degreewise and almost preserves id-cartesian cubes, and hence is "determined" by its value on ordinary rings.

2.2.1 Calculating homotopy limits over $\mathcal{RF}_p$

Consider the two subcategories $\mathcal{F}_p$ and $\mathcal{R}_p$ of $\mathcal{RF}_p$, namely the ones with only the F (Frobenius = inclusion of fixed points) maps or only the R (restriction) maps. We will typically let

$$TR(A;X;p) = \underset{\overleftarrow{p^n\in\mathcal{R}_p}}{\operatorname{holim}} T\langle p^n\rangle(A;X) \qquad \text{and} \qquad TF(A,X;p) = \underset{\overleftarrow{p^n\in\mathcal{F}_p}}{\operatorname{holim}} T\langle p^n\rangle(A;X)$$

and similarly for the spectra and the related functors of $\Gamma\mathcal{S}_*$-categories.

Let L be any functor from $\mathcal{RF}_p$ to spaces, and let $\langle x, y, \dots\rangle$ be the monoid generated by some number of commuting letters. If $\langle x\rangle$ acts on a space Y, we write $\operatorname{holim}_{\overleftarrow{\langle x\rangle}} Y$ as Y^{hx}, in analogy with the group case, and it may be calculated as the homotopy limit

$$Y^{hx} = \underset{\leftarrow}{\operatorname{holim}} \begin{pmatrix} & Map(I_+, Y) \\ & \big\downarrow{\scriptstyle f\mapsto(f(0),f(1))} \\ Y \xrightarrow{(1,x)\Delta} & Y\times Y \end{pmatrix}$$

We see that $\langle R, F\rangle$ acts on $\prod_{p^n\in\mathcal{RF}_p} L(p^n)$, and writing out the cosimplicial replacement carefully, we see that

$$\begin{aligned}\underset{\overleftarrow{\mathcal{RF}_p}}{\operatorname{holim}} L \cong & Tot(q\mapsto \prod_{N_q\langle F,R\rangle}(\prod_{p^n\in\mathcal{RF}_p} L(p^n))) \\ \cong & \underset{\overleftarrow{\langle R,F\rangle}}{\operatorname{holim}}(\prod_{p^n\in\mathcal{RF}_p} L(p^n)) \cong \underset{\overleftarrow{\langle R\rangle\times\langle F\rangle}}{\operatorname{holim}}(\prod_{p^n\in\mathcal{RF}_p} L(p^n))\end{aligned}$$

We may choose to take the homotopy limit over the product $\langle R\rangle\times\langle F\rangle$ in the order we choose. If we take the R map first we get

Lemma 2.2.2 *Let L be a functor from $\mathcal{RF}_p$ to spaces. Then*

$$\begin{aligned}\underset{\overleftarrow{\mathcal{RF}_p}}{\operatorname{holim}} L &\cong \underset{\overleftarrow{\langle F\rangle}}{\operatorname{holim}}(\underset{\overleftarrow{\langle R\rangle}}{\operatorname{holim}}(\prod_{p^n\in\mathcal{RF}_p} L(p^n))) \\ &\cong \underset{\overleftarrow{\langle F\rangle}}{\operatorname{holim}}\,\underset{\overleftarrow{p^n\in\mathcal{R}_p}}{\operatorname{holim}} L(p^n) = (\underset{\overleftarrow{p^n\in\mathcal{R}_p}}{\operatorname{holim}} L(p^n))^{hF}\end{aligned}$$

Similarly we may take the F map first and get the same result with R and F interchanged.

For our applications we note that

$$TC(A;X,p)\simeq TR(A;X,p)^{hF}\simeq TF(A;X,p)^{hR}$$

Lemma 2.2.3 *The spectrum $\underline{TC}(-;p)$ is -2 connected, and likewise for the other variants.*

Proof: Consider the short exact sequence

$$0 \to \lim_{\overleftarrow{p^n \in \mathcal{R}_p}}{}^{(1)} \pi_{k+1}\underline{T}\langle p^n\rangle(\mathcal{C};X) \to \pi_k \underline{TR}(\mathcal{C};X,p) \to \lim_{\overleftarrow{p^n \in \mathcal{R}_p}} \pi_k \underline{T}\langle p^n\rangle(\mathcal{C};X) \to 0$$

of the tower defining TR. Since $\pi_0\underline{T}\langle p^n\rangle(\mathcal{C};X) \to \pi_0\underline{T}\langle p^{n-1}\rangle(\mathcal{C};X)$ is always surjective (its cokernel is $\pi_{-1}sd_{p^n}\underline{T}\langle p^n\rangle(\mathcal{C};X)_{C_{p^n}} = 0$), the $\lim_{\overleftarrow{R}}{}^{(1)}$ part vanishes, and $\underline{TR}$ is always -1-connected (alternatively, look at the spectral sequence of the R tower, and note that all the fibers are -1-connected). Hence the pullback $\underline{TC}(\mathcal{C};p) \simeq TR(\mathcal{C};p)^{hF}$ cannot be less than -2-connected. ∎

TCpdegwise}

Lemma 2.2.4 *If A is a simplicial* $\mathbf{S}$*-algebra, then $\underline{TC}(A;X,p)$, may be calculated degreewise in that*

$$diag^*\{[q] \mapsto \underline{TC}(A_q;X,p)\} \simeq \underline{TC}(diag^*A;X,p)$$

Proof: This is true for THH (lemma III.1.3.1), and so, by the fundamental cofibration sequence (ref) it is true for all $sd_{p^n}THH(A;X)^{C_{p^n}}$. By corollary A1.9.? of the appendix, homotopy limits of towers of connective simplicial spectra may always be computed degreewise, so

$$\underline{TR}(A;p) = \operatorname{holim}_{\overleftarrow{R}} sd_{p^n}\underline{T}(A)^{C_{p^n}}$$

is naturally equivalent to $diag^*\{[q] \mapsto \underline{TR}(A_q;p)\}$. Now, $\underline{TC}(A;p) \simeq \underline{TR}(A;p)^{hF}$, a homotopy pullback construction which may be calculated degreewise. ∎

ma:VTCconn}

Lemma 2.2.5 *Let $f\colon A \to B$ be a k-connected map of* $\mathbf{S}$*-algebras and X an l-connected space. Then*

$$\underline{TC}(A;X,p) \to \underline{TC}(B;X,p)$$

is $k+l-1$-connected.

Proof: Since $THH(-;X)$, and hence the homotopy orbits of $THH(-;X)$, rise connectivity by l, we get by the tower defining TR that $\underline{TR}(-;X,p)$ also rises connectivity by l. We may loose one when taking the fixed points under the F action to get $\underline{TC}(-;X,p)$. ∎

When restricted to simplicial rings, there is a cute alternative to this proof using the fact that any functor from simplicial rings to n-connected spectra which preserves equivalences and may be computed degreewise, sends $k \geq 0$-connected maps to $n+k+1$-connected maps.

Lemma 2.2.6 *Assume $\mathcal{A}$ is a cube of* $\mathbf{S}$*-algebras such that $\underline{T}(\mathcal{A};X)$ is id-cartesian. Then $\underline{TR}(\mathcal{A};X,p)$ is also id cartesian.*

Proof: Choose a big k such that $THH(\mathcal{A},S^k \wedge X)$ is $id+k$ cartesian Let $\mathcal{X}$ be any m subcube and $\mathcal{X}^l = sd_{p^l}\mathcal{X}^{C_{p^l}}$. We are done if we can show that $\operatorname{holim}_{\overleftarrow{R}} \mathcal{X}^l$ is $(m+k)$-cartesian. Let Z^l be the iterated fiber of $\mathcal{X}^l$ (i.e., the homotopy fiber of $\mathcal{X}^l_\emptyset \to \operatorname{holim}_{\overleftarrow{S \neq \emptyset}} \mathcal{X}^l_S$). Then $Z = \operatorname{holim}_{\overleftarrow{R}} Z^l$ is the iterated fiber of $\operatorname{holim}_{\overleftarrow{R}} \mathcal{X}^l$, and we must show that Z is $m+k-1$ connected. Since homotopy orbits preserve connectivity and homotopy colimits,

$THH(\mathcal{A}, S^k \wedge X)_{hC_{p^l}}$ must be $id+k$ cartesian, and so the fiber of $R\colon \mathcal{X}^l \to \mathcal{X}^{l-1}$ is $id+k$ cartesian. Hence $\pi_q Z^l \to \pi_q Z^{l-1}$ is surjective for $q = m+k$ and an isomorphism for $q < m+k$, and so $\pi_q Z \cong \lim_{\leftarrow} R^{(1)} \pi_{q+1} Z^l \times \lim_{\leftarrow} R\ \pi_q Z^l = 0$ for $q < m+k$. ∎

{prop:VTC}

Proposition 2.2.7 *Assume $\mathcal{A}$ is a cube of* **S***-algebras such that $\underline{T}(\mathcal{A}; X)$ is id cartesian. Then $\underline{TC}(\mathcal{A}; X, p)$ is $id-1$ cartesian.*

Proof: This follows from the lemma, plus the interpretation of $\underline{TC}(-,p) \simeq \underline{TR}(-,p)^{hF}$ as a pullback. ∎

When applying this to the canonical resolution of **S**-algebras by $H\mathbf{Z}$-algebras of III.3.1.8, we get the result saying essentially that TC is determined by its value on simplicial rings:

{theo:VTC}

Theorem 2.2.8 *Let A be an* **S***-algebra and X a space. Let $S \mapsto (A)_S$ be the cubical diagram of III.3.1.8. Then*

$$\underline{TC}(A; X, p) \xrightarrow{\sim} \operatorname{holim}_{\overleftarrow{S \neq \emptyset}} \underline{TC}(\mathcal{A}_S; X, p)$$ ☺

2.2.9 The Frobenius maps

The reason the inclusion of fixed points map F now often is called the Frobenius map is that Hesselholt and Madsen [50] has shown that if A is a commutative ring, then $\pi_0 TR(A;p)$ is isomorphic to the p-typical Witt vectors $W(A,p)$, and that F corresponds to the Frobenius map.

Even better, they prove that there is an isomorphism

$$\pi_0 THH(A)^{\mathsf{C}_{p^n}} \cong W_n(A)$$

where $W_n(A)$ is the ring of truncated p-typical Witt vectors, i.e., it is A^n as an abelian group, but with multiplication gotten by requiring that the "ghost map"

$$w\colon W_n(A) \to A^n, \qquad (a_0, \dots, a_{n-1}) \mapsto (w_0, \dots, w_{n-1})$$

where

$$w_i = a_0^{p^i} + p a_1^{p^{i-1}} + \dots + p^i a_i$$

is a ring map. If A has no p-torsion the ghost map is injective.

The map

$$R\colon W_{n+1}(A) \to W_n(A) \qquad (a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1}),$$

is called the restriction and the isomorphisms

$$\pi_0 THH(A)^{C_{p^n}} \cong W_n(A)$$

respect the restriction maps.

On the Witt vectors the Frobenius and Verschiebung are given by

$$F, V\colon W(A) \to W(A)$$
$$F(w_0, w_1, \dots) = (w_1, w_2, \dots)$$
$$V(a_0, a_1, \dots) = (0, a_0, a_1, \dots).$$

. satisfying the relations

$$x \cdot V(y) = V(F(x) \cdot y), \quad FV = p, \quad VF = mult_{V(1)}$$

(if A is an $\mathbf{F}_p$-alg. then $V(1) = p$).

2.2.10 $TC(-;p)$ of spherical group rings

Let G be a simplicial group. We briefly sketch the argument of [6] giving $TC(\mathbf{S}[G];p)$ (see also [103]). Recall from corollary 1.4.3 that

$$|THH(\mathbf{S}[G])|^{C_{p^n}} \xrightarrow{\sim} \prod_{j=0}^{n} |THH(\mathbf{S}[G])|_{hC_{p^j}},$$

and that the restriction map corresponds to the projection

$$\prod_{j=0}^{n} |THH(\mathbf{S}[G])|_{hC_{p^j}} \to \prod_{j=0}^{n-1} |THH(\mathbf{S}[G])|_{hC_{p^j}}.$$

What is the inclusion of fixed point map $|THH(\mathbf{S}[G])|^{C_{p^n}} \subseteq |THH(\mathbf{S}[G])|^{C_{p^{n-1}}}$ in this factorization? Write T as shorthand for $|THH(\mathbf{S}[G])|$, and consider the diagram

$$\begin{array}{ccccc} & & T_{hC_p} \vee T & & \\ & & \downarrow & \searrow^{\text{proj}} & \\ T_{hC_p} & \longrightarrow & T^{C_p} & \underset{S}{\overset{R}{\rightleftarrows}} & T \\ & \searrow_{\text{trf}} & \downarrow F & & \\ & & T & & \end{array}$$

where S is the section of R defined in the proof of lemma 1.4.1.

The trf in the diagram above is the composite (in the homotopy category)

$$T_{hC_p} = (E\mathbb{T}_+ \wedge T)_{C_p} \simeq (E\mathbb{T}_+ \wedge T)^{C_p} \xrightarrow{F} E\mathbb{T}_+ \wedge T \simeq T$$

and is called the *transfer*. Generally we will let the transfer be any (natural) map in the stable homotopy category making

$$\begin{array}{ccccc} T_{hC_{p^n}} & \simeq & (E\mathbb{T}_+ \wedge T)^{C_{p^n}} & \longrightarrow & T^{C_{p^n}} \\ {\scriptstyle \text{trf}}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ T_{hC_{p^{n-1}}} & \simeq & (E\mathbb{T}_+ \wedge T)^{C_{p^{n-1}}} & \longrightarrow & T^{C_{p^{n-1}}} \end{array}$$

commute.

Hence, the inclusion of fixed points $F\colon T^{C_{p^{n+1}}} \to T^{C_{p^n}}$ acts as $FS\colon T \to T$ on the zero'th factor, and as $\mathrm{trf}\colon T_{hC_{p^n}} \to T_{hC_{p^{n-1}}}$ on the others

$$\begin{array}{ccccc} T & \times & T_{hC_p} & \times & T_{hC_{p^2}} \\ \downarrow{\scriptstyle FS} & \swarrow{\scriptstyle \mathrm{trf}} & & \swarrow{\scriptstyle \mathrm{trf}} & \\ T & \times & T_{hC_p} & & \\ \downarrow{\scriptstyle FS} & \swarrow{\scriptstyle \mathrm{trf}} & & & \\ T & & & & \end{array}$$

$$TC(\mathbf{S}[G],p) = \operatorname*{holim}_{\overleftarrow{\mathcal{RF}_p}} T\langle p^n\rangle(\mathbf{S}[G]) \simeq \left(\operatorname*{holim}_{\overleftarrow{R}} T\langle p^n\rangle(\mathbf{S}[G])\right)^{hF}$$

which is the homotopy equalizer of the map

$$\begin{array}{ccccccc} T & \times & T_{hC_p} & \times & T_{hC_{p^2}} & \times & \cdots \\ \downarrow{\scriptstyle FS} & \swarrow{\scriptstyle \mathrm{trf}} & & \swarrow{\scriptstyle \mathrm{trf}} & & \swarrow{\scriptstyle \mathrm{trf}} & \\ T & \times & T_{hC_p} & \times & T_{hC_{p^2}} & \times & \cdots \end{array}$$

and the identity; or equivalently, the "diagram"

$$\begin{array}{ccc} TC(\mathbf{S}[G],p) & \longrightarrow & \operatorname*{holim}_{\overleftarrow{\mathrm{trf}}} |THH(\mathbf{S}[G])|_{hC_{p^n}} \\ \downarrow & & \downarrow \\ |THH(\mathbf{S}[G])| & \xrightarrow{FS-1} & |THH(\mathbf{S}[G])| \end{array}$$

is homotopy cartesian (in order to make sense of this, one has to have chosen models for all the maps, see e.g., [103]).

We will identify these terms more closely in VII.3

2.2.11 Relation to the homotopy fixed points

Lemma 2.2.12 *The natural maps* $(\underline{T}(A;X)^{h\mathbf{S}^1})^{\widehat{}}_p \to \operatorname{holim}_{\overleftarrow{\mathcal{F}_p}} \underline{T}(A;X)^{hC_{p^r}}{}^{\widehat{}}_p$ *is an equivalence.*

Proof: This follows from the general fact A.1.9.8.5 that $(Y^{h\mathbb{T}})^{\widehat{}}_p \xrightarrow{\sim} \operatorname{holim}_{\overleftarrow{\mathcal{F}_p}} Y^{hC_{p^r}}{}^{\widehat{}}_p$ for any $\mathbb{T}$-spectrum Y. ∎

2.3 The definition and properties of TC

Definition 2.3.1 We define $\underline{TC}$ to be the functor from **S**-algebras, or more generally $\Gamma\mathcal{S}_*$-categories, to spectra to be the homotopy limit of

$$\begin{array}{ccc} & & \underline{T}(-)^{h\mathbb{T}} \\ & & \downarrow \\ \prod_{p \text{ prime}} \underline{TC}(-;p)^{\wedge}_p & \longrightarrow & \prod_{p \text{ prime}} \underset{\overleftarrow{p^r \in \mathcal{F}_p}}{\operatorname{holim}} \underline{T}(-)^{hC_{p^r}}{}^{\wedge}_p \end{array}$$

where the lower map is given by the projection onto $\mathcal{F}_p \subseteq \mathcal{RF}_p$

$$\underline{TC}(-;p) = \underset{\overleftarrow{p^r \in \mathcal{RF}_p}}{\operatorname{holim}} \underline{T}\langle p^r\rangle(-) \to \underset{\overleftarrow{p^r \in \mathcal{F}_p}}{\operatorname{holim}} \underline{T}\langle p^r\rangle(-)$$

followed by the map from the fixed points to the homotopy fixed points

$$\underline{T}\langle p^r\rangle(-) \cong \underline{T}(-)^{C_{p^r}} \to \underline{T}(-)^{hC_{p^r}}.$$

More useful than the definition is the characterization given by the following lemma.

emma:VsqTC}

Lemma 2.3.2 *All the squares in*

$$\begin{array}{ccccc} \underline{TC}(-) & \longrightarrow & \underline{T}(-)^{h\mathbb{T}} & \longrightarrow & (\underline{T}(-)_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{TC}(-)^{\wedge} & \longrightarrow & (\underline{T}(-)^{\wedge})^{h\mathbb{T}} & \longrightarrow & (\underline{T}(-)^{\wedge}_{(0)})^{h\mathbb{T}} \end{array}$$

are homotopy cartesian.

Proof: The rightmost square is cartesian as it is an arithmetic square to which $-^{h\mathbb{T}}$ is applied, and the leftmost square is cartesian by the definition of TC since

$$\underline{TC}(-)^{\wedge} \simeq \prod_{p \text{ prime}} \underline{TC}(-)^{\wedge}_p \simeq \prod_{p \text{ prime}} \underline{TC}(-,p)^{\wedge}_p$$

and by the equivalence

$$\underset{\overleftarrow{p^r \in \mathcal{F}_p}}{\operatorname{holim}} \underline{T}(-)^{hC_{p^r}}{}^{\wedge}_p \longleftarrow \underline{T}(-)^{h\mathbb{T}}{}^{\wedge}_p.$$

∎

Csq srings}

Corollary 2.3.3 *Let A be a simplicial ring. Then*

$$\begin{array}{ccc} \underline{TC}(A;X) & \longrightarrow & (\underline{HH}(A;X)_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \underline{TC}(A;X)^{\wedge} & \longrightarrow & (\underline{HH}(A;X)^{\wedge}_{(0)})^{h\mathbb{T}} \end{array}$$

is homotopy cartesian.

Proof: This follows from lemma 2.3.2 by extending the square to the right with the $\mathbb{T}$-homotopy fixed point spectra of the square

$$\begin{array}{ccc} \underline{T}(-)_{(0)} & \xrightarrow{\simeq} & \underline{HH}(-)_{(0)} \\ \downarrow & & \downarrow \\ \underline{T}(-)^{\widehat{}}_{(0)} & \xrightarrow{\simeq} & \underline{HH}(-)^{\widehat{}}_{(0)} \end{array}$$

which is cartesian by lemma IV.1.3.8 which says that the horizontal maps are equivalences. ∎

{theo:VTC

Theorem 2.3.4 *Let A be an $\mathbf{S}$-algebra and X a space. Let $\mathcal{A}$ be the cubical diagram of III.3.1.8. Then*

$$\underline{TC}(A;X) \xrightarrow{\sim} \operatorname{holim}_{\overleftarrow{S\neq\emptyset}} \underline{TC}(\mathcal{A}_S;X).$$

Proof: By theorem 2.2.8 this is true for $TC(-;X)^{\widehat{}}$ (products and completions of spectra commute with homotopy limits). Since $\underline{T}(\mathcal{A};X)$ is id-cartesian, so are $\underline{T}(\mathcal{A};X)_{(0)}$ and $\underline{T}(\mathcal{A};X)^{\widehat{}}_{(0)}$, and hence

$$\underline{T}(A;X)_{(0)} \xrightarrow{\sim} \operatorname{holim}_{\overleftarrow{S\neq\emptyset}}(\underline{T}(\mathcal{A}_S;X)_{(0)})$$

and

$$\underline{T}(A;X)^{\widehat{}}_{(0)} \xrightarrow{\sim} \operatorname{holim}_{\overleftarrow{S\neq\emptyset}}(\underline{T}(\mathcal{A}_S;X)^{\widehat{}}_{(0)})$$

Since homotopy fixed points commute with homotopy limits we are done since we have proved the theorem for all the theories but TC in the outer homotopy cartesian square of lemma 2.3.2. ∎

3 The homotopy $\mathbb{T}$-fixed points and the connection to cyclic homology of simplicial rings

Theorem 2.3.4 tells us that we can obtain much information about TC from our knowledge of simplicial rings. We have seen (corollary 2.3.3) that, when applied to a simplicial ring A, TC fits into the cartesian square

$$\begin{array}{ccc} \underline{TC}(A;X) & \longrightarrow & (\underline{HH}(A;X)_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \underline{TC}(A;X)^{\widehat{}} & \longrightarrow & (\underline{HH}(A;X)^{\widehat{}}_{(0)})^{h\mathbb{T}} \end{array}$$

We can say something more about the right hand column, especially in some relative cases. As a matter of fact, it is calculated by *negative cyclic homology*, a theory which we will recall the basics about shortly, and which much is known about.

If A is a simplicial ring, we let $(C_*(A), b)$ be the chain complex associated to the bisimplicial abelian group. $HH(A)$

To make the comparison easier we first describe spectral sequences computing the homotopy groups of the homotopy fixed and orbit spectra.

3.1 On the spectral sequences for the $\mathbb{T}$- homotopy fixed point and orbit spectra

Let $\mathbb{T}$ be the circle group and let X be a $\mathbb{T}$-spectrum. The collapse maps give an isomorphism

$$\pi_*(\Sigma^\infty \mathbb{T}_+) \cong \pi_*(\Sigma^\infty S^1) \oplus \pi_*(\Sigma^\infty S^0),$$

and let σ, η be the elements in $\pi_1(\Sigma^\infty \mathbb{T}_+)$ projecting down to the identity class

$$\{S^{n+1} = S^{n+1}\} \in \pi_1(\Sigma^\infty S^1) \cong \mathbf{Z}$$

and the stable Hopf map

$$\{S^n \wedge S^3 \to S^n \wedge \mathbf{C}\mathrm{P}^1 \cong S^n \wedge S^2\} \in \pi_1(\Sigma^\infty S^0) \cong \mathbf{Z}/2\mathbf{Z}$$

respectively. The spectral sequences coming from the homotopy limit and colimit spectral sequences have interesting E^1-terms. The following is shown in [44], and for identification of the differential, see [47, 1.4.2].

3S1fixptSS}

Lemma 3.1.1 *Let $\mathbb{T}$ be the circle group and let X be a $\mathbb{T}$-spectrum. The E^2 sheet of the spectral sequence for $X_{h\mathbb{T}}$, comes from an E^1 sheet with*

$$E^1_{s,t}(X_{h\mathbb{T}}) = \pi_{t-s}X, \qquad t \geq s \geq 0$$

and where the differentials

$$d^1_{s,t}\colon E^1_{s,t} = \pi_{t-s}X \to \pi_{t-s+1}X = E^1_{s-1,t}$$

are induced by the map $\sigma + s \cdot \eta\colon \pi_{t-s}X \to \pi_{t-s+1}(\mathbb{T}_+ \wedge X)$ composed with $\pi_{t-(s-1)}$ of the $\mathbb{T}$-action $\mathbb{T}_+ \wedge X \to X$.

Likewise, the E^2 sheet of the spectral sequence for $X^{h\mathbb{T}}$, comes from an E^1 sheet with

$$E^1_{s,t}(X^{h\mathbb{T}}) = \pi_{t-s}X, \qquad t \geq s \leq 0$$

and where the differentials are the same as for the homotopy orbit spectral sequence.

The picture is as follows:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & . \\
\xleftarrow{d^1_{2,-1}} & & \pi_{2-(-1)}X & \xleftarrow{d^1_{2,0}} & \pi_{2-0}X & \xleftarrow{d^1_{2,1}} & \pi_{2-1}X & \xleftarrow{d^1_{2,2}} & \\
\xleftarrow{d^1_{1,-1}} & & \pi_{1-(-1)}X & \xleftarrow{d^1_{1,0}} & \pi_{1-0}X & \xleftarrow{d^1_{1,1}} & \pi_{1-1}X & & \\
\xleftarrow{d^1_{0,-1}} & & \pi_{0-(-1)}X & \xleftarrow{d^1_{0,0}} & \pi_{0-0}X & & & & \\
\xleftarrow{d^1_{-1,-1}} & & \pi_{-1-(-1)}X & & & & & &
\end{array}
$$

Note that for rational $\mathbb{T}$-spectra – like the ones we are talking about in connection with TC – or more generally, Eilenberg-MacLane $\mathbb{T}$-spectra the Hopf map η is trivial, and so the differentials are simply the $\mathbb{T}$-action:

{cor:V Ta

Corollary 3.1.2 *Let X be a $\mathbb{T}$-spectrum such that $\eta\colon \pi_*X \to \pi_{*-1}X$ is trivial, then the differential*

$$d^1_{s,t}\colon E^1_{s,t} = \pi_{t-s}X \to \pi_{t-s+1}X = E^1_{s-1,t}$$

is induced by $S^1 \wedge X \subseteq S^1 \wedge X \vee S^0 \wedge X \simeq \mathbb{T}_+ \wedge X \to X$ where the latter map is the $\mathbb{T}$-action.

For convenience we reconstruct the bare essentials of the spectral sequence in a pedestrian language.

There is a particularly convenient model for $E(\mathbb{T})$ given as follows. Consider S^{2n-1} as the subspace of vectors in $\mathbf{C}^n$ of length 1. $\mathbb{T} = S^1 \subset \mathbf{C}$ acts on S^{2n-1} by complex multiplication in each coordinate, and this action is (unbased) free. The inclusion $\mathbf{C}^{n-1} \subseteq \mathbf{C}^n$ into the first coordinates gives an $\mathbb{T}$ inclusion $S^{2n-3} \subseteq S^{2n-1}$. So, taking the union of all S^{2n-1} as n varies we get a contractible free $\mathbb{T}$-space which we call $E(\mathbb{T})$. This space comes with a filtration, namely by the S^{2n-1}s, and this filtration is exactly the one giving rise to the above mentioned spectral sequence. In order to analyze the spectral sequence we need to know the subquotients of the filtration, but this is easy enough: there is a $\mathbb{T}$-isomorphism $S^{2n+1}/S^{2n-1} \cong S^1_+ \wedge S^{2n}$ given by considering S^{2n} as $\mathbf{C}\mathrm{P}^n/\mathbf{C}\mathrm{P}^{n-1}$ (with trivial $\mathbb{T}$ action) and sending the class of $(z_0, \dots z_n)$ in S^{2n+1}/S^{2n-1} to the class of $\left(\frac{z_n}{|z_n|} \wedge [z_0, \dots z_n]\right)$ in $S^1_+ \wedge \mathbf{C}\mathrm{P}^n/\mathbf{C}\mathrm{P}^{n-1}$.

The spectral sequence for $X_{h\mathbb{T}}$ comes from this filtration, and there are no convergence issues associated to this spectral sequence.

The spectral sequence for $X^{h\mathbb{T}}$ is from the point of view of [44] simply dual, but for the more pedestrian users we note that this actually makes sense even if you are very naïve about things. Using that $X^{h\mathbb{T}} = Map_*(E\mathbb{T}, X)^{\mathbb{T}}$, and $E\mathbb{T} = \lim_{\overrightarrow{k}} S^{2k+1}$ we can write $X^{h\mathbb{T}}$ as the limit

$$X^{h\mathbb{T}} \cong \lim_{\overleftarrow{n}} Map_*(S^{2n+1}_+, -)^{\mathbb{T}}$$

Recall the Bousfield–Kan spectral sequence of a tower of fibrations

$$\lim_{\overleftarrow{k}}(M^k) \twoheadrightarrow \ldots \twoheadrightarrow M^{n+1} \twoheadrightarrow M^n \twoheadrightarrow M^{n-1} \twoheadrightarrow \ldots M^1 \twoheadrightarrow M^0 = *.$$

It is a first quadrant spectral sequence, with $E_1^{s,t} = \pi_{t-s}F^s$ where F^s is the fiber of M^s. The differential $E_1^{s,t} \to E^{s+1,t}$ is induced given by $\pi_{t-s}F^s \to \pi_{t-s}M^s \to \pi_{t-s-}F^{s+1}$.

This means that $F^s = Map_*(S^1_+ \wedge S^{2s}, X)^{\mathbb{T}} \cong \Omega^{2s}X$, and we get that $E^1_{s,t} = \pi_{t-s}\Omega^{2s}X \cong \pi_{t+s}X$.

Strictly speaking, the Bousfield-Kan spectral sequence is set to zero outside $0 \leq s \leq t$, but in our case this restriction is not really relevant since we are going to apply the spectral sequence to (connective) $\mathbb{T}$-spectra. By reindexing, we shall think of this as a homologically indexed spectral sequence in the left half plane with

$$E^1_{s,t}(X^{h\mathbb{T}}) = \pi_{t-s}X, \qquad t \geq s \leq 0.$$

These spectral sequences fit into a bigger spectral sequence (corresponding to the Tate spectrum of [44])

$$E^1_{s,t}(\mathrm{Tate}) = \pi_{t-s}X, \qquad t \geq s$$

and there is a short exact sequence of spectral sequences

$$0 \to E^1_{s,t}(X^{h\mathbb{T}}) \to E^1_{s,t}(\mathrm{Tate}) \to E^1_{s-1,t-1}(X_{h\mathbb{T}}) \to 0$$

and the norm map induces the edge homomorphism. This has the following consequence:

quivalence}

Lemma 3.1.3 *Let X be an $\mathbb{T}$-spectrum. If the Tate spectral sequence converges to zero, then the norm map*

$$S^1 \wedge X_{h\mathbb{T}} \to X^{h\mathbb{T}}$$

is a stable equivalence.

3.2 Cyclic homology and its relatives

Let Z be a cyclic module, i.e., a functor from Λ^o to simplicial abelian groups. Let $B\colon Z_q \to Z_{q+1}$ be Connes' operator

$$Z_q \xrightarrow{N=\sum(-1)^{qj}t^j} Z_q \xrightarrow{(-1)^q s_q} Z_{q+1} \xrightarrow{(1+(-1)^q t)} Z_{q+1}$$

satisfying $B \circ B = 0$ and $B \circ b + b \circ B = 0$ where $b = \sum(-1)^j d_j$. Due to these relations, the B operator defines a complex

$$(\pi_* Z, B) = (\pi_0 Z \xrightarrow{B} \pi_1 Z \xrightarrow{B} \ldots \xrightarrow{B} \pi_q Z \xrightarrow{B} \ldots)$$

whose homology $H_*^{dR}(Z) = H_*(\pi_* Z, B)$ we call the *deRham homology* of Z.

Using the b and the B, and their relations one can form bicomplexes (called (b, B)-bicomplexes, see e.g., [74, 5.1.7] for more detail) with s, t-entry Z_{t-s} connected by the bs vertically and the Bs horizontally (the relations guarantee that this becomes a bicomplex).

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\dots & \xleftarrow{B} & Z_2 & \xleftarrow{B} & Z_1 & \xleftarrow{B} & \dots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\dots & \xleftarrow{B} & Z_1 & \xleftarrow{B} & Z_0 & & \\
 & & \downarrow{\scriptstyle b} & & & & \\
\dots & \xleftarrow{B} & Z_0 & & & &
\end{array}$$

If you allow $t \geq s$ you get the so-called *periodic* (b, B)*-bicomplex* $B^{per}(Z)$ (called $\mathcal{B}Z^{\mathrm{per}}$ in [74]), if you allow $t \geq s \geq 0$ you get the *positive* (b, B)*-bicomplex* $B^+(Z)$ and if you allow $t \geq s \leq 0$ you get the *negative* (b, B)*-complex* $B^-(Z)$.

If $-\infty \leq m \leq n \leq \infty$ we let $T^{m,n}Z$ be the total complex of the part of the normalized (b, B)–bicomplex which is between the mth and nth column: $T_q^{m,n}Z = \prod_{k=m}^{n} C_{k,q-k}^{\mathrm{norm}}(Z)$ (where C^{norm} denotes the normalized chains defined in A.1.7.0.7). The associated homologies $H_*(T^{-\infty,0}Z)$, $H_*(T^{-\infty,\infty}Z)$ and $H_*(T^{0,\infty}Z)$ are called *periodic*, *negative* and (simply) *cyclic* homology, and often denoted $HP_*(Z)$, $HC^-(Z)$ and $HC(Z)$. The associated short exact sequence of complexes

$$0 \to T^{-\infty,0}Z \to T^{-\infty,\infty}Z \to T^{1,\infty}Z \to 0$$

together with the isomorphism $T_q^{1,\infty}Z \cong T_{q-2}^{0,\infty}Z$ gives rise to the well-known long exact sequence

$$\dots \longrightarrow HC_{q-1}(Z) \longrightarrow HC_q^-(Z) \longrightarrow HP_q(Z) \longrightarrow HC_{q-2}(Z) \longrightarrow \dots,$$

and similarly one obtains the sequence

$$\dots \longrightarrow HC_{q-1}(Z) \xrightarrow{B} HH_q(Z) \xrightarrow{I} HC_q(Z) \xrightarrow{S} HC_{q-2}(Z) \longrightarrow \dots$$

(the given names of the maps are the traditional ones, and we will have occasion to discuss the S-map a bit further).

Notice that $T^{-\infty,n}Z = \lim_{\overleftarrow{m}} T^{m,n}Z$, and so if $\dots \twoheadrightarrow Z^{k+1} \twoheadrightarrow Z^k \twoheadrightarrow \dots$ is a sequence of surjections of cyclic modules with $Z = \lim_{\overleftarrow{k}} Z^k$, then $T^{-\infty,n}Z \cong \lim_{\overleftarrow{k}} T^{-\infty,n}Z^k$, and you have $\lim_{\leftarrow}{}^{(1)}$-$\lim_{\leftarrow}$ exact sequences, e.g.,

$$0 \to \lim_{\overleftarrow{k}}{}^{(1)} HC_{q+1}^-(Z^k) \to HC_q^-(Z) \to \lim_{\overleftarrow{k}} HC_q^-(Z^k) \to 0.$$

Also, from the description in terms of bicomplexes we see that we have a short exact sequence

$$0 \to \lim_{\overleftarrow{k}}{}^{(1)} HC_{q+1+2k}(Z) \to HP_q(Z) \to \lim_{\overleftarrow{k}} HC_{q+2k}(Z) \to 0$$

describing the periodic homology of a cyclic complex Z in terms of the cyclic homology and the S-maps connecting them.

We see that filtering $B^{\mathrm{per}}(Z)$ by columns we get a spectral sequence for $HP_*(Z)$ with E^2 term given by $H^{dR}(Z)$.

These homology theories have clear geometrical meaning in terms of orbit and fixed point spectra as is apparent from theorem 3.2.5 below.

Connes' used B to define cyclic homology, and probably documented a variant of the following fact somewhere. The only available source we know is Jones [59], see also the closely related statements in [37] and [74, chap. 7]. Note that the identification between the $\mathbb{T}$-action and the differentials in the Tate spectral sequence follows by 3.1.2 since $|HM|_\Lambda$ is an Eilenberg-MacLane spectrum.

ma:A3Bisd1}

Lemma 3.2.1 *Let M be a cyclic abelian group. Then the Eilenberg-MacLane spectrum $|HM|_\Lambda$ is a $\mathbb{T}$-spectrum, and the $\mathbb{T}$-action and the B-maps agree on homotopy groups in the sense that the diagram*

$$\begin{array}{ccc} \pi_* M & \xrightarrow{\cong} & \pi_*|HM|_\Lambda \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle d^1} \\ \pi_{*+1} M & \xrightarrow{\cong} & \pi_{*+1}|HM|_\Lambda \end{array}$$

commutes where the horizontal isomorphisms are the canonical isomorphisms relating the homotopy groups of simplicial abelian groups and their related Eilenberg-MacLane spectra, and d^1 is induced by the $\mathbb{T}$-action (and so is the differential in the spectral sequences of the homotopy orbit and fixed point spectra of $|HM|$ in lemma 3.1.1).

Notice that if we filter by columns, lemma 3.2.1 says that the resulting E^1 sheet agrees with the Tate, orbit and fixed point spectrum spectral sequences of lemma 3.1.1. For the record:

s periodic}

Corollary 3.2.2 *Let M be a cyclic abelian group. Then E^1(Tate) (resp. $E^1(|HM|_{h\mathbb{T}})$, resp. $E^1(|HM|^{h\mathbb{T}})$) equals the E^1 term of the spectral sequence given by filtering the periodic (resp. positive, resp. negative) (b, B)-bicomplex by columns.*

As a matter of fact, there is a natural isomorphism between the periodic (resp. negative, resp. cyclic) homology of M and the homotopy groups of the Tate spectrum (resp. $\mathbb{T}$-fixed point spectrum, resp. $\mathbb{T}$-orbit spectrum). See [44] in general, or [74] for the cyclic homology part. We won't need all that much, but only the following fact.

VHP0normeq}

Corollary 3.2.3 *Let M be a cyclic abelian group. If the periodic homology of M vanishes, then the $\mathbb{T}$-norm map $S^1 \wedge |HM|_{h\mathbb{T}} \to |HM|^{\mathbb{T}}$ is a stable equivalence.*

Proof: Since the periodic homology of M vanishes, so the spectral sequence gotten by filtering by columns must converge to zero. Hence by corollary 3.2.2 the Tate spectral sequence converges to zero, and we get the result from 3.1.1. ∎

3.2.4 Negative cyclic homology and fixed point spectra

The above statements say that the spectral sequences coming from the cyclic actions have E^1-sheets have E^1-sheets that are naturally isomorphic to the E^1-sheets we get by filtering the associated cyclic homology theories by column. This is all we need to make our argument work, but it is satisfactory to know that these natural isomorphisms come from spectrum level equivalences.

By a *spectrum* M of simplicial abelian groups, we mean a sequence $\{n \mapsto M^n\}$ of simplicial abelian groups together with homomorphisms $\tilde{\mathbf{Z}}[S^1] \otimes M^n \to M^{n+1}$ (according to the notion of spectra in any simplicial model category). Maps between spectra are as usual. There is a correspondence between spectra of simplicial abelian groups and (unbounded) chain complexes (probably a Quillen equivalence in some model structure) given by

$$C_*^{\mathrm{spt}} M = \varinjlim_{n} C_*^{\mathrm{norm}}(M^n)[n]$$

where the maps are given by the adjoint of the structure map $M^n \to \mathcal{S}_*(S^1, M^{n+1}) \cong s\mathcal{A}b(\tilde{\mathbf{Z}}[S^1], M^{n+1})$ followed by the isomorphism

$$C_*^{\mathrm{norm}} s\mathcal{A}b(\tilde{\mathbf{Z}}[S^1], M^{n+1}) \cong C_*^{\mathrm{norm}} M^{n+1}[1][0,\infty)$$

and the inclusion $C_*^{\mathrm{norm}} M[1][0,\infty) \subseteq C_*^{\mathrm{norm}} M[1]$.

{theo:HCi

Theorem 3.2.5 *Let $M \colon \Lambda^o \to s\mathcal{A}b$ be a cyclic simplicial abelian group. There are natural chains of weak equivalences*

$$C_*^{spt} \sin |HM|_{h\mathbb{T}} \simeq T^{0,\infty} M$$
$$C_*^{spt} \sin |HM|^{h\mathbb{T}} \simeq T^{-\infty,0} M.$$

Proof: The first statement follows from the corresponding statement in Loday's book [74] which shows that there is a natural chain of weak equivalences between $C^{\mathrm{norm}} \sin |M|_{h\mathbb{T}}$ and $T^{0,\infty} M$ and the fact that $\sin |HM|_{h\mathbb{T}}$ is a connected Ω-spectrum.

Both statements can be proved hands on by the standard filtration on $E\mathbb{T}$: Choose as your model for the contractible free $\mathbb{T}$-space $E\mathbb{T}$ in the definition of the homotopy fixed points to be the colimit of

$$|S^1| \to |S^3| \to \cdots \to |S^{2n+1}| \to \ldots$$

The maps in question are the inclusions gotten by viewing $|S^{2n+1}|$ as the space you get by attaching a free $\mathbb{T}$-cell to $|S^{2n-1}|$ along the action:

$$\begin{array}{ccc} \mathbb{T} \times |S^{2n-1}| & \xrightarrow{id \times \mathrm{inclusion}} & \mathbb{T} \times D^{2n} \\ {\scriptstyle \mathrm{action}} \downarrow & & \downarrow \\ |S^{2n-1}| & \longrightarrow & |S^{2n+1}| \end{array}$$

Then we have a natural equivalence

$$\sin|HM|^{h\mathbb{T}}(X) \xleftarrow{\sim} \lim_{\overleftarrow{n}}(\mathbb{T}-Top_*)(|S^{2n+1}|_+, |M\otimes\tilde{\mathbf{Z}}[X]|)$$

(this is only a natural equivalence and not an isomorphism since the definition of the homotopy fixed points of a topological spectrum involves a fibrant replacement, which is unnecessary since HM is an Ω-spectrum).

Hence we are done once we have shown that there is a natural (in n and M) chain of maps connecting

$$C_*^{\text{norm}}(\mathbb{T}-Top_*)(|S^{2n+1}|_+, |M|), \text{ and } T^{-n,0}M$$

inducing an isomorphism in homology in positive dimensions.

This is done by induction on n taking care to identify all maps in question. In particular, you need a refinement of the statement in lemma 3.2.1 to a statement about natural homotopies between the $\mathbb{T}$-action and the B-map. ∎

An important distinction for our purposes between homotopy orbits and fixed points is that homotopy orbits may be calculated degreewise. This is false for the homotopy fixed points.

VHCdegwise}

Lemma 3.2.6 *Let X be a simplicial $\mathbb{T}$-spectrum. Then $diag^*(X_{h\mathbb{T}})$ is naturally equivalent to $(diag^*X)_{h\mathbb{T}}$. In particular, if A is a simplicial ring, then $HC(A)$ can be calculated degreewise.*

Proof: True since homotopy colimits commute, and the diagonal may be calculated as $\operatorname{holim}_{\overrightarrow{[q]\in\Delta}} X_q$. ∎

3.2.7 Derivations

The following is lifted from [37], and we skip the gory calculations. Let A be a simplicial ring. A *derivation* is a simplicial map $D\colon A\to A$ satisfying the usual Leibniz relation $D(ab) = D(a)b + aD(b)$. A derivation $D\colon A\to A$ induces an endomorphism of cyclic modules $L_D\colon HH(A)\to HH(A)$ by sending $a = a_0\otimes\dots a_q\in A_p^{\otimes q+1}$ to

$$L_D(a) = \sum_{i=0}^{q} a_0\otimes\dots a_{i-1}\otimes D(a_i)\otimes a_{i+1}\otimes\dots a_q$$

From [37] we get that there are maps

$$e_D\colon C_q(A)\to C_{q-1}(A), \text{ and } E_D\colon C_q(A)\to \bar{C}_{q+1}(A)$$

satisfying

ns formula}

Lemma 3.2.8 *Let $D\colon A\to A$ be a derivation. Then*

$$e_Db + be_D = 0,$$

$$e_D B + B e_D + E_D b + b E_D = L_D$$

and

$$E_D B + B E_D$$

is degenerate.

To be explicit, the maps are given by sending $a = a_0 \otimes \dots a_q \in A_p^{\otimes q+1}$ to

$$e_D(a) = (-1)^{q+1} D(a_q) a_0 \otimes a_1 \otimes \dots \otimes a_{q-1}$$

and

$$E_D(a) = \sum_{1 \le i \le j \le q} (-1)^{iq+1} \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{j-1} \otimes D(a_j) \otimes a_{j+1} \otimes \dots \otimes a_q \otimes a_0 \otimes a_{i-1}$$
$$+ \text{ degenerate terms} \quad (3.2.8)$$

The first equation of lemma 3.2.8 is then a straightforward calculation, but the second is more intricate (see [37] or [74]).

{cor:Vderi

Corollary 3.2.9 ([37]) *Let D be a derivation on a flat ring A. Then*

$$L_D S \colon HC_* A \to HC_{*-2} A$$

is the zero map.

Proof: Collecting the relations in lemma 3.2.8 we get that $(E_D + e_D)(B + b) + (B + b)(E_D + e_D) = L_D$ on the periodic complex. However, this does not respect the truncation to the positive part of the complex. Hence we shift once and get the formula $((E_D + e_D)(B + b) + (B + b)(E_D + e_D))S = L_D S$ which gives the desired result. ∎

{cor:VHPva

Corollary 3.2.10 ([37]) *Let $f \colon A \to B$ be a map of simplicial rings inducing a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel. Let X be the homotopy fiber of $HH(A) \to HH(B)$. Then $HP_*(X_{(0)}) = HP_*(\widehat{X}_{(0)}) = 0$.*

Proof: By considering the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0 A & \longrightarrow & \pi_0 B \end{array}$$

we see that it is enough to prove the case where f is a surjection with nilpotent kernel and f is a surjection with connected kernel separately.

Let P be completion followed by rationalization or just rationalization. The important thing is that P is an exact functor with rational values.

The basic part of the proof, which is given in [37, II.5], is the same for the connected and the nilpotent case. In both situations we end up by proving that the shift map S is nilpotent on the relative part, or more precisely: for every q and every $k > q$ the map

$$S^k\colon HC_{q+2k}Y \to HC_qY$$

is zero, where Y is the homotopy fiber of $P(HH(A)) \to P(HH(B))$ (actually in this formulation I have assumed that the kernel was square zero in the nilpotent case, but we will see that this suffices for giving the proof). From this, and from the fact that periodic homology sits in a $\lim_{\overleftarrow{S}}{}^{(1)}$-$\lim_{\overleftarrow{S}}$ short exact sequence, we conclude the vanishing of periodic homology.

The main difference between our situation and the rational situation of [?]Goodfreeloop is that we can not assume that our rings are flat. That means that HH is not necessarily calculated by the Hochschild complex.

In the connected case, this is not a big problem, since the property of being connected is a homotopy notion, and so we can replace everything in sight by degreewise free rings and we are in business as explained in [37, IV.2.1]. Being nilpotent is not a homotopy notion, and so must be handled with a bit more care. First, by considering

$$A \to A/I^n \to A/I^{n-1} \to \cdots \to A/I^2 \to A/I = B$$

we see that it is enough to do the sqare zero case. Let $X \overset{\sim}{\twoheadrightarrow} B$ be a free resolution of B and consider the pullback

$$\begin{array}{ccc} P & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & B \end{array}.$$

Since the vertical maps are equivalences we have reduced to the case where $A \to B$ is a surjection of simplicial rings with discrete square zero kernel I and where B is free in every degree. But since cyclic homology can be calculated degreewise by lemma 3.2.6, it is enough to prove this in every degree, but since B is free in every degree it is enough to prove it when $A \to B$ is a split surjection of discrete rings with square zero kernel I. Choosing a splitting we can write $A \cong B \ltimes I$, where I is a B-bimodule with square zero multiplication. Let $J \overset{\sim}{\twoheadrightarrow} I$ be a free resolution of I as B-bimodules. Then we have an equivalence $B \ltimes J \xrightarrow{\sim} B \ltimes I$, and again since cyclic homology can be calculated degreewise we have reduced to the case $B \ltimes I \to B$ where B is free and I is a free B module.

Hence we are in the flat case, and can prove our result in this setting.

First consider the case where f is split with square zero kernel. Then the distributive law provides a decomposition of $A^{\otimes q+1} = (B \oplus I)^{\otimes q+1}$, and if we let F^k_q consist of the summands with k or more I-factors we get a filtration

$$0 = F^\infty = \bigcap_n F^n \subset \cdots \subset F^2 \subset F^1 \subset F^0 = HH(A)$$

(it is of finite length in each degree, in fact $F^n_k = 0$ for all $n-1 > k$). Note that we have isomorphisms of cyclic modules $HH(B) = F^0/F^1$, $HH(A) = F^0 \cong \oplus_{k\geq 0} F^k/F^{k+1}$.

We must show that for every q and every $k > q$ the map

$$S^k\colon HC_{q+2k}(P(F^1)) \to HC_q(P(F^1))$$

is zero. Since $F^n_k = 0$ for all $n-1>k$ we have that $HC_q(P(F^n)) = 0$ for all $q < n-1$. Hence it is enough to show that for every q

$$S^k\colon HC_{q+2k}(P(F^1/F^{k+1})) \to HC_q(P(F^1/F^{k+1}))$$

is zero.

The projection $D\colon B \ltimes I \to I$ is a derivation, and it acts as multiplication by m on F^m/F^{m+1}. Therefore we have by corollary 3.2.9 (whose proof is not affected by the insertion of P) that

$$m \cdot S = L_D S = 0$$

on $HC_*(P(F^m/F^{m+1}))$. Since $m \geq 0$ is invertible in $\mathbf{Q}$, we get that $S = 0$ on $HC_*(P(F^m/F^{m+1}))$, and by induction $S^k = 0$ on $HC_*(P(F^1/F^{k+1}))$.

The proof of the connected case is similar: first assume that I is *reduced* (has just one zero-simplex: this is obtained by the lemma 3.2.11 we have cited below). Use the "same" filtration as above (it no longer splits), and the fact that F^k is zero in degrees less than k since I is reduced.

Filter A by the powers of I:

$$\cdots \subseteq I^m \subseteq \cdots \subseteq I^1 \subseteq I^0 = A$$

This gives rise to a filtration of the Hochschild homology

$$0 = F^\infty = \bigcap_n F^n \subset \cdots \subset F^2 \subset F^1 \subset F^0 = HH(A)$$

by defining

$$F^k_q = im\left\{ \bigoplus_{\sum k_i = k} \bigotimes_{i=0}^{q} I^{k_i} \to HH(A)_q \right\}.$$

Consider the associated graded ring $gr(A)$ with $gr_k A = I^k/I^{k+1}$. Note that we have isomorphisms of cyclic modules $HH(B) = F^0/F^1$, $HH(A) = F^0$ and $HH(grA) \cong \oplus_{k\geq 0} F^k/F^{k+1}$.

We define a derivation D on grA by letting it be multiplication by k in degree k. Note that L_D respects the filtration and acts like k on F^k/F^{k+1}. The proof then proceeds as in the nilpotent case.

∎

In the above proof we used the following result of Goodwillie [39, I.1.7]:

{lem:Vconn

Lemma 3.2.11 *Let $f\colon A \twoheadrightarrow B$ be a k-connected surjection of simplicial rings. Then there is a diagram*

$$\begin{array}{ccc} R & \xrightarrow{\sim} & A \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ S & \xrightarrow{\sim} & B \end{array}$$

of simplicial rings such that the horizontal maps are equivalences, the vertical maps surjections, and the kernel of g is k-reduced (i.e., its $(k-1)$-skeleton is a point). If A and B were flat in every degree, then we may choose R and S flat too.

rop:VHHrel}

Proposition 3.2.12 *Let $f\colon A \to B$ be a map of simplicial rings inducing a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then the diagrams induced by the norm map*

$$\begin{array}{ccc} S^1\wedge(THH(A)_{(0)})_{h\mathbb{T}} & \longrightarrow & (THH(A)_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ S^1\wedge(THH(B)_{(0)})_{h\mathbb{T}} & \longrightarrow & (THH(B)_{(0)})^{h\mathbb{T}} \end{array}$$

and

$$\begin{array}{ccc} S^1\wedge(THH(A)\hat{}_{(0)})_{h\mathbb{T}} & \longrightarrow & (THH(A)\hat{}_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ S^1\wedge(THH(B)\hat{}_{(0)})_{h\mathbb{T}} & \longrightarrow & (THH(B)\hat{}_{(0)})^{h\mathbb{T}} \end{array}$$

are homotopy cartesian.

Proof: Recall that by lemma IV.1.3.8 THH is equivalent to HH after rationaliztion, or profinite completion followed by rationalization, and so can be regarded as the Eilenberg-MacLane spectrum associated with a cyclic module.

By corollary 3.2.3 lemma C.3.1.3 and corollary C.3.2.2 we are done if the corresponding periodic cyclic homology groups vanish, and this is exactly the contents of corollary 3.2.10.

∎

Remark 3.2.13 *A priori $(\underline{T}_{(0)})^{h\mathbb{T}}$ should not preserve connectivity, and does not do so (look e.g., at the zero-connected map $\mathbf{Z} \twoheadrightarrow \mathbf{Z}/p\mathbf{Z}$: $(\underline{T}(\mathbf{Z})_{(0)})^{h\mathbb{T}}$ is not connective (its homotopy groups are the same as rational negative cyclic homoloy of the integers and so have a $\mathbf{Q}$ in in every even nonpositive dimension), but $(\underline{T}(\mathbf{Z}/p\mathbf{Z})_{(0)})^{h\mathbb{T}}$ vanishes.*

However, since homotopy colimits preserve connectivity proposition 3.2.12 gives that we do have the following result.

xptandconn}

Corollary 3.2.14 *If $A \to B$ is $k > 0$-connected map of simplicial rings, and let X be either ${THH_{(0)}}^{h\mathbb{T}}$ or ${THH\hat{}_{(0)}}^{h\mathbb{T}}$ considered as a functor from simplicial rings to spectra. Then $X(A) \to X(B)$ is $k+1$ connected If $A \to B$ induces a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then $X(A) \to X(B)$ is -1-connected.*

3.3 Structural properties for integral TC

The importance of the results about the $\mathbb{T}$-homotopy fixed point spectra in the section above comes, as earlier remarked, from the homotopy cartesian square of lemma 2.3.2

$$\begin{array}{ccc} \underline{TC}(-) & \longrightarrow & (\underline{T}(-)_{(0)})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \underline{TC}(-)\hat{} & \longrightarrow & (\underline{T}(-)\hat{}_{(0)})^{h\mathbb{T}} \end{array}.$$

So combining these facts with the properties of $TC(-,p)$ exposed in section 2 we get several results on TC quite for free.

Proposition 3.3.1 *If $A \to B$ is $k > 0$-connected map of* **S***-algebras, then $TC(A) \to TC(B)$ is $(k-1)$-connected. If $A \to B$ induces a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then $TC(A) \to TC(B)$ is -1-connected.*

Proof: Consider the cubical approximation in III.3.1.8. In this construction the conditions on the maps of **S**-algebras are converted to conditions on homomorphisms of simplicial rings (that the maps in the cubes are not themselves homomorphisms does not affect the argument). Hence by theorem 2.3.4. Follows from the homotopy cartesian square, corollary 3.2.14 and lemma 2.2.5. ∎

In fact, for the same reason this applies equally well to higher dimensional cubes:

Proposition 3.3.2 *Let $\mathcal{A}$ be cubical diagram of positive dimension of* **S***-algebras, and assume that all maps are k-connected and induce surjections with nilpotent kernel on π_0. Assume that we have shown that $\underline{T}(\mathcal{A})$ is $id - k$ cartesian. Then $\underline{TC}(\mathcal{A})$ is $id - k - 1$ cartesian.*

Proof: Again we do the proof for each of the vertices in the cartesian square giving TC. For $TC(-)\hat{} \simeq \prod_{p \text{ prime}} TC(-,p)\hat{}_p$ this is proposition 2.2.7. For the two other vertices we again appeal to theorem 2.3.4 which allow us to prove it only for simplicial rings, and then to proposition 3.2.12 which tells us that the cubes involving $(\underline{T}_{(0)})^{h\mathbb{T}}$ and $(\underline{T}\hat{}_{p(0)})^{h\mathbb{T}}$ are as (co)cartesian as the corresponding cubes, $\Sigma(\underline{HH}(\mathcal{A})_{(0)})_{h\mathbb{T}}$ and $\Sigma(\underline{HH}(\mathcal{A})\hat{}_{(0)})_{h\mathbb{T}}$. Thus we are done since homotopy colimits preserve cocartesianness. ∎

Notice that this is slightly stronger than what we used in theorem 2.3.4 to establish the approximation property for TC: There we went all the way in the limit, obtaining stable equivalences before taking the homotopy fixed point construction. Here we actually establish that the connectivity grows as expected in the tower, not just that it converges.

3.3.3 Summary of results

In addition to the above results depending on the careful analysis of the homotopy fixed points of topological hochschild homology we have the following more trivial results following from our previous analyses of $TC(-,p)$ and the general properies of homotopy fixed points:

- TC is an Ω-spectrum.
- TC can be calculated degreewise in certain relative situations
- TC preserves $\Gamma\mathcal{S}_*$-equivalences
- TC is Morita-equivariant
- TC preserves products
- TC of triangular matrices give the same result as products.
- TC satisfies strict cofinality
- TC depends only on its value on simplicial rings

4 The trace.

4.1 Lifting the trace to topological cyclic homology

In this section we prove the

Lemma 4.1.1 *Let $\mathcal{C}$ be a $\Gamma\mathcal{S}_*$-category. Then the Bökstedt-Dennis trace map $ob\mathcal{C} \to \underline{T}(\mathcal{C})(S^0)$ factors through $\underline{TC}(\mathcal{C};S^0)$.*

Proof: Remember that the Bökstedt-Dennis trace map was defined as the composite

$$ob\mathcal{C} \to \bigvee_{c\in ob\mathcal{C}} \mathcal{C}(c,c)(S^0) \to THH(\mathcal{C})_0 \to THH(\mathcal{C})$$

where the first map assign to every object its identity map, and the last map is the inclusion by degeneracies. With some care, using the simplicial replacement description of the homotopy colimit, one sees that this inclusion can be identified with the inclusion $\lim_{\overleftarrow{\Lambda^o}} THH(\mathcal{C}) \subseteq THH(\mathcal{C})$ (see lemma 1.1.4.), and so is invariant under the circle action on $|THH(\mathcal{C})|$. In particular, the Bökstedt-Dennis trace map yields maps $ob\mathcal{C} \to sd_r THH(\mathcal{C})^{C_r}$. To see that it commutes with the restriction maps, one chases an object $c \in ob\mathcal{C}$ through $ob\mathcal{C} \to sd_{rs}THH(\mathcal{C})^{C_{rs}}@>R>> sd_r THH(\mathcal{C})^{C_r}$, and see that it coincides with its image under $ob\mathcal{C} \to sd_r THH(\mathcal{C})^{C_r}$ ∎

Recall the notion of the K-theory $K(\mathcal{C},w)$ of a symmetric monoidal $\Gamma\mathcal{S}_*$-category $\mathcal{C}$ with weak equivalences from II.?. The letter combination $\underline{TC}(K(\mathcal{C},w))$ is supposed to signify the bispectrum $m,n \mapsto \underline{TC}(K(\mathcal{C},w)(S^n);S^m)$. In order to have the following definition well defined, we consider K-theory as a bispectrum in the trivial way: $m,n \mapsto obK(\mathcal{C},w)(S^n)\wedge S^m$ without changing the notation.

Definition 4.1.2 Let $\mathcal{C}$ be a symmetric monoidal $\Gamma\mathcal{S}_*$-category with weak equivalences. Then the cyclotomic trace is the lifting of the Bökstedt-Dennis trace

$$obK(\mathcal{C},w)\to \underline{TC}(K(\mathcal{C},w))$$

considered as a map of bispectra.

This section must be (re)written. Put the Γ space stuff in II.

In the case of rings we have already produced a satisfactory "trace map" to THH, namely

$$\mathbf{K}(A)=ob\S\mathcal{P}_A=\mathbf{T}(A)^{\mathbb{T}}\to\mathbf{T}(A)$$

This defines a map to **TC** as follows. The diagram

$$\begin{array}{ccc}\mathbf{T}(A)^{\mathbb{T}} & \longrightarrow & \mathbf{T}(A)^{h\mathbb{T}}\\ \downarrow & & \downarrow\\ \prod_{p\text{prime}}\mathbf{TC}(A;p)^{\widehat{}}_{p} & \longrightarrow & \prod_{p\text{prime}}\underset{\overleftarrow{p^r\in\mathcal{F}_p}}{\text{holim}}\mathbf{T}(A)^{hC_{p^r}}{}^{\widehat{}}_{p}\end{array}$$

commutes, and so we have a map $\mathbf{T}(A)^{\mathbb{T}}\to\mathbf{TC}(A)$, which we may call the trace. The unfortunate thing is of course that as we have defined it, this only works for rings. There are many approaches to defining the trace map for **S**-algebras in general We will follow the outline of [120]. This is only a weak transformation, in the sense that we will encounter weak equivalences going the wrong way, but this will cause no trouble in our context.

For any **S**-algebra A we will construct a weak map from $BA^*=B\widehat{GL}_1(A)$, the classifying space of the monoid of homotopy units of A, to $TC(A)$. Applying this to the **S**-algebras Mat_nA, we get weak maps from $B\widehat{GL}_n(A)$ to $TC(Mat_nA)\simeq TC(A)$. Then we have a choice of strategies:

Either we stabilize with respect to n and take the plus construction on both sides to get a weak transformation from $B\widehat{GL}(A)^+$ to $\lim_{\overrightarrow{n}}TC(M_nA)^+\simeq TC(A)$.

Or, we insists upon having a transformation on the spectrum level. Then we may choose the Γ space approach. Let $\mathbf{K}_\Gamma(A)$ and and $\mathbf{TC}_\Gamma(A)$ be two Γ spaces defined on objects by sending the finite set K to the space

$$\coprod_{f\in Map_*(K,\mathbf{N}_0)}B(\prod_{k\in K}Mat_{f(k)}(A)^*)\times E\Sigma_{\sum_{k\in K}f(k)}$$

and

$$\coprod_{f\in Map_*(K,\mathbf{N}_0)}TC(\prod_{k\in K}Mat_{f(k)}(A))\times E\Sigma_{\sum_{k\in K}f(k)}$$

The action on the morphisms are far from obvious, and we refer the reader to [6] for the details. The transformation we have defined give rise to a map of Γ spaces, and hence a spectrum-level transformation $\mathbf{K}_\Gamma(A)\to\mathbf{TC}_\Gamma(A)$.

We will only need the former, and have to prove that it is compatible with the definition we already have given for rings.

Note that the inclusion $BGM\to N^{cy}NGM$ is onto $N^{cy}(NGM)^{S^1}$, and is fixed by the restriction maps, and so our trace factors through a weak transformation $K(A)\to TC(A)$.

5 Split square zero extensions and the trace

Let A be an $\mathbf{S}$-algebra and P an A bimodule. Then we define $A \vee P$ as before, and recall that we have for every $\mathbf{x} \in \mathcal{I}^{q+1}$ a decomposition of $V(A \vee P)(\mathbf{x})$ by letting

$$V^{(j)}(A,P)(\mathbf{x}) = \bigvee_{\phi\in\Delta_m([j-1],[q])} \bigwedge_{0\leq i\leq q} F_{i\phi}(x_i)$$

where

$$F_{i,\phi}(x) = \begin{cases} A(S^x) & \text{if } i \notin im\phi \\ P(S^x) & \text{if } i \in im\phi \end{cases}$$

Then $V(A\vee P)(\mathbf{x}) \cong \bigvee_{j\geq 0} V^{(j)}(A,P)(\mathbf{x})$ and $THH(A\vee P;X)@>\sim>> \prod_{j\geq 0} THH^{(j)}(A,P;X)$ where

$$THH^{(j)}(A,P;X)_q = \underrightarrow{\operatorname{holim}}_{\mathbf{x}\in\mathcal{I}^{q+1}} \Omega^{\sqcup\mathbf{x}}(X\wedge V^{(j)}(A,P)(\mathbf{x}))$$

We want to get control over the various actions, to get a description of TC of a split square zero extension.

Lemma 5.0.3 *For every prime power $a = p^r$*

$$sd_a THH(A\vee P;X)^{C_a} \xrightarrow{\sim} \prod_{j\geq 0} sd_a THH^{(j)}(A,P;X)^{C_a}$$

is an equivalence.

Proof: Note that, for every a

$$X\wedge V(A\vee P)(\mathbf{x}^a) \cong \bigvee_{j\geq 0}(X\wedge V^{(j)}(A,P)(\mathbf{x}^a))$$

is a C_a isomorphism, and the action respects the wedge decomposition. We note that

$$V^{(j)}(A,P)(\mathbf{x}^a)^{C_a} \cong \begin{cases} V^{(j/a)}(A,P)(\mathbf{x}) & \text{if } j\equiv 0 \mod a \\ * & \text{otherwise} \end{cases}$$

If $a = p^n$ where p is a prime, we have maps of fibrations

$$\begin{array}{ccc} Map(S^{\sqcup\mathbf{x}^{ap}}, X\wedge\bigvee_{j\geq 0} V^{(j)}(A,P)(\mathbf{x}^{ap}))^{C_{ap}} & \longrightarrow & Map(S^{\sqcup\mathbf{x}^{a}}, X\wedge\bigvee_{j\geq 0} V^{(j)}(A,P)(\mathbf{x}^{a}))^{C_{a}} \\ \downarrow & & \downarrow \\ \prod_{j\geq 0} Map(S^{\sqcup\mathbf{x}^{ap}}, X\wedge V^{(j)}(A,P)(\mathbf{x}^{ap}))^{C_{ap}} & \longrightarrow & \prod_{j\geq 0} Map(S^{\sqcup\mathbf{x}^{a}}, X\wedge V^{(j)}(A,P)(\mathbf{x}^{a}))^{C_{a}} \end{array}$$

with homotopy fiber

$$\begin{array}{c} \underrightarrow{\operatorname{holim}}_{k}\Omega^k Map(S^{\sqcup\mathbf{x}^{ap}}, S^k\wedge X\wedge \bigvee_{j\geq 0} V^{(j)}(A,P)(\mathbf{x}^{ap}))_{hC_{ap}} \\ \downarrow \\ \prod_{j\geq 0}\underrightarrow{\operatorname{holim}}_{k}\Omega^k Map(S^{\sqcup\mathbf{x}^{ap}}, S^k\wedge X\wedge V^{(j)}(A,P)(\mathbf{x}^{ap}))_{hC_{ap}} \end{array}$$

The map of fibers factors through

$$\underset{\overrightarrow{k}}{\operatorname{holim}}\Omega^k(\prod_{j\geq 0} Map(S^{\sqcup\mathbf{x}^{ap}}, S^k\wedge X\wedge V^{(j)}(A,P)(\mathbf{x}^{ap})))_{hC_{ap}}$$

By Blakers–Massey the map into this space is an equivalence, and also the map out of this space (virtually exchange the product for a wedge to tunnel it through the orbits).

Hence

$$\begin{array}{ccc} sd_{ap}THH(A\vee P;X)^{C_{ap}} & \xrightarrow{R} & sd_aTHH(A\vee P;X)^{C_a} \\ \downarrow & & \downarrow \\ \prod_{j\geq 0} sd_{ap}THH^{(j)}(A\vee P;X)^{C_{ap}} & \xrightarrow{R} & \prod_{j\geq 0} sd_aTHH^{(j)}(A\vee P;X)^{C_a} \end{array}$$

is cartesian. But the right vertical map is an equivalence for $a=1$, and hence by induction for all $a=p^n$. ∎

Assume from now on that P is $n-1$ connected, and X is $m-1$ connected. We want to study the maps

$$sd_{ap}THH^{(jp)}(A,P;X)^{C_{ap}} \xrightarrow{R} sd_aTHH^{(j)}(A,P;X)^{C_a}$$

a bit closer. The fiber is

$$\underset{\overrightarrow{k}}{\operatorname{holim}}(\Omega^k sd_{ap}THH^{(jp)}(A,P;S^k\wedge X)_{hC_{ap}})$$

and is by assumption $jpn+m-1$ connected. If p does not divide j, the base space is equivalent to

$$\underset{\overrightarrow{k}}{\operatorname{holim}}(\Omega^k sd_aTHH^{(j)}(A,P;S^k\wedge X)_{hC_a})$$

which is $jn+m-1$ connected.

So consider

$$sd_{p^r}THH^{(lp^s)}(A,P;X)^{C_{p^r}}$$

where p does not divide l. If $s\geq r$ the R maps will compose to an $lp^{s-r+1}n+m$ connected map to $THH^{(lp^{s-r})}(A,P;X)$ which is $lp^{s-r}n+m-1$ connected. If $r\geq s$ the R maps will compose to an $lpn+m$ connected map to to $sd_{p^{r-s}}THH^{(l)}(A,P;X)^{C_{p^{r-s}}}\simeq \operatorname{holim}_{\overrightarrow{k}}(sd_{p^{r-s}}THH^{(l)}(A,P;S^k\wedge X))_{hC_{p^{r-s}}}$ which is $ln+m-1$ connected.

Hence

$$\underset{\overleftarrow{p^t\in\mathcal{R}_p}}{\operatorname{holim}} sd_{p^{r+t}}THH^{(lp^{s+t})}(A,P;X)^{C_{p^{r+t}}}$$

will be $max(ln+m-1, lp^{s-r}n+m-1)$ connected. This means that there is a $2n+m$ connected map

$$\begin{aligned} TR(A\vee P;X,p) &\to TR(A;X,p)\times\prod_{r\geq 0}\underset{\overleftarrow{p^t\in\mathcal{R}_p}}{\operatorname{holim}} sd_{p^{t+r}}THH^{p^t}(A,P;X)^{C_{p^{t+r}}} \\ &\to TR(A;X,p)\times\prod_{r\geq 0} sd_{p^r}THH^{(1)}(A,P;X)^{C_{p^r}} \end{aligned}$$

(the last map is $pn+m \geq 2n+m$ connected as all maps in the homotopy limit on the first line are $pn+m$ connected).

Lemma 5.0.4 *Let A, P, X and $a = p^r$ be as before. Then*

$$|THH^{(1)}(A,P;X)|^{C_a} \xrightarrow{\simeq} |THH^{(1)}(A,P;X)|^{hC_a}.$$

Proof: Let $\mathbf{x} \in \mathcal{I}^q$, and set $S(\mathbf{x}) = S^{\sqcup \mathbf{x}^a}$, $Y(\mathbf{x}) = X \wedge V^{(1)}(A,P)(\mathbf{x}^a)$ and $U(\mathbf{x}) = S(\mathbf{x})/S(\mathbf{x})^{C_p}$. Notice that $U(\mathbf{x})$ is finite and free, $S(\mathbf{x})^{C_p}$ is $a(\sqcup\mathbf{x})/p$ dimensional, and $Y(\mathbf{x})$ is $a(\sqcup\mathbf{x})+m-1$ connected and $Y^{C_p} = *$. Consider the map of fibrations

$$\begin{array}{ccccc}
Map(U(\mathbf{x}),Y(\mathbf{x}))^{C_a} & \longrightarrow & Map(S(\mathbf{x}),Y(\mathbf{x}))^{C_a} & \longrightarrow & Map(S(\mathbf{x})^{C_p},Y(\mathbf{x}))^{C_a} \\
\downarrow & & \downarrow & & \downarrow \\
Map(U(\mathbf{x}),QY(\mathbf{x}))^{hC_a} & \longrightarrow & Map(S(\mathbf{x}),QY(\mathbf{x}))^{hC_a} & \longrightarrow & Map(S(\mathbf{x})^{C_p},QY(\mathbf{x}))^{hC_a}
\end{array}$$

The left vertical map is highly connected, and goes to infinity with $\sqcup\mathbf{x}$ (ref). The upper right hand corner is isomorphic to $Map_{C_{a/p}}(S(\mathbf{x})^{C_p}, Y(\mathbf{x})^{C_p}) = *$. By the approximation lemma, $\operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}}$ commutes with finite homotopy limits, and so

$$\operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}} Map(S(\mathbf{x})^{C_p},QY(\mathbf{x}))^{hC_a} \xrightarrow{\sim} (\operatorname{holim}_{\overrightarrow{\mathbf{x}\in\mathcal{I}^q}} Map(S(\mathbf{x})^{C_p},QY(\mathbf{x})))^{hC_a},$$

but as $S(\mathbf{x})^{C_p}$ is only $a(\sqcup\mathbf{x})/p$ dimensional, and $QY(\mathbf{x})$ is $a(\sqcup\mathbf{x})+m-1$ connected, this means that the homotopy colimit is contractible. ∎

Collecting the information so far we get

Lemma 5.0.5 *There is a $2n+m$ connected map*

$$\underline{TC}(A;X,p) \to \underline{TC}(A;X,p) \times \operatorname{holim}_{\overleftarrow{p^r\in\mathcal{F}_p}} |\underline{T}^{(1)}(A,P;X)|^{hC_{p^r}}.$$

Proof: Take $-^{h\langle F\rangle}$ of the TR expression, and insert the lemma. ∎

Recall that $\underline{T}^{(1)}(A,P;X)@ <<< N^{cy}\underline{T}(A,P;X) \to S^1_+ \wedge \underline{T}(A,P;X)$ are equivalences. Let Q be the endofunctor of spectra sending X to the equivalent Ω spectrum $QX = \{m \mapsto \operatorname{holim}_{\overrightarrow{x\in\mathcal{I}}} \Omega^x X_{x+k}\}$.

Theorem 5.0.6 (Hesselholt) *Let A, P, X and p as above. The "composite"*

$$\begin{array}{ccccccc}
\tilde{\underline{TC}}(A\vee P;X,p) & \longrightarrow & \tilde{\underline{T}}(A\vee P;X) & \longleftarrow & N^{cy}\underline{T}(A,P;X) & \xrightarrow{\sim} & S^1_+ \wedge \underline{T}(A,P;X) \\
 & & & & & & \downarrow \\
 & & & & & & S^1 \wedge \underline{T}(A,P;X)
\end{array}$$

is $2n+m-?$ connected after p completion.

Proof: In a $2n+m-?$ range, this looks like

$$\begin{array}{ccccccc}
\underset{\overleftarrow{p^r\in\mathcal{F}_p}}{\mathrm{holim}}(N^{cy}\underline{T}(A,P;X))^{hC_{p^r}} & \longrightarrow & QN^{cy}\underline{T}(A,P;X) & \overset{\sim}{\longleftarrow} & N^{cy}\underline{T}(A,P;X) & \overset{\sim}{\longrightarrow} & S^1_+\wedge\underline{T}(A,P;X) \\
 & & & & & & \downarrow \\
 & & & & & & S^1\wedge\underline{T}(A,P;X)
\end{array}$$

but the diagram

$$\begin{array}{ccccc}
\underset{\overleftarrow{p^r\in\mathcal{F}_p}}{\mathrm{holim}}(N^{cy}\underline{T}(A,P;X))^{hC_{p^r}} & \overset{\sim}{\longrightarrow} & \underset{\overleftarrow{p^r\in\mathcal{F}_p}}{\mathrm{holim}}(S^1_+\wedge\underline{T}(A,P;X))^{hC_{p^r}} & \longleftarrow & (S^1_+\wedge\underline{T}(A,P;X))^{hS^1} \\
\downarrow & & \downarrow & & \simeq\downarrow \\
QN^{cy}\underline{T}(A,P;X) & \overset{\sim}{\longrightarrow} & Q(S^1_+\wedge\underline{T}(A,P;X)) & \longrightarrow & Q(S^1\wedge\underline{T}(A,P;X))
\end{array}$$

gives the result as the upper right hand map is an equivalence after p completing (ref). The right hand vertical map is an equivalence (refapp) ∎

Corollary 5.0.7 *Let A be a simplicial ring and P a simplicial A bimodule. The trace induces an equivalence*

$$D_1K(A\ltimes -)(P)\to D_1TC(A\ltimes -)(P).$$

Proof: We have seen that this is so after profinite completion, and we must study what happens for the other corners in the definition of $\underline{TC}$. But here we may replace the S^1 homotopy fixed points by the negative cyclic homology, and as we are talking about square zero extensions, even by shifted cyclic homology. But as cyclic homology respects connectivity we see that the horizontal maps in

$$\begin{array}{ccc}
(N^{cy}\underline{T}(A,P;X)_{(0)})^{hS^1} & \longrightarrow & (\underline{\tilde{T}}(A\ltimes P;X)_{(0)})^{hS^1} \\
\downarrow & & \downarrow \\
(N^{cy}\underline{T}(A,P;X)\widehat{\ }_{(0)})^{hS^1} & \longrightarrow & (\underline{\tilde{T}}(A\ltimes P;X)\widehat{\ }_{(0)})^{hS^1}
\end{array}$$

are both $2k+m$ connected if P is $k-1$ connected and X is $m-1$ connected.

Summing up: both maps going right to left in

$$\begin{array}{ccccccc}
\tilde{\mathbf{K}}(A\ltimes P) & \longrightarrow & \tilde{\mathbf{TC}}(A\ltimes P) & \longleftarrow & (N^{cy}\mathbf{T}(A,P))^{hS^1} & \overset{\sim}{\longrightarrow} & (S^1_+\wedge\mathbf{T}(A,P))^{hS^1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \tilde{\mathbf{T}}(A\ltimes P) & \longleftarrow & QN^{cy}\mathbf{T}(A,P) & \overset{\sim}{\longrightarrow} & Q(S^1_+\wedge\mathbf{T}(A,P)) \\
 & & & & \simeq\uparrow & & \simeq\uparrow \\
 & & & & N^{cy}\mathbf{T}(A,P) & \overset{\sim}{\longrightarrow} & S^1_+\wedge\mathbf{T}(A,P) \\
 & & & & & & \downarrow \\
 & & & & & & S^1\wedge\mathbf{T}(A,P)
\end{array}$$

are $2k$ connected, and all composites from top to the bottom are $2k$ connected. ∎

Chapter VII

The comparison of K-theory and TC

At long last we come to the comparison between algebraic K-theory and topological cyclic homology. The reader should be aware, that even though we propose topological cyclic homology as an approximation to algebraic K-theory, there are marked differences between the two functors. This is exposed by a number of different formal properties, as well as the fact that in most cases they give radically different output. {VI}

However, the local structure is the same. We have seen that this is the case if we use the myopic view of stabilizing, but we will now see that they have the same local structure even with the eyes of deformation theory.

More precisely, we prove the integral version of the Goodwillie conjecture

Theorem 0.0.8 *Let $A \to B$ be a map of $\mathbf{S}$-algebras inducing a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then* {GC}

$$\begin{array}{ccc} K(A) & \longrightarrow & TC(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & TC(B) \end{array}$$

is homotopy cartesian.

The version where the map is in fact 1-connected was proposed as a conjecture by the second author at the ICM in Kyoto 1990. The current proof was found in 1996.

Some history

Something about consequences and related results.

We prove it in two steps. In section 1 we prove the result for the case where $A \to B$ is a split surjection of simplicial rings with square zero kernel. This case is possible to attack by means of a concrete cosimplicial resolution calculating the loops of the classifying space of the fiber. Some connectedness bookkeeping then gives the result. In section 2 we get rid of the square zero condition and the condition that $A \to B$ is split. This last point requires some delicate handling made possible by the fact that we know that in the relative situations both K-theory and TC can be calculated degreewise. Using the "denseness" of simplicial rings in $\mathbf{S}$-algebras, and the "continuity" of K and TC we are finished.

1 K-theory and cyclic homology for split square zero extensions of rings

Recall that if A is a ring and P is an A-bimodule we write $A \times P$ for the ring whose underlying abelian group is $A \oplus P$ and whose multiplication is defined by $(a_1, p_1) \cdot (a_2, p_2) = (a_1 a_2, a_1 p_2 + p_1 a_2)$. Then the projection $A \ltimes P \to A$ is a surjection of rings whose kernel is a the square zero ideal which we identify with P..

{def:VI1.1}

Definition 1.0.9 For A a simplicial ring and P an A bimodule, let $\mathbf{F}_A P$ be the iterated fiber of

$$\begin{array}{ccc} \mathbf{K}(A \ltimes P) & \longrightarrow & \mathbf{TC}(A \ltimes P) \\ \downarrow & & \downarrow \\ \mathbf{K}(A) & \longrightarrow & \mathbf{TC}(A) \end{array}$$

regarded as a functor from A bimodules to spectra.

{theo:VI1.2}

Theorem 1.0.10 *Let A be a simplicial ring. Then $\mathbf{F}_A P \simeq *$ for all A-bimodules P.*

That is, the diagram in definition 1.0.9 is homotopy cartesian.

The proof of this theorem will occupy the rest of this section. In the next, we will show how the theorem extends to **S**-algebras to prove Goodwillie's ICM90 conjecture.

We may without loss of generality assume that A is discrete. We know by ref that if P is k-connected, then $\mathbf{F}_A P$ is $2k$-connected; so for general P it is natural to study $\Omega^k \mathbf{F}_A(B^k P)$ (whose connectivity goes to infinity with k), or more precisely, the map

$$\mathbf{F}_A P \xrightarrow{\eta_P^k} \Omega^k \mathbf{F}_A(B^k P)\cdot$$

This map appears naturally as the map of fibers of $\mathbf{F}_A$ applied to the (co)cartesian square

$$\begin{array}{ccc} P & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & BP \end{array}$$

Since $\mathbf{F}_A$ is not a priori linear, we don't know that η_P is an equivalence, but we will show that it is as connected as $\mathbf{F}_A(BP)$ is. This means that $\mathbf{F}_A P$ is as connected as $\Omega \mathbf{F}_A(BP)$, and by induction $\mathbf{F}_A P$ must be arbitrarily connected, and we are done.

To see this, model $P \simeq \Omega BP$ by means of the cosimplicial object

$$\omega(*, BP, *) = \{[q] \mapsto \underline{\mathcal{S}_*}(S^1_q, BP) \cong BP^{\times q}\}$$

(refappend)NBNB. We coaugment this by $P \xrightarrow{\sim} \omega(*, BP, *)$, and let $S \mapsto \mathcal{P}^n_S$ be the composite

$$\mathcal{P}\mathbf{n} \xrightarrow{\subset} \Delta \cup \emptyset \xrightarrow{\omega(*,-,*)\cup P} \text{simplicial } A \text{ bimodules.}$$

For any n-cube $\mathcal{X}$, we have a tower

$$\begin{array}{ccccccccc} F_{n+1} & \longrightarrow & F_n & \longrightarrow & \dots & \longrightarrow & F_1 & \longrightarrow & F_0 \\ & & \downarrow & & & & \downarrow & & \downarrow \\ & & \Phi_n & & \dots & & \Phi_1 & & \Phi_0 \end{array}$$

where

$$\begin{aligned} F_0 &= \mathcal{X}_\emptyset, \\ \Phi_0 &= \mathcal{X}_{\{1\}}, \\ F_j &= fib\{F_{j-1} \to \Phi_{j-1}\} \text{ and} \\ \Phi_j &= fib\left\{\mathcal{X}_j \to \operatorname{holim}_{\leftarrow} \mathcal{X}|\mathcal{P}\mathbf{n}[j] - \{j\}\right\} \end{aligned}$$

where $\mathcal{P}\mathbf{n}[j] = \{S \in \mathcal{P}\mathbf{n} | j \in S, \{j+1, \dots, n\} \cap S = \emptyset\}$. This is nothing more than a specific choice of path for computing the iterated fiber F_{n+1} of $\mathcal{X}$.

In the case $\mathcal{X} = \mathbf{F}_A\mathcal{P}^n$ we get $F_1 = F_0 = \mathbf{F}_A P$ and $\Phi_1 = \Omega\mathbf{F}_A BP$. Theorem 1.0.10 follows from the claim that F_2 is as connected as $\Omega\mathbf{F}_A BP$ is. This will again follow if we know this to be true for the Φ_is and for F_{n+1}.

We first consider the question for the Φ_js. Note that the maps in $\mathcal{P}\mathbf{n}[j]$ always preserve j, which is the last element. Translated to Δ, for all $0 < l < j$, it has all the inclusions $d^i\colon [l] \to [l+1]$ but the one omitting $l+1$. This leaves som room for a change of base isomorphism of j cubes $\mathcal{P}^n|\mathcal{P}\mathbf{n}[j] \cong \mathcal{Q}^j$ given by sending d^i to δ^i which omits the $i+1$th coordinate, and is the identity on the vertices of cardinality ≤ 1. Here we have used that the cubes are strongly (co)cartesian. The important outcome is that $\mathcal{Q}^j$ can be constructed iteratively by taking products with BP.

Thus $\Phi_j \cong$ iterated fiber$\mathbf{F}_A\mathcal{Q}^j$, which can be analyzed as follows. Let $P_0, \dots P_n$ be A bimodules, and define

$$\mathbf{F}_A(P_0; P_1 \dots, P_j)$$

inductively by letting $\mathbf{F}_A(P_0)$ be as before, and setting

$$\mathbf{F}_A(P_0; P_1, \dots, P_j) = fiber\{\mathbf{F}_A(P_0; P_1, \dots, P_{j-1}) \longrightarrow \mathbf{F}_A(P_0 \times P_j; P_1, \dots, P_{j-1})\}$$

We see that

$$\mathbf{F}_A(*; BP, \dots, BP) \simeq \Phi_j$$

Now, assume that we know that $\mathbf{F}_A(*; BP) \simeq \Omega\mathbf{F}_A BP$ is m-connected for all A and P. We will show that $\Phi_j \simeq \mathbf{F}_A(*; BP, \dots, BP)$ is also m-connected.

This will follow from the more general statement, that if all the P_i are 1-reduced, then $\mathbf{F}_A(-; P_1, \dots, P_j)$ is m-connected. For $j = 1$ this is immediate as

$$\mathbf{F}_A(P_0; P_1) = fiber\{\mathbf{F}_A(P_0) \to \mathbf{F}_A(P_0 \times P_1)\} \simeq \Omega\mathbf{F}_{A \ltimes P_0}(P_1)\}$$

so assume that $\mathbf{F}_A(-;P_1,\dots,P_{j-1})$ is m-connected. In particular $\mathbf{F}_A(P_0;P_1,\dots,P_{j-1})$ and $\mathbf{F}_A(P_0\times(P_j)_q;P_1,\dots,P_{j-1})$ are m-connected, and using that $\mathbf{F}_A(-)$ may be calculated degreewise we see that

$$\mathbf{F}_A(P_0;P_1,\dots,(P_j)_q) \text{ is } \begin{cases} 0 & \text{if } q=0 \\ m-1 & \text{connected} \end{cases}$$

and hence the conclusion follows.

We are left with showing that the iterated fiber of $\mathbf{F}_A\mathcal{P}^n$ is as highly connected as we need. In fact, we will show that $\mathbf{F}_A\mathcal{P}^n$ is $(n-3)$-cartesian, and so choosing n big enough we are done. In order to prove this, and so to prove the triviality of $\mathbf{F}_AP$, it is enough to prove the two following lemmas.

{lem:VI1.3}

Lemma 1.0.11 *$K(A\ltimes\mathcal{P}^n)$ is n-cartesian.*

Proof: Follows from NBNBref with $k=0$. ∎

and

{lem:VI1.4}

Lemma 1.0.12 *$TC(A\ltimes\mathcal{P}^n)$ is $n-3$ cartesian.*

Proof: This proof occupies the rest of the section and involves several sublemmas.

For each $0\le i\le n$, let P_i be an A bimodule, and let $\underline{T}(A;P_0,\dots,P_n)=\{k\mapsto \underline{T}(A;P_0,\dots,P_n;S^k)\}$ be the n reduced simplicial spectrum given by

$$[q]\mapsto \underset{\overrightarrow{\mathbf{x}\in I^{q+1}}}{\operatorname{holim}}\,\Omega^{\sqcup\mathbf{x}}(S^k\wedge\bigvee_{\phi\in\Delta_m([n],[q])}\bigwedge_{0\le i\le q}F^j\otimes\tilde{\mathbf{Z}}S^{x_i})$$

for $q\ge n$, where $\Delta_m([n],[q])$ is the set of injective order preserving maps $\phi\colon\{0<\dots<n\}\to\{0<\dots<q\}$, and where $F^j=A$ if $j\notin im\phi$ and $F^j=P_{\phi^{-1}(j)}$ otherwise. The simplicial operations are the ordinary Hochschild ones, where the Ps multiply trivially. This is a functor from simplicial A bimodules to spectra, and restricted to each factor it preserves cartesian diagrams. We let $\underline{T}^{(n+1)}(A,P)$ be the composite with the diagonal. We see that this agrees with our earlier definition.

{lem:VI1.5}

Lemma 1.0.13 *Let $\mathcal{M}$ be a strongly (co)cartesian S-cube of simplicial A bimodules. Then $\underline{T}^{(n)}(A,\mathcal{M})$ is cartesian if $|S|>n$.*

Proof: The proof follows Goodwille's argument in [40, proposition 3.4]. We define a new S-cube $\mathcal{Z}$ as follows. If $T\subseteq S$, let $S_T\subseteq\mathcal{P}S$ be the full subcategory with objects U containing T and with $|S-U|\le 1$.

$$\mathcal{Z}_T=\underset{\overleftarrow{U_1\in S_T}}{\operatorname{holim}}\dots\underset{\overleftarrow{U_n\in S_T}}{\operatorname{holim}}\,\underline{T}(A,\{\mathcal{M}_{U_i}\})$$

As $\mathcal{M}$ was strongly cartesian $\mathcal{M}|S_T$ is cartesian, and so the map $\underline{T}^{(n)}(A,\mathcal{M}_T)\to\mathcal{Z}_T$ is an equivalence for each T. The homotopy limits may be collected to be over $S_T^{\times n}$ which may be written as $\cap_{s\in T}\mathcal{A}_s$ where $\mathcal{A}_s$ is the full subcategory of $\mathcal{A}=S_\emptyset^{\times n}$ such that s is in every factor. As $|S|>n$ the $\mathcal{A}_s$ cover $\mathcal{A}$ as in the hypothesis of [40, lemma 1.9], and so $\mathcal{Z}$ is cartesian. ∎

{lem:VI1.6}

Lemma 1.0.14 *If P is $(k-1)$-connected, then*

$$\underline{T}(A\ltimes P)\xleftarrow{\sim}\bigvee_{0\le j<\infty}\underline{T}^{(j)}(A,P)\longrightarrow\bigvee_{0\le j<n}\underline{T}^{(j)}(A,P)$$

is $(n(k+1)-1)$-connected.

Proof: NBNB(ref) gives that

$$\underline{T}(A\ltimes P)\xleftarrow{\sim}\bigvee_{0\le j<\infty}\underline{T}^{(j)}(A,P).$$

If P is $(k-1)$-connected each simplicial dimensions contains smash of j copies of P:

$$\underline{T}(A,P)_q \text{ is } \begin{cases}0 \text{ if } q<j-1\\ kj-1 \text{ connected}\end{cases}$$

and so $\underline{T}^{(j)}(A,P)$ is $j-1+kj-1=(k+1)j-2$ connected. ∎

{lem:VI1.7}

Lemma 1.0.15 *$\underline{T}(A\ltimes\mathcal{P}^n)$ is $id-2$ cartesian.*

Proof: Consider

$$\begin{array}{ccc}\underline{T}(A\ltimes P) & \xrightarrow{a} & \bigvee_{0\le j<n}\underline{T}^{(j)}(A,P)\\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle b}\\ \underset{\leftarrow}{\mathrm{holim}} S\neq\emptyset\,\underline{T}(A\ltimes\mathcal{P}^n_S) & \xrightarrow{c} & \underset{\leftarrow}{\mathrm{holim}} S\neq\emptyset\bigvee_{0\le j<n}\underline{T}^{(j)}(A,\mathcal{P}^n_S)\end{array}$$

By lemma 1.0.14 a is $(n-1)$-connected. By lemma 1.0.13 $\underline{T}^{(j)}(A,-)$ is n-excisive for $j<n$, and so

$$\bigvee_{0\le j<n}\underline{T}^{(j)}(A,P)\xrightarrow{\sim}\bigvee_{0\le j<n}\underset{\overleftarrow{S\neq\emptyset}}{\mathrm{holim}}\,\underline{T}^{(j)}(A,\mathcal{P}^n_S).$$

Since $\vee\sim\prod$ for spectra, this implies that b is an equivalence.

Again, by lemma 1.0.14

$$\underline{T}(A\ltimes\mathcal{P}^n_S)\xrightarrow{a}\bigvee_{0\le j<n}\underline{T}^{(j)}(A,\mathcal{P}^n_S)$$

is $(2n-1)$-connected for $S\neq\emptyset$, with fiber, say $\mathcal{F}_S$, $(2n-2)$-connected. The fiber of c equals $\mathrm{holim}_{\overleftarrow{S\neq\emptyset}}\mathcal{F}_S$, and must then be $2n-2-n+1=n-1$ connected (an n cube consisting of l connected spaces must have $(l-n)$-connected iterated fiber: by induction). Hence c is n-connected.

This means that d must be $(n-2)$-connected. Likewise for all subcubes (some are id-cartesian). ∎

Applying proposition NBNB5.? with $k=n-2$, this concludes the proof of lemma 1.0.12 ∎

This also concludes the proof of theorem 1.0.10. ∎

2 Goodwillie's ICM'90 conjecture.

In this section we will prove

Theorem 2.0.16 (Goodwillie's ICM'90 conjecture) *Let $A \to B$ be a map of* **S**-*algebras inducing a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then*

$$\begin{array}{ccc} K(A) & \longrightarrow & TC(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & TC(B) \end{array}$$

is homotopy cartesian.

Following the procedure of [39] we first prove it in the case $A \to B$ is a simplicial ring map, and then use the density argument to extend it to **S**-algebras.

We start up with some consequences of the split square zero case.

{VI2.2}

Lemma 2.0.17 *Let $f\colon B \to A$ be a simplicial ring map such that each f_q is an epimorphism with nilpotent kernel. Then*

$$\begin{array}{ccc} K(B) & \longrightarrow & TC(B) \\ \downarrow & & \downarrow \\ K(A) & \longrightarrow & TC(A) \end{array}$$

is homotopy cartesian.

Proof: Let $I = \ker(f)$. As we may calculate the K-theory and TC of a simplicial radical extension degreewise, the statement will follow if we can prove the it for $f_q\colon B_q \to A_q$ for each q. That is, we may assume that B and A are discrete, and that $I = \ker(f)$ satisfies $I^n = 0$. Note that each of the maps

$$B = B/I^n \to B/I^{n-1} \to \dots B/I^2 \to B/I = A$$

are square zero extensions, so it will be enough to show the corollary when $I^2 = 0$.

Let $F \overset{\sim}{\twoheadrightarrow} A$ be a free resolution of A, and perform pullback

$$\begin{array}{ccc} P & \longrightarrow & F \\ \simeq\downarrow & & \simeq\downarrow \\ B & \xrightarrow{f} & A \end{array}$$

Using again that we may calculate the K-theory and TC of a simplicial radical extension degreewise, the result will follow for $P \to F$ (and hence for f) if we can prove the statement for $P_q \to F_q$ for each q. But as each F_q is a free ring $P_q \to F_q$ must be a split square zero extension, for which the theorem is guaranteed by 1.0.10. ∎

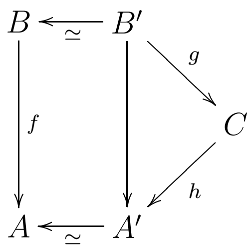
{lem:VI2.3}

Lemma 2.0.18 *Let $f\colon B \to A$ be a 1-connected epimorphism of simplicial rings, then*

$$\begin{array}{ccc} K(B) & \longrightarrow & TC(B) \\ \downarrow & & \downarrow \\ K(A) & \longrightarrow & TC(A) \end{array}$$

is homotopy cartesian.

Proof: Note that if $R \to S$ is a k-connected map of simplicial rings, then $K(R) \to K(S)$ and $\mathbf{TC}(R) \to \mathbf{TC}(S)$ will be (at least) $(k-1)$-connected. We will clearly be done if we can show that any $k \geq 1$ connected map $f\colon A \to B$ has a diagram of the following sort



where g is a $(k+1)$-connected epimorphism and h is a square zero extension. The horizontal maps are simply the replacement of I by a k-reduced ideal $I' \subseteq B'$ described in ??. We set g to be the projection $B' \to B'/(I')^2 = C$. We have a short exact sequence of simplicial abelian groups

$$0 \longrightarrow \ker(m) \longrightarrow I' \otimes_{\mathbf{Z}} I' \xrightarrow{m} (I')^2 \longrightarrow 0.$$

As I' is k-reduced, so is $\ker(m)$, and $I' \otimes_{\mathbf{Z}} I'$ is $(2k-1)$-connected, and accordingly $\ker(g) = (I')^2$ must be at least k-connected. ∎

{theo:VI2

Theorem 2.0.19 *Let $f\colon A \to B$ be a map of simplicial rings inducing a surjection $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then*

$$\begin{array}{ccc} K(A) & \longrightarrow & TC(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & TC(B) \end{array}$$

is homotopy cartesian.

Proof: Consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ \pi_0 B & \longrightarrow & \pi_0 A \end{array}$$

The theorem will follow for f if it is true for the three other maps. This follows for the vertical maps by lemma 2.0.18, and for the horizontal map by lemma **??**. ∎

Proof: (Proof of theorem **??**. As in the preceding proof, it is enough to consider the maps $A \to \pi_0 A$. We know that for every $S \in ob\mathcal{P}$, the diagram

$$\begin{array}{ccc} K(A_S) & \longrightarrow & TC(A_S) \\ \downarrow & & \downarrow \\ K(\pi_0 A) & \longrightarrow & TC(\pi_0 A) \end{array}$$

is cartesian, and furthermore, that

$$K(A) \xrightarrow{\sim} \underset{\overleftarrow{S \in \mathcal{P} - \emptyset}}{\operatorname{holim}} K(A_S)$$

and

$$TC(A) \xrightarrow{\sim} \underset{\overleftarrow{S \in \mathcal{P} - \emptyset}}{\operatorname{holim}} TC(A_S)$$

and the result follows. ∎

3 Some hard calculations and applications

{calc}

To be written. Must contain reviews of the relevant results of Bökstedt, Brun, Hesselholt, Hsiang, Madsen, Rognes and Tsalidis. (will contain no original mathematics)

Appendix A

Simplicial techniques

{A1}

For the convenience of the reader, we give a short review of simplicial techniques. This is meant only as a reference, and is far from complete. Most results are referred away, and we only provide proofs when no convenient single reference was available, or when the proofs have some independent interest. Most of the material in this appendix can be found in Bousfield and Kan's book [14] or in Goerss and Jardine's book [36].

0.1 The category Δ

{subsec:De

Let Δ be the category consisting of the finite ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for every nonnegative integer n, and monotone maps. In particular, for $0 \leq i \leq n$ we have the maps

$$d^i\colon [n-1] \to [n], \qquad d^i(j) = \begin{cases} j & j < i \\ j+1 & i \leq j \end{cases} \qquad \text{“skips } i\text{”}$$

$$s^i\colon [n+1] \to [n], \qquad s^i(j) = \begin{cases} j & j \leq i \\ j-1 & i < j \end{cases} \qquad \text{“hits } i \text{ twice”}$$

Every map in Δ has a factorization in terms of these maps. Let $\phi \in \Delta([n],[m])$. Let $\{i_1 < i_2 < \cdots < i_k\} = [m] - im(\phi)$, and $\{j_1 < j_2 < \cdots < j_l\} = \{j \in [n] | \phi(j) = \phi(j+1)\}$. Then

$$\phi(j) = d^{i_k} d^{i_{k-1}} \cdots d^{i_1} s^{j_1} s^{j_2} \cdots s^{j_l}(j)$$

This factorization is unique, and hence we could describe Δ as being generated by the maps d^i and s^i subject to the “cosimplicial identities” :

$$d^j d^i = d^i d^{j-1} \quad \text{for } i < j$$

$$s^j s^i = s^{i-1} s^j \quad \text{for } i > j$$

and

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ d^{i-1} s^j & \text{for } i > j+1 \end{cases}$$

0.2 Simplicial and cosimplicial objects

If $\mathcal{C}$ is a category, the *opposite* category, $\mathcal{C}^o$ is the same category, but where you have reversed the direction of all arrows. A functor from $\mathcal{C}^o$ is sometimes called a contravariant functor.

If $\mathcal{C}$ is any category, a *simplicial* $\mathcal{C}$ *object* is a functor $\Delta^o \to \mathcal{C}$, and a *cosimplicial* $\mathcal{C}$ *object* is a functor $\Delta \to \mathcal{C}$.

If X is a simplicial object, we let X_n be the image of $[n]$, and for a map $\phi \in \Delta$ we will often write ϕ^* for $X(\phi)$. For the particular maps d^i and s^i, we write simply d_i and s_i for $X(d^i)$ and $X(s^i)$, and call them *face* and *degeneracy maps*. Note that the face and degeneracy maps satisfy the "simplicial identities" which are the duals of the cosimplicial identities. Hence a simplicial object is often defined in the literature to be a subcategory of $\mathcal{C}$, consisting of a sequence of objects X_n and maps d_i and s_i satisfying these identities.

Dually, for a cosimplicial object X, we let $X^n = X([n])$, $\phi_* = X(\phi)$, and the coface and codegeneracy maps are written d^i and s^i.

A map between two (co)-simplicial $\mathcal{C}$ objects is a natural transformation. Generally, we let $s\mathcal{C}$ and $c\mathcal{C}$ be the categories of simplicial and co-simplicial $\mathcal{C}$ objects.

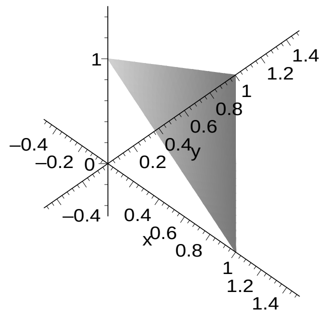
Functor categories like $s\mathcal{C}$ and $c\mathcal{C}$ inherits limits and colimits from $\mathcal{C}$ (and in particular sums and products), when these exist. We say that (co)limits are formed degreewise.

Example 0.2.1 (the topological standard simplices) There is an important cosimplicial topological space $[n] \mapsto \Delta^n$, where Δ^n is the *standard topological n-simplex*

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1} | \sum x_i = 1, x_i \geq 0\}$$

with

$$d^i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_{n-1})$$
$$s^i(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n+1})$$



The standard topological 2-simplex $\Delta^2 \in \mathbf{R}$.

{A103}

0.3 Resolutions from adjoint functors

Adjoint functors are an important source of (co)simplicial objects. Let

$$\mathcal{D} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

be a pair of adjoint functors: we have a natural bijection of morphism sets

$$\mathcal{C}(F(d), c) \cong \mathcal{D}(d, U(c))$$

induced by the unit $\sigma_d\colon d \to UF(d)$ (corresponding to $id_{F(d)} \in \mathcal{D}(F(d), F(d))$) and counit $\delta_c\colon FU(c) \to c$ (corresponding to $id_{U(c)} \in \mathcal{C}(U(c), U(c))$). Then

$$[q] \mapsto (FU)^{q+1}(c)$$

defines a simplicial $\mathcal{C}$ object with structure maps defined by

$$d_i = (FU)^i \delta_{(FU)^{q-i+1}}\colon (FU)^{q+2}(c) \to (FU)^{q+1}(c)$$

and

$$s_i = (FU)^i F\sigma_{U(FU)^{q-i}}\colon (FU)^q(c) \to (FU)^{q+1}(c).$$

Dually, $[q] \mapsto (UF)^{q+1}(d)$ defines a cosimplicial $\mathcal{D}$ object. These (co)simplicial objects are called the *(co)simplicial resolutions associated with the adjoint pair.*

The coposite $T = UF$ (together with the associated natural transformations $1 \to T$ and $TT \to T$) is occasionally referred to as a *triple* or *monad* (probably short for "monoid in the monoidal category of endofunctors and composition"), and likewise FU a *cotriple* or *comonad*, but never mind: the important thing to us are the associated (co)simplicial resolutions,

1 Simplicial sets

Let $\mathcal{E}ns$ be the category of sets (when we say "sets" they are supposed to be small in some fixed universe). Let$\mathcal{S} = s\mathcal{E}ns$ the category of simplicial sets. Since all (co)limits exist in $\mathcal{E}ns$, all (co)limits exist in $\mathcal{S}$. The category of simplicial sets has close connections with the category $\mathcal{T}op$ of topological spaces. In particular, the realization and singular functors (see 1.1) induce equivalences between their respective "homotopy categories" (see 1.3.3 below).

In view of this equivalence, we let a "*space*" mean a simplicial set (unless explicitly called a *topological* space). We also have a pointed version. A pointed set is a set with a preferred element, called the base point, and a pointed map is a map respecting base points. The category of pointed spaces (= pointed simplicial sets = simplicial pointed sets) is denoted $\mathcal{S}_*$. Also $\mathcal{S}_*$ has (co)limits. In particular we let

$$X \vee Y = X \coprod_* Y$$

and

$$X \wedge Y = X \times Y / X \vee Y$$

If $X \in \mathcal{S}$ we can add a disjoint basepoint and get the pointed simplicial set

$$X_+ = X \coprod *$$

1.1 Simplicial sets vs. topological spaces

{subsec:A

There are adjoint functors

$$\mathcal{T}op \overset{|-|}{\underset{\sin}{\leftrightarrows}} \mathcal{S}$$

defined as follows. For $Y \in \mathcal{T}op$, the *singular functor* is defined as

$$\sin Y = \{[n] \mapsto \mathcal{T}op(\Delta^n, Y)\}$$

(the set of unbased continuous functions from the topological standard simplex to Y). As $[n] \mapsto \Delta^n$ was a cosimplicial space, this becomes a simplicial set. For $X \in \mathcal{S}$, the *realization functor* is defined as

$$|X| = \left(\coprod_n X_n \times \Delta^n\right) / (\phi^* x, u) \sim (x, \phi_* u).$$

The realization functor is left adjoint to the singular functor, i.e. there is a bijection

$$\mathcal{T}op(|X|, Y) \leftrightarrow \mathcal{S}(X, \sin Y)$$

The bijection is induced by the adjunction maps

$$X \to \sin |X|$$
$$x \in X_n \mapsto (\Delta^n \overset{u \mapsto (x,u)}{\to} X_n \times \Delta^n \to |X|) \in \sin |X|_n$$

and

$$|\sin Y| \to Y$$
$$(y, u) \in \sin(Y)_n \times \Delta^n \mapsto y(u) \in Y$$

If $\mathcal{S}_*$ is the category of simplicial pointed sets(i.e. "pointed spaces"), then the singular and realization functor also define adjoint functors between $\mathcal{S}_*$ and the category of pointed topological spaces, $\mathcal{T}op_*$.

If $X \in \mathcal{S}_*$ we define

$$\pi_*(X) = \pi_*(|X|), \text{ and } H_*(X) = H(|X|).$$

1.2 Simplicial abelian groups

Let $\mathcal{A}b$ be the category of abelian groups. Since $\mathcal{A}b$ has all (co)limits, so has the category $\mathcal{A} = s\mathcal{A}b$ of simplicial abelian groups. A simplicial abelian group M may be regarded as a chain complex:

$$C(M) = \{M_0 \xleftarrow{d_0 - d_1} M_1 \xleftarrow{d_0 - d_1 + d_2} M_2 \xleftarrow{d_0 - d_1 + d_2 - d_3} \dots\}$$

We define the *homotopy groups* of M by

$$\pi_*(M) = H_*(C(M))$$

There are free/forgetful functors

$$\mathcal{A}b \underset{U}{\overset{\mathbf{Z}[-]}{\leftrightarrows}} \mathcal{E}ns$$

where $\mathbf{Z}[X]$ is the free abelian group on the set X. One can prove that the definition of homotopy groups of simplicial abelian groups agree with the definition for simplicial sets: $\pi_*(|U(M)|) \cong \pi_*(M) = H_*(C(M))$ see e.g. [36, ?]. Note that the *singular homology* of a topological space Y is defined as $H_*(Y) = \pi_*(\mathbf{Z}[\sin Y])$. For $X \in \mathcal{S}$ we define $H_*(X) = \pi_*(\mathbf{Z}[X])$. The map $X = 1 \cdot X \subseteq U\mathbf{Z}[X]$ induces the *Hurewicz map* $\pi_*(X) \to H_*(X)$ on homotopy groups.

1.3 The standard simplices, and homotopies

We define a cosimplicial space (cosimplicial simplicial set)

$$[n] \mapsto \Delta[n] = \{[q] \mapsto \Delta([q], [n])\}.$$

The spaces $\Delta[n]$ are referred to as the standard simplices. Note that the realization $|\Delta[n]|$ of the standard simplex equals Δ^n, the *topological* standard simplex. The standard simplices are in a precise way, the building blocks (representing objects) for all simplicial sets: if X is a simplicial set, then there is a functorial isomorphism

$$\mathcal{S}(\Delta[n], X) \cong X_n, \qquad f \mapsto f([n] = [n])$$

A *homotopy* between two maps $f_0, f_1 \colon X \to Y \in \mathcal{S}$ is a map $H \colon X \times \Delta[1] \to Y$ such that the composites

$$X \cong X \times \Delta[0] \xrightarrow{id \times d_i} X \times \Delta[1] \xrightarrow{H} Y, \qquad i = 0, 1$$

are f_1 and f_0. Since $|X \times \Delta[1]| \cong |X| \times |\Delta[1]|$, we see that the realization of a homotopy is a homotopy in $\mathcal{T}op$. The pointed version is a map

$$H \colon X \wedge \Delta[1]_+ \to Y$$

(the subscript $_+$ means a disjoint basepoint added).

We say that f_0 and f_1 are *strictly homotopic* if there is a homotopy between them, and *homotopic* if there is a chain of homotopies which connect f_0 and f_1. In this way, "homotopic" forms an equivalence relation.

Another way to say this is that two maps $f_0, f_1 \colon X \to Y$ are homotopic if there is a map

$$H \colon X \times I \to Y, \text{ or in the pointed case } H \colon X \wedge I_+ \to Y$$

which is equal to f_0 and f_1 at the "ends" of I, where I is a finite number of $\Delta[1]$s glued together at the endpoints, i.e. for some sequence of numbers $i_j \in \{0,1\}$, $1 \leq j \leq n$, I is the colimit of

$$\begin{array}{ccccccccccc} \Delta[1] & & & & \Delta[1] & & & & \cdots & & & & \Delta[1] \\ & \nwarrow^{d^1} & & \nearrow^{d^{i_1}} & & \nwarrow^{d^{1-i_1}} & & \nearrow^{d^{i_2}} & & & \nwarrow^{d^{1-i_n}} & & \nearrow^{d^0} \\ & & \Delta[0] & & & & \Delta[0] & & & & & \Delta[0] & \end{array}$$

We still denote the two end inclusions $d^0, d^1 \colon * = \Delta[0] \to I$.

We note that elements in $\pi_1(X)$ can be represented by maps $\alpha\colon I \to X$ such that $\alpha d^0 = \alpha d^1 = 0$.

1.1.4 Function spaces

In analogy with the mapping space, we define the simplicial function space of maps from X to Y to be the simplicial set

$$\underline{\mathcal{S}}(X,Y) = \{[q] \mapsto \mathcal{S}(X \times \Delta[q], Y)\}$$

the cosimplicial structure of the standard simplices makes this into a simplicial set. In the pointed case we set

$$\underline{\mathcal{S}_*}(X,Y) = \{[q] \mapsto \mathcal{S}_*(X \wedge \Delta[q]_+, Y)$$

We reserve the symbol Y^X for the pointed case: $Y^X = \underline{\mathcal{S}_*}(X,Y)$, and so $Y^{X_+} = \underline{\mathcal{S}}(X,Y)$. Unfortunately, these definition are not homotopy invariant; for instance, the weak equivalence $B\mathbf{N} \to \sin|B\mathbf{N}|$ does not induce an equivalence $\underline{\mathcal{S}_*}(S^1, B\mathbf{N}) \to \underline{\mathcal{S}_*}(S^1, \sin|B\mathbf{N}|)$ (on π_0 it is the inclusion $\mathbf{N} \subset \mathbf{Z}$). To remedy this we define

$$Map(X,Y) = \underline{\mathcal{S}}(X,Y)$$

and

$$Map_*(X,Y) = \underline{\mathcal{S}_*}(X, \sin|Y|)$$

In fact, using the adjointness of the singular and realization functor, we see that

$$\begin{aligned} Map(X,Y) &\cong \{[q] \mapsto \mathcal{T}op(|X| \times |\Delta[q]|, |Y|\} \cong \{[q] \mapsto \mathcal{T}op(\Delta^q, \mathcal{T}op(|X|, |Y|)\} \\ &= \sin(\mathcal{T}op(|X|, |Y|)) \end{aligned}$$

and likewise in the pointed case.

These function spaces still have some sort of adjointness properties, in that

$$Map(X \times Y, Z) \cong \mathcal{S}(X, Map(Y,Z)) \xrightarrow{\sim} Map(X, Map(Y,Z))$$

and

$$Map_*(X \wedge Y, Z) \cong \underline{\mathcal{S}_*}(X, Map_*(Y,Z)) \xrightarrow{\sim} Map_*(X, Map_*(Y,Z))$$

where the equivalences have canonical left inverses.

1.1.5 The nerve of a category

{A115}

Let $\mathcal{C}$ be a small category. For every $n \geq 0$, regard $[n] = \{0 < 1 < \cdots < n\}$ as a category (if $a \leq b$ there is a unique map $a \leftarrow b$: beware that many authors let the arrow point in the other direction. The choice of convention does not matter to the theory). Furthermore, we identify the maps in Δ with the corresponding functors, so that Δ sits as a full subcategory of the category of (small) categories.

{Def:nerv

Definition 1.1.5.1 The *nerve* $N\mathcal{C}$ of the small category $\mathcal{C}$ is the simplicial category

$$[q] \mapsto obN_q\mathcal{C} = \{\text{category of functors } [q] \to \mathcal{C}\}$$

The nerve is a functor from the category of small categories to simplicial categories.

We see that the set of objects $obN_q\mathcal{C}$, is the set of all chains $c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q$ in $\mathcal{C}$, and in particular $obN_0\mathcal{C} = ob\mathcal{C}$.

Frequently, the underlying simplicial set $obN\mathcal{C}$ is also referred to as the nerve of $\mathcal{C}$. Note that $obN[q] \cong \Delta[q]$.

The nerve obN, as a functor from categories to spaces, has a left adjoint given by sending a simplicial set X to the category CX defined as follows. The set of objects is X_0. The set of morphisms is generated by X_1, where $y \in X_1$ is regarded as an arrow $y\colon d_0y \to d_1y$, subject to the relations that $s_0x = 1_x$ for every $x \in X_0$, and for every $z \in X_2$

$$\begin{array}{ccc} d_0d_0z & \xrightarrow{d_0z} & d_1d_0z \\ & {\scriptstyle d_1z}\searrow \quad \swarrow{\scriptstyle d_2z} & \\ & d_1d_1z & \end{array}$$

commutes. The adjunction map $CobN\mathcal{D} \to \mathcal{D}$ is an isomorphism, and the nerve obN is a full and faithful functor.

1.1.5.2 Natural transformations and homotopies

The nerve takes natural transformations to homotopies: if $\eta\colon F_1 \to F_0$ is a natural transformation of functors $\mathcal{C} \to \mathcal{D}$, regard it as a functor $\eta\colon \mathcal{C} \times [1] \to \mathcal{D}$ by sending $(c \leftarrow c', 0 < 1)$ to

$$\begin{array}{ccc} F_1(c) & \longleftarrow & F_1(c') \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} \\ F_0(c) & \longleftarrow & F_0(c') \end{array}$$

Thus we have defined a homotopy between F_0 and F_1:

$$obN\mathcal{C} \times \Delta[1] \cong obN\mathcal{C} \times obN[1] \cong obN(\mathcal{C} \times [1]) \to obN\mathcal{D}$$

1.1.5.3 Over and under categories

If $\mathcal{C}$ is a category and c an object in $\mathcal{C}$, the category over c, written $\mathcal{C}/c$, is the category whose objects are maps $f\colon d \to c \in \mathcal{C}$, and a morphism from f to g is a factorization $f = g\alpha$. Dually the under category is defined. Over and under categories are frequently referred to as *comma categories* in the literature.

1.1.6 Subdivisions and Kan's Ex^∞

Consider the subcategory $\Delta_m \subset \Delta$ with all objects, but just monomorphisms. For any $n \geq 0$ we consider the *subdivision* of the standard n-simplex $\Delta[n]$. To be precise, it is $N(\Delta_m/[n])$, the nerve of the category of order preserving monomorphisms into $[n]$. For every $\phi\colon [n] \to [m] \in \Delta$ we get a functor $\phi_*\colon \Delta_m/[n] \to \Delta_m/[m]$ sending $\alpha\colon [q] \subseteq [n]$ to the unique monomorphism $\phi_*(\alpha)$ such that $\phi\alpha = \phi_*(\alpha)\phi$ where ϕ is an epimorphism (see section 0.1). This means that $N(\Delta_m/-)$ is a cosimplicial space, and the functor $\Delta_m/[n] \to [n]$ sending $\alpha\colon [p] \subseteq [n]$ to $\alpha(p) \in [n]$ defines a cosimplicial map to the standard simplices $\{[n] \mapsto \Delta[n] = N[n]\}$.

For any simplicial set X Kan then defines

$$Ex(X) = \{[q] \mapsto \mathcal{S}(N(\Delta_m/[q]), X)\}$$

This is a simplicial set, and $N(\Delta_m/[q]) \to \Delta[n]$, defines an inclusion $X \subseteq Ex(X)$. Set

$$Ex^\infty X = \varinjlim_{k} Ex^{(k)}(X).$$

The inclusion $X \subseteq Ex^\infty X$ is a weak equivalence, and $Ex^\infty X$ is always a "Kan complex", that is a *fibrant* object in the sense of section 1.3. In fact, they give the possibility of defining the homotopy groups without reference to topological spaces via

$$\pi_q X = \pi_0 \underline{\mathcal{S}_*}(S^q, Ex^\infty X)$$

where π_0 component classes.

1.1.7 Filtered colimits in $\mathcal{S}_*$

Filtered colimits are colimits over *filtered categories* (see [79, p. 207]). Filtered colimits of sets especially nice because they commute with finite limits (see [79, p. 211]). This fact has an analog for simplicial sets, and this is one of the many places we should be happy for not considering general topological spaces. Recall that a filtered category J is a nonempty category such that for any $j, j' \in obJ$ there are maps $j \to k$, $j' \to k$ to a common object, and such that if $f, g\colon j \to j'$, then there is an $h\colon j' \to k$ such that $hf = hg$.

Given a space Y, its *N-skeleton* is the subspace $sk_N Y \subseteq Y$ generated by simplices in dimension less than or equal to N.

A space Y is *finite* if it has only finitely many non degenerate base points. Alternatively finiteness can be spelled out as, $Y = sk_N Y$ for some N, Y_0 is finite, and its q-skeleton for $q \leq N$ is formed by iterated pushouts over finite sets D_q

$$\begin{array}{ccc} \bigvee_{D_q} \partial\Delta[q]_+ & \longrightarrow & \bigvee_{D_q} \Delta[q]_+ \\ \downarrow & & \downarrow \\ sk_{q-1} Y & \longrightarrow & sk_q Y \end{array}$$

{lemma:A1:

Lemma 1.1.7.1 *Let J be a filtered category and Y finite.Then the canonical map*

$$\varinjlim_{J} \underline{\mathcal{S}_*}(Y, X) \to \underline{\mathcal{S}_*}(Y, \varinjlim_{J} X)$$

is an isomorphism

Proof: Since $\underline{\mathcal{S}_*}(Y, -)_q = \mathcal{S}_*(Y \wedge \Delta[q]_+, -)$ and $Y \wedge \Delta[q]_+$ is finite, it is clearly enough to prove that

$$\varinjlim_{J} \mathcal{S}_*(Y, X) \cong \mathcal{S}_*(Y, \varinjlim_{J} X).$$

Remember that filtered colimits commute with finite limits. Since Y is a finite colimit of diagrams made out of $\Delta[q]$'s, this means that it is enough to prove the lemma for $Y = \Delta[q]$, which is trivial since $\mathcal{S}_*(\Delta[q], X) = X_q$ and colimits are formed degreewise. ∎

Lemma 1.1.7.2 *If J is a filtered category, then the canonical map*

$$\varinjlim_{J} Ex^\infty X \to Ex^\infty \varinjlim_{J} X$$

is an isomorphism.

Proof: Since colimits commute with colimits, it is enough to prove that Ex commute with filtered colimits, but this is clear since $Ex(X)_n = \mathcal{S}_*(N(\Delta_m/[n]), X)$ and $N(\Delta_m/[n])$ is a simplicial finite set equal to its n-skeleton. ∎

{lemma:pi

Proposition 1.1.7.3 *Homotopy groups commute with filtered colimits.*

Proof: Let $X \in ob\mathcal{S}_*$ and J be a filtered category. First note that π_0, being a colimit itself, commutes with arbitrary colimits. For $q \geq 0$ we have isomorphisms

$$\begin{aligned} \pi_q \varinjlim_{J} X &\cong \pi_0 \underline{\mathcal{S}_*}(S^q, Ex^\infty \varinjlim_{J} X) \\ &\cong \pi_0 \underline{\mathcal{S}_*}(S^q, \varinjlim_{J} Ex^\infty X) \\ &\cong \varinjlim_{J} \pi_0 \underline{\mathcal{S}_*}(S^q, Ex^\infty X) \cong \varinjlim_{J} \pi_q X \end{aligned}$$

∎

1.1.8 The classifying space of a group

Let G be a group, and regard it as a one point category whose morphisms are the group elements. Then $N_qG = G^{\times q}$, and $BG = NG$ is called the *classifying space* of G. The homotopy groups of BG are given by

$$\pi_i(BG) = \begin{cases} G & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

If G is abelian, BG becomes a simplicial abelian group. Hence we may apply the construction again, and get a bisimplicial abelian group, and so on. Taking the diagonal, we get a sequence G, BG, $diag^*BBG$, ... The nth term $diag^*B^nG$ is isomorphic to $\tilde{\mathbf{Z}}[S^n] \otimes G$, and is often written $H(G, n)$, and is characterized up to homotopy by having only one nonzero homotopy group $\pi_n = G$, and such spaces are called *Eilenberg-MacLane spaces*. We call any space (weakly) equivalent to a simplicial abelian group, an Eilenberg-MacLane space. Note that there is a map $S^1 \wedge H(G, n) \to \tilde{\mathbf{Z}}[S^1] \otimes H(G, n) \cong H(G, n + 1)$, and so they are examples of spectra (see section 1.2 below).

If G is a simplicial group there is an alternative construction for the homotopy type of $diag^*BG$, called $\bar{W}G$, which is most easily described as follows: We have a functor $\sqcup\colon \Delta \times \Delta \to \Delta$ sending two ordered sets S and T to the naturally ordered disjoint union $S \sqcup T$. For any simplicial set X we may consider the bisimplicial set sd_2X gotten by precomposing X with $\sqcup$ (so that the $(sd_2X)_{p,q} = X_{p+q+1}$: the diagonal of this construction was called the (second) *edgewise subdivision* in chapter VI). We define $\bar{W}G$ to be space with q-simplices

$$\bar{W}_qG = \{\text{bisimplicial maps } sd_2\Delta[q] \to BG\},$$

where the simplicial structure is induced by the cosimplicial structure of $[q] \mapsto \Delta[q]$. This description is isomorphic to the one given in [82] (with reversed orientation).

1.1.9 Kan's loop group

{A1Kanloop}

The classifying space BG of a group G is a reduced space (i.e. it only has one zero simplex). On the category of reduced spaces X there is a particularly nice model GX for the loop functor due to Kan [61], see [82, p. 118] or [36]. If $q \geq 0$ we have that G_qX is the free group generated by X_{q+1} modulo contracting the image of s_0 to the base point. The degeneracy and face maps are induced from X except the extreme face map (which extreme depends on your choice of orientation, see [82, definition 26.3] for one choice). The Kan loop group is adjoint to the $\bar{W}$-construction described above. As a matter of fact, $\bar{W}$ and G form a "Quillen equivalence" which among other things implies that the homotopy category of reduced spaces is equivalent to the homotopy category of simplicial groups.

1.1.10 Path objects

pathobject}

Let Y be a simplicial object in a category $\mathcal{C}$. There is a convenient combinatorial model mimicking the path space Y^I. Let $\sqcup\colon \Delta \times \Delta \to \Delta$ be the ordered disjoint union.

Definition 1.1.10.1 Let Y be a simplicial object in a category $\mathcal{C}$. Then the *path object* is the simplicial object PY given by precomposing Y with $[0]\sqcup ?\colon \Delta \longrightarrow \Delta$.

Hence $P_qX = X_{q+1}$.

The map $PY \to Y$ corresponding to evaluation is given by the natural transformation $d^0\colon id \to [0] \sqcup id$ (concretely: it is $P_qX = X_{q+1} \xrightarrow{d_0} X_q$).

Lemma 1.1.10.2 *The maps $Y_0 \to PY \to Y_0$ induced by the natural maps $[0] \to [0] \sqcup [q] \to [0]$ are simplicial homotopy equivalences.*

Proof: That $PY \to Y_0 \to PY$ is simplicially homotopic to the identity follows from the natural transformation of functors $(\Delta/[1])^o \to \Delta^o$ sending $\phi\colon [q] \to [1]$ to $\phi_*\colon [0] \sqcup [q] \to [0] \sqcup [q]$ with $\phi_*(0) = 0$ and

$$\phi_*(j+1) = \begin{cases} 0 & \text{if } \phi(j) = 0 \\ j+1 & \text{if } \phi(j) = 1 \end{cases}.$$

∎

The connection to the path-space is the following: considering Δ as a subcategory of the category of small categories in the usual way, there is a (non-naturally) split projection $[1] \times [q] \to [0] \sqcup [q]$ collapsing everything in $\{0\} \times [q]$ through the natural $[0] \subseteq [0] \sqcup [q]$, and sending $\{1\} \times [q]$ isomorphically to the image of the natural $[q] \subseteq [0] \sqcup [q]$. If X is a simplicial set, the usual path space is

$$X^{I_+} = \underline{\mathcal{S}}(\Delta[1], X) = \{[q] \mapsto \mathcal{S}(\Delta[1] \times \Delta[q], X) = \mathcal{S}(N([1] \times [q]), X)\}$$

whereas $PX = \{[q] \mapsto \mathcal{S}(N([0] \sqcup [q]), X)\}$, and the injection $PX \subseteq X^{I_+}$ is induced by the above projection.

1.2 Spectra

{A12spt}

A *spectrum* is a sequence of spaces $X = \{X^0, X^1, X^2, \dots\}$ together with (structure) maps $S^1 \wedge X^k \to X^{k+1}$ for $k \geq 0$. A map of spectra $f\colon X \to Y$ is a sequence of maps $f^k\colon X^k \to Y^k$ compatible with the structure maps: the diagrams

$$\begin{array}{ccc} S^1 \wedge X^k & \longrightarrow & X^{k+1} \\ \downarrow{\scriptstyle f^k} & & \downarrow{\scriptstyle f^{k+1}} \\ S^1 \wedge Y^k & \longrightarrow & Y^{k+1} \end{array}$$

We let $\mathcal{S}pt$ be the resulting category of spectra. This category is enriched in $\mathcal{S}_*$, with morphism spaces given by

$$\underline{\mathcal{S}pt}^0(X,Y) = \{[q] \mapsto \mathcal{S}pt(X \wedge \Delta[q]_+, Y)\}$$

In fact, this is the zero space of a function spectrum

$$\underline{\mathcal{S}pt}(X,Y) = \{k \mapsto \underline{\mathcal{S}pt}^0(X, Y^{(k+?)})\}$$

There is a specially important spectrum, namely the *sphere spectrum*

$$\underline{\mathbf{S}} = \{k \mapsto S^k = S^1 \wedge \ldots \wedge S^1\}$$

whose structure maps are the identity. Note that there is a natural isomorphism $\underline{\mathcal{S}pt}(\underline{\mathbf{S}}, X) \cong X$.

Recall that if $Y \in \mathcal{S}_*$, then $\Omega^k Y = \underline{\mathcal{S}_*}(S^k, \sin|Y|)$. Let $\eta_X^n \colon \sin|X^n| \to \Omega^1 X^{n+1}$ be the adjoint of $S^1 \wedge \sin|X^n| \to \sin|S^1 \wedge X^n| \to \sin|X^{n+1}|$. We say that a spectrum X is an Ω-*spectrum* if η_X^n is an equivalence for all n.

For $X \in ob\mathcal{S}pt$ we define the spectrum

$$QX = \{n \mapsto \varinjlim_{k} \Omega^k X^{k+n}\}$$

where the colimit is taken over the maps

$$\Omega^k X^{k+n} \xrightarrow{(\eta_X^{n+k})_*} \Omega^{k+1} X^{k+n+1}.$$

This is an Ω-spectrum. The homotopy groups of X are set to

$$\pi_q X = \varinjlim_{k} \pi_{q+k} X^k$$

where the colimit is over the maps $\pi_{q+k} X^k \to \pi_{q+k} \Omega X^{k+1} \cong \pi_{q+k+1} X^{k+1}$ for $k > q$. In effect, this means that a map of spectra $X \to Y$ induces an isomorphism in homotopy if and only if $QX^k \to QY^k$ is a weak equivalence for every k.

We say that a spectrum X is *cofibrant* if all the structure maps $S^1 \wedge X^k \to X^{k+1}$ are cofibrations (i.e., inclusions). We say that a spectrum X is n-*connected* if $\pi_q X = 0$ for $q \leq n$. We then get the trivial observation:

connective}

Lemma 1.2.0.3 *There is a canonical pointwise equivalence* $\bigcup_{n \geq 0} C_n(X) \xrightarrow{\sim} X$ *such that for given* n, $C_n(X)$ *is a cofibrant* $-n-1$*-connected spectrum.*

Proof: By induction we use the functorial factorizations in $\mathcal{S}_*$ to build a pointwise equivalence (and fibration) $C(X) \xrightarrow{\sim} X$ such that $C(X)$ is cofibrant. Let

$$C_n(X) = \{C(X)^0, C(X)^1, \ldots, C(X)^n, S^1 \wedge C(X)^n, \ldots\}$$

with the obvious structure maps, and we see that $C(X) = \bigcup_{n \geq 0} C_n(X)$. ∎

1.3 Homotopical algebra

{Sect3HA}

Categories like $s\mathcal{C}$ or $cs\mathcal{C}$ sometimes have structure like we are used to in homotopy theory in $\mathcal{T}op$. Technically speaking, they often form what is called a *simplicial closed model category*, see either one of [100], [54], [53] or [36].

In homotopy theory there are three important concepts: **fibrations, cofibrations** and **weak equivalences**. The important thing is to know how these concepts relate to each other: Consider the (solid) commuting diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ X & \longrightarrow & B \end{array}$$

where i is a cofibration and f is a fibration. If either i or f are weak equivalences, then there exists a (dotted) map $s\colon X \to E$ making the resulting diagram commutative. The map s will in general only be unique up to homotopy (there is a general rule in this game which says that "existence implies uniqueness", meaning that the existence property also can be used to prove that there is a homotopy between different liftings).

Note that there may be many meaningful choices of weak equivalences, fibrations and cofibration on a given category.

1.3.1 Examples

{A411}

1. **Spaces**. In $\mathcal{S}$ the weak equivalences are the maps $f\colon X \to Y$ which induce an isomorphism on homotopy groups $\pi_*(X, x) \to \pi_*(Y, f(x))$ for all $x \in X_0$. The cofibrations are simply the injective maps, and the fibrations are all maps which have the lifting property described above with respect to the cofibrations which are weak equivalences. These are classically called *Kan fibrations*.

   These notions also pass over to the subcategory $\mathcal{S}_*$ of pointed simplicial sets. The inclusion of the basepoint is always a cofibration (i.e. all spaces are *cofibrant*), but the projection onto a one point space is not necessarily a fibration (i.e. not all spaces are *fibrant*). The fibrant spaces are also called *Kan spaces* (or *Kan complex*). {A412}

2. **Topological spaces**. In $\mathcal{T}op$ and $\mathcal{T}op_*$, weak equivalences are still those which induce isomorphism on homotopy. The fibrations are the Serre fibrations, and the cofibrations are those which satisfy the lifting property with respect to the Serre fibrations which are weak equivalences. All topological spaces are fibrant, but not all are cofibrant. CW-complexes are cofibrant. Both the realization functor and the singular functor preserve weak equivalences, fibrations and cofibrations. {A413}

3. **Simplicial groups, rings, monoids, abelian groups**. In $s\mathcal{G}$, the category of simplicial groups, a map is a weak equivalence or a fibration if it is in $\mathcal{S}_*$, and the cofibrations are the maps which have the lifting property with respect to the

fibrations which are weak equivalences. Note that this is much more restrictive, than just requiring it to be a cofibration (inclusion) in $\mathcal{S}_*$ (the lifting is measured in different categories). However, if $X \to Y \in \mathcal{S}_*$ is a cofibration, then $F(X) \to F(Y)$ is also a cofibration, where $F\colon \mathcal{E}ns_* \to \mathcal{G}$ is the free functor, which sends a pointed set X to the free group on X modulo the basepoint.

Likewise in $\mathcal{A}$, the category of simplicial abeilan groups, and *sRing*, the category of simplicial rings.

{A414}

4. **Functor categories**. Let I be any small category, and let $[I, \mathcal{S}_*]$ be the category of functors from I to $\mathcal{S}_*$. This is a closed simplicial model category in the "pointwise" structure: a map $X \to Y$ (natural transformation) is a weak equivalence (resp. fibration) if $X(i) \to Y(i)$ is a weak equivalence (resp. fibration) of simplicial sets, and it is a cofibration if it has the left lifting property with respect to all maps that are both weak equivalences and fibrations. An important example is the pointwise structure on Γ-spaces (see chapter II.2.1.5), G-spaces (see below).

   Generally, it is the pointwise structure which is used for the construction of homotopy (co)limits (see section 1.9 below).

{A415}

5. **The pointwise structure on spectra**. A map $X \to Y$ of spectra is a pointwise equivalence (resp. fibration) if for every k it gives an equivalence (resp. pointwise fibration) $X^k \to Y^k$ of pointed simplicial sets. A map is a cofibration if it has the lifting property with respect to maps that are both pointwise fibrations and pointwise equivalences. A spectrum X is cofibrant if all the structure maps $S^1 \wedge X^k \to X^{k+1}$ are cofibrations (i.e., inclusions).

{A416}

6. **The stable structure on spectra**. A map $X \to Y$ of spectra is a stable equivalence if it induces an isomorphism on homotopy groups, and a (stable) cofibration if it is a cofibration in the pointwise structure. The map is a fibration if it has the lifting property with respect to maps that are both cofibrations and stable equivalences.

{A417}

7. **G-spaces and G-spectra**. Let G be a simplicial monoid. The category of G-spaces (see CNBNBref) is a closed simplicial model category with the following structure: a map is a G-equivalence (resp. G-fibration) if it is an equivalence (resp. fibration) of spaces, and a cofibration if it has the left lifting property with respect to all maps that are both G-equivalences and G-fibrations. Also, the category of *G-spectra* (see CNBNBref) has a pointwise and a stable structure giving closed simplicial model categories. Pointwise fibrations and pointwise equivalences (resp. stable fibrations and stable equivalences) er given by forgetting down to spectra, and pointwise (resp. stable) cofibrations are given by the left lifting property.

The examples 1.3.1.1–1.3.1.3 can be summarized as follows: Consider the diagram

$$\mathcal{T}op_* \underset{\sin}{\overset{|-|}{\leftrightarrows}} \mathcal{S}_* \underset{U}{\overset{F}{\rightleftarrows}} s\mathcal{G} \underset{U}{\overset{H_1(-)}{\rightleftarrows}} \mathcal{A} \underset{U}{\overset{T_{\mathbf{Z}}(-)}{\rightleftarrows}} sRing$$

where the U are forgetful functors, $T_{\mathbf{Z}}(A)$ the tensor ring on an abelian group, and $H_1(-) = -/[-,-]\colon \mathcal{G} \to \mathcal{A}b$ (applied degreewise). We know what weak equivalences in $\mathcal{T}op_*$ are, and we define them everywhere else to be the maps which are sent to weak equivalences in $\mathcal{T}op_*$. We know what cofibrations are in $\mathcal{S}_*$, and use the axiom to define the fibrations. We then define fibrations everywhere else to be the maps that are sent to fibrations in $\mathcal{S}_*$, and use the axioms to define the cofibrations.

The proof that 1.3.1.1–1.3.1.4 define closed simplicial model categories is contained in [100, II4], and the proof 1.3.1.5 and 1.3.1.6 is in [13]. None of these proofs state explicitly the functoriality of the factorizations of the axiom CM5 below, but for each of the cases it may be easily reconstructed from a "small object" kind of argument. For a discrete group G, the case of G-spaces is a special case of 1.3.1.4 but a direct proof in the general case is fairly straight forward, and the same proof works the pointwise structure on G-spectra. The proof for the stable structure then follows from the pointwise structure by the same proof as in [13] for the case $G = *$.

1.3.2 The axioms

{Def:CMC}

For convenience we list the axioms for a closed simplicial model category $\mathcal{C}$. It is a category enriched in $\mathcal{S}$, it is tensored and cotensored (see B.?? and B.??). We call the function spaces $\underline{\mathcal{C}}(-,-)$. Furthermore $\mathcal{C}$ has three classes of maps called *fibrations*, *cofibrations* and *weak equivalences* satisfying the following axioms

{CM1}

CM1 $\mathcal{C}$ is closed under finite limits and colimits.

{CM2}

CM2 (The saturation axiom) For two composable morphisms

$$b \xrightarrow{f} c \xrightarrow{g} d \in \mathcal{C},$$

if any two of f, g and gf are weak equivalences, then so is the third.

{CM3}

CM3 (Closed under retracts) If a map f is a retract of g (in the arrow category), and g is a weak equivalence, a fibration or a cofibration, then so is f.

{CM4}

CM4 Given a solid diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow \\ X & \longrightarrow & B \end{array}$$

where i is a cofibration and f is a fibration. If either i or f are weak equivalences, then there exists a (dotted) map $s\colon X \to E$ making the resulting diagram commutative.

{CM5}

CM5 (Functorial factorization axiom) Any map f may be functorially factored as $f = ip = jq$ where i is a cofibration, p a fibration and a weak equivalence, j a cofibration and a weak equivalence, and q a fibration.

{SM7}

SM7 If $i\colon A \rightarrowtail B$ is a cofibration and $p\colon X \to Y$ is a fibration, then the canonical map

$$\underline{\mathcal{C}}(B,X) \xrightarrow{(i^*,p_*)} \underline{\mathcal{C}}(A,X) \prod_{\underline{\mathcal{C}}(A,Y)} \underline{\mathcal{C}}(B,Y)$$

is a fibration of simplicial sets. If either i or p are weak equivalences, then (i^*, p_*) is also a weak equivalence.

An object X is a retract of Y if there are maps $X \to Y \to X$ whose composite is id_X. Note that the demand that the factorizations in CM5 should be functorial is not a part of Quillen's original setup, but is true in all examples we will encounter, and is sometimes extremely useful. Furthermore, with the exception of $\mathcal{S}$ and $\mathcal{T}op$, all our categories will be $\mathcal{S}_*$-categories, that is the function spaces have preferred basepoints.

1.3.3 The homotopy category

bsec:HoCat{A133}

It makes sense to talk about the homotopy category $Ho(\mathcal{C})$ of a closed simplicial model category $\mathcal{C}$. These are the categories where the weak equivalences are formally inverted (see e.g. [100]).

The realization and singular functor induce equivalences

$$Ho(\mathcal{S}_*) \simeq Ho(\mathcal{T}op_*)$$

This has the consequence that for all practical purposes we can choose whether we rather want to work with simplicial sets or topological spaces. Both categories have their drawbacks, and it is useful to know that all theorems which are proven for either homotopy category holds for the other.

1.4 Fibrations in $\mathcal{S}_*$

Let $f\colon E \to B \in \mathcal{S}_*$ be a fibration. We call $F = * \prod_B E$ the *fiber* of f. Recall that we get a long exact sequence

$$\cdots \to \pi_{q+1}E \to \pi_{q+1}B \to \pi_q F \to \pi_q E \to \pi_q B \to \dots.$$

The π_is are groups for $i > 0$ and abelian groups for $i > 1$, and $\pi_2 E$ maps into the center of $\pi_1 F$.

1.4.1 Actions on the fiber

{A141}

If π and G are groups and $\pi \to Aut(G)$ is a group homomorphism from π to the group os automorphisms on G we say that π *acts* on G. If $H \subset G$ is a normal subgroup we have an action $G \to Aut(H)$ via $g \mapsto \{h \mapsto g^{-1}hg\}$. In particular, any groups acts on itself in this fashion, and these automorphisms are called the *inner automorphisms*.

Let $f\colon E \twoheadrightarrow B$ be a fibration and assume B is fibrant. Let $i\colon F = E\prod_B * \subseteq E$ be the inclusion of the fiber. Then there are group actions

$$\pi_1 E \to Aut(\pi_* F)$$

and

$$\pi_1 B \to Aut(H_* F)$$

and the actions are compatible in the sense that the obvious diagram

$$\begin{array}{ccc} \pi_*F \times \pi_1 E & \longrightarrow & \pi_* F \\ \downarrow & & \downarrow \\ H_*F \times \pi_1 B & \longrightarrow & H_*F \end{array}$$

commutes. For future reference, we review the construction.

The spaces F, E, and B are fibrant, so function spaces into these spaces are homotopy invariant. For instance is $B^{S^1} = \underline{\mathcal{S}_*}(S^1, B)$ a model for the loops on B. We write X^I for the free path space $X^{\Delta[1]_+}$.

Consider the map $p\colon X \to F \times B^{S^1}$ defined by

$$\begin{array}{ccccccccccc} X & = & \lim_{\leftarrow}\{F & \xrightarrow{i} & E & \xleftarrow{d_0} & E^I & \xrightarrow{d_1} & E & \xleftarrow{i} & F\} \\ {\scriptstyle p}\downarrow & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle f^I}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow \\ F \times B^{S^1} & = & \lim_{\leftarrow}\{F & \xrightarrow{*} & B & \xleftarrow{d_0} & B^I & \xrightarrow{d_1} & B & \longleftarrow & *\} \end{array}$$

We see that p is both a fibration and a weak equivalence.

Hence there exists splittings $F \times B^{S^1} \to X$, unique up to homotopy, which by adjointness give rise to a homotopy class of maps $B^{S^1} \to Hom(F, X)$. Via the projection onto the last factor

$$X = F \times_E E^I \times_E F \xrightarrow{pr_3} F,$$

this gives rise to a homotopy class of maps $B^{S^1} \to Hom(F, F)$. For every such we have a commuting diagram

$$\begin{array}{ccc} B^{S^1} & \longrightarrow & Hom(F, F) \\ \downarrow & & \downarrow \\ \pi_0 B^{S^1} = \pi_1 B & \longrightarrow & \pi_0 Hom(F, F) \to End(H_*(F)) \end{array}$$

and the lower map does not depend on the choice of the upper map. As F is fibrant $\pi_0 Hom(F, F)$ is the monoid of homotopy classes of unbased self maps. Any homotopy class of unbased self maps defines an element in $End(H_*(F))$, and the map from $\pi_1 B$ is a monoid map, and giving rise to the desired group action $\pi_1 B \to Aut(H_*(F))$.

For the pointed situation, consider the (solid) diagram

$$\begin{array}{ccccc} F \vee E^{S^1} & \xrightarrow{in_1+j} & X & \xrightarrow{pr_3} & F \\ \downarrow & \nearrow & \downarrow p & & \\ F \times E^{S^1} & \longrightarrow & F \times B^{S^1} & & \end{array} \qquad (1.4.1.0)$$ {eq:A1410}

where $in_1 \colon F \to X = F \times_E E^I \times_E F$ is inclusion of the first factor, and j is the inclusion $E^{S^1} = * \times_E E^I \times_E * \subseteq F \times_E E^I \times_E F = X$. Again there is a homotopy class of liftings, and since the top row in the diagram is trivial, the composites

$$F \times E^{S^1} \to X \xrightarrow{pr_3} F$$

all factor through $F \wedge E^{S^1}$. So, this time the adjoints are pointed: $E^{S^1} \to Hom_*(F, F)$, giving rise to a unique

$$\pi_1 E = \pi_0 E^{S^1} \to \pi_0(Hom_*(F, F)) \to End(\pi_*(F)).$$

Again the map is a map of monoids, and so factors through the automorphisms, and we get the desired group action $\pi_1 E \to Aut(\pi_*(F))$, compatible with the homology operation.

1.4.2 Actions for maps of grouplike simplicial monoids

If $j \colon G \subseteq M$ is the inclusion of a subgroup in a monoid, then $j/1$ is the *over category* of j considered as a functor of categories. Explicitly, it has the elements of M as objects, and a map from m to m' is a $g \in G$ such that $m'g = m$.

We have an isomorphism $M \times G^{\times q} \xrightarrow{\cong} N_q(j/1)$ given by

$$(m, g_1, \dots, g_q) \mapsto m \xleftarrow{g_1} mg_1 \xleftarrow{g_2} \dots \xleftarrow{g_q} mg_1 g_2 \cdots g_q$$

and $B(M, G, *) = \{[q] \mapsto M \times G^{\times q}\}$ with the induced simplicial structure, is called the one-sided bar construction.

GinMaction}

Theorem 1.4.2.1 *Let M be a group-like simplicial monoid, and $j \colon G \subseteq M$ a (simplicial) subgroup. Then*

$$N(j/1) \to NG \to NM$$

is a fiber sequence, and the action

$$\Omega NG \times N(j/1) \to N(j/1) \in Ho(\mathcal{S}_*)$$

may be identified with the conjugate action

$$\begin{aligned} G \times N(j/1) &\to N(j/1) \\ (g, (m, g_1, \dots, g_q)) \in G_q \times N_q(j/1) &\mapsto (gmg^{-1}, gg_1g^{-1}, \dots gg_qg^{-1}) \end{aligned}$$

Proof: That the sequence is a fiber sequence follows from e.g. Waldhausen's theorem B' (NBNBref-can't we find a better reference?). As to the action, replace the fiber sequence with the equivalent fiber sequence

$$F \xrightarrow{i} E \xrightarrow{f} B$$

defined by $B = \sin|NM|$, and the pullback diagrams

$$\begin{array}{ccc} E & \longrightarrow & NG \\ \downarrow & & \downarrow \\ B^I & \xrightarrow{d_0} & B \end{array}, \text{ and } \begin{array}{ccc} F & \xrightarrow[i]{} & E \\ \downarrow & & \downarrow{\scriptstyle f} \\ * & \longrightarrow & B \end{array}$$

where f is the composite

$$E = NG \times_B B^I \xrightarrow{pr_2} B^I \xrightarrow{d_1} B\cdot$$

To describe the action, consider the diagram (the maps will be described below)

$$\begin{array}{ccccccc} F \vee G & \xrightarrow{\sim} & F \vee (NG)^{S^1} & \xrightarrow{\sim} & F \vee E^{S^1} & \xrightarrow{in_1 + j} & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ F \times G & \xrightarrow{\sim} & F \times (NG)^{S^1} & \xrightarrow{\sim} & F \times E^{S^1} & \longrightarrow & B^{S^1} \times F \end{array}$$

The rightmost square is the same as the lifting square in 1.4.1.0. The horizontal weak equivalences are induced by

$$NG \longrightarrow E = NG \times_B B^I, \qquad x \mapsto (x, f(x)) \text{ (the constant map at } f(x))$$

$$G \longrightarrow (NG)^{S^1} = \underline{\mathcal{S}_*}(S^1, NG), \text{ adjoint to the canonical inclusion } S^1 \wedge G \longrightarrow NG$$

By the uniqueness of liftings, any lifting $F \times G \to X$ is homotopic to $F \times G \xrightarrow{\sim} G \times E^{S^1}$ composed with a lifting $F \times E^{S^1} \to X$. Hence we may equally well consider lifings $F \times G \to X$. We will now proceed to construct such a lifting by hand, and then show that the constructed lifting corresponds to the conjugate action.

We define a map $NG \times G \times \Delta[1] \to (NG)$ by sending $(x, g, \phi) = (g, (x_1, \dots, x_q), \phi) \in G_q \times N_qG \times \Delta([q], [1])$ to

$$H^g(x)(\phi) = (g^{\phi(0)} x_1 g^{-\phi(1)}, g^{\phi(1)} x_2 g^{-\phi(2)}, \dots, g^{\phi(q-1)} x_q g^{-\phi(q)})$$

(where $g^0 = 1$ and $g^1 = g$). Note that, if 1 is the constant map $[q] \to [1]$ sending everything to 1, then $H^g(x)(1) = (gx_1g^{-1}, \dots, gx_qg^{-1})$. We let $H\colon NG \times G \to NG^I$ be the adjoint, and by the same formula we have a diagram

$$\begin{array}{ccc} NG \times G & \xrightarrow{H} & (NG)^I \\ \downarrow & & \downarrow \\ NM \times G & \xrightarrow{\bar{H}} & (NM)^I \\ \downarrow & & \downarrow \\ B \times G & \xrightarrow{\bar{H}} & B^I \end{array}$$

This extends to a map $E\times G \to E^I$ by sending $(x,\alpha)\in NG\times_B B^I = E$ to $g\mapsto H^g(x,\alpha) = (H^g(x),\bar{H}^g(\alpha))$. Since

$$G\subset B\times G \xrightarrow{\bar{H}} B^I \xrightarrow{d_i} B$$

is trivial for $i=0,1$ (and so, if $(x,\alpha)\in F$, we have $H^g(x,\alpha)(i) = (H^g(x)(i),\bar{H}^g(\alpha)(i))\in F$ for $i=0,1$), we get that, upon restricting to $F\times G$ this gives a lifting

$$F\times G\to F\times_E E^I\times_E F = X$$

Composing with

$$X = F\times_E E^I\times_E F \xrightarrow{pr_3} F$$

we have the "conjugate" action

$$F\times G\to F,\qquad (g,(x,\alpha))\mapsto c^g(x,\alpha) = H^g(x,\alpha)(1) = (H^g(x)(1),\bar{H}^g(\alpha)(1))$$

is equivalent to the action of G on the fiber in the fiber sequence of the statement of the theorem.

Let $C = \sin|N(M/1)|\times_B B^I$ and $\tilde{F} = C\times_B E$. Since C is contractible, $F\xrightarrow{\sim}\tilde{F}$ is an equivalence. We define a conjugate action on C using the same formulas, such that $C\to B$ is a G-map, and this defines an action on $\tilde{F}$ such that

$$\begin{array}{ccc} F\times G & \longrightarrow & F \\ \downarrow & & \downarrow \\ \tilde{F}\times G & \longrightarrow & \tilde{F} \\ \uparrow & & \uparrow \\ N(j/1)\times G & \longrightarrow & N(j/1) \end{array}$$

commutes, where the lower map is the action in the theorem. As the vertical maps are equivalences by the first part of the theorem, this proves the result. ∎

1.5 Bisimplicial sets

A *bisimplicial set* is a simplicial object in $\mathcal{S}$, that is, a simplicial space. From the functors

$$\Delta \xrightarrow{diag} \Delta\times\Delta \underset{pr_2}{\overset{pr_1}{\longrightarrow}} \Delta$$

we get functors

$$\mathcal{S} \xleftarrow{diag^*} s\mathcal{S} \longleftarrow \mathcal{S}$$

where the leftmost is called the diagonal and sends X to $diag^*(X) = \{[q]\mapsto X_{q,q}\}$, and the two maps to the right reinterpret a simplicial space X as a bisimplicial set by letting it be constant in one direction (e.g. $pr_1^*(X) = \{[p],[q]\mapsto X_p\}$).

There are important criteria for when information about each X_p may be sufficient to conclude something about $diag^*X$. We cite some useful facts. Proofs may be found either in the appendix of [13] or in [36].

eewise equ}

Theorem 1.5.0.2 *Let $X \to Y$ be a map of simplicial spaces inducing a weak equivalence $X_q \xrightarrow{\sim} Y_q$ for every $q \geq 0$. Then $diag^*X \to diag^*Y$ is a weak equivalence.*

Definition 1.5.0.3 (The π_*-Kan condition, [13]) Let

$$X = \{[q] \mapsto X_q\} = \{[p], [q] \mapsto X_{p,q}\}$$

be a simplicial space. For $a \in X_q$, consider the maps

$$d_i \colon \pi_p(X_q, a) \to \pi_p(X_{q-1}, d_i a), \qquad 0 \leq i \leq q$$

We say that X satisfies the π_*-Kan condition at $a \in X_q$ if for every tuple of elements

$$(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots x_q) \in \prod_{\substack{0 \leq i \leq m \\ i \neq k}} \pi_p(X_{q-1}, d_i a)$$

such that $d_i x_j = d_{j-1} x_i$ for $k \neq i < j \neq k$, there is an

$$x \in \pi_p(X_q, a)$$

such that $d_i x = x_i$ for $i \neq k$.

For an alternative description of the π_*-Kan condition see [36].

Examples of simplicial spaces which satisfies the π_*-Kan condition are bisimplicial groups and simplicial spaces $\{[q] \mapsto X_q\}$ where each X_q is connected, see [13].

Recall that a square is (homotopy) catesian if it is equivalent to a categorically cartesian square of fibrations.

{theo:A1B

Theorem 1.5.0.4 *[[13]] Let*

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

be a commutative diagram of simplicial spaces, such that

$$\begin{array}{ccc} V_p & \longrightarrow & X_p \\ \downarrow & & \downarrow \\ W_p & \longrightarrow & Y_p \end{array}$$

is homotopy cartesian for every p. If X and Y satisfy the π_*-Kan condition and if $\{[q] \mapsto \pi_0(X_q)\} \to \{[q] \mapsto \pi_0(Y_q)\}$ is a fibration, then

$$\begin{array}{ccc} diag^*V & \longrightarrow & diag^*X \\ \downarrow & & \downarrow \\ diag^*W & \longrightarrow & diag^*Y \end{array}$$

is homotopy cartesian.

As an immediate corollary we have the important result that loops can often be calculated degreewise. Recall that if X is a space, then the loop space of Y is $\Omega X = \underline{\mathcal{S}_*}(S^1, \sin|X|)$.

gwise loop}

Corollary 1.5.0.5 *Let X be a simplicial space such that X_p is connected for every $p \geq 0$. Then there is a natural equivalence between $\Omega diag^* X$ and $diag^*\{[p] \mapsto \Omega X_p\}$.*

Proof: Let $Y_p = \sin|X_p|$, and consider the homotopy cartesian square

$$\begin{array}{ccc} \underline{\mathcal{S}_*}(S^1, Y_p) & \longrightarrow & \underline{\mathcal{S}_*}(\Delta[1], Y_p) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y_p \end{array}.$$

Now, since each Y_p is connected, this diagram satisfies the conditions of theorem 1.5.0.4, and so

$$\begin{array}{ccc} diag^*\{[p] \mapsto \underline{\mathcal{S}_*}(S^1, Y_p)\} & \longrightarrow & diag^*\{[p] \mapsto \underline{\mathcal{S}_*}(\Delta[1], Y_p)\} \\ \downarrow & & \downarrow \\ * & \longrightarrow & diag^*\{[p] \mapsto Y_p\} \end{array}$$

is homotopy cartesian. Since $\underline{\mathcal{S}_*}(\Delta[1], Y_p)$ is contractible for every p we get a natural equivalence between $\Omega diag^* X \simeq \Omega diag^*\{[p] \mapsto Y_p\}$ and $diag^*\{[p] \mapsto \Omega X_p\} = diag^*\{[p] \mapsto \underline{\mathcal{S}_*}(S^1, Y_p)\}$. ∎

heo:A1BFSS}

Theorem 1.5.0.6 *[[13]] Let X be a pointed simplicial space satisfying the π_*-Kan condition. Then there is a first quadrant convergent spectral sequence*

$$E^2_{pq} = \pi_p([n] \mapsto \pi_q(X_n)) \Rightarrow \pi_{p+q}(diag^* X)$$

As an application we prove two corollaries

Corollary 1.5.0.7 *Let G be a simplicial group, and let BG be the diagonal of $[n] \mapsto BG_n$. Then*

$$\pi_q BG \cong \pi_{q-1} G$$

Proof: Note that BG_n is connected for each n, and so BG satisfies the π_*-Kan condition. Now

$$E^2_{pq} = \pi_p([n] \mapsto \pi_q(BG_n)) = \begin{cases} 0 & \text{if } q \neq 1 \\ \pi_p G & \text{if } q = 1 \end{cases}$$

and the result follows. ∎

Corollary 1.5.0.8 *Let X be a simplicial space. Then there is a convergent spectral sequence*

$$E^2_{pq} = H_p([n] \mapsto H_q(X_n)) \Rightarrow H_{p+q}(diag^* X)$$

Proof: Apply the spectral sequence of the theorem to the bisimplicial abelian group $\mathbf{Z}X$. ∎

1.5.0.9 Linear simplicial spaces

ial object} **Definition 1.5.0.10** A simplicial object X in a model category is *linear* if the natural maps

$$X_p \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are weak equivalences, where the i'th component is induced from $[1] \cong \{i-1, i\} \subseteq [p]$ for $0 < i \leq p$.

This is inspired by categories where a space X is a nerve of a category exactly if the said map is an isomorphism. A slicker way of formulating this is to say that X is linear if it takes the pushouts of monomorphisms that exist in Δ to homotopy pullbacks.

Note that if $X_0 = *$, this gives a "weak multiplication" on X_1:

$$X_1 \times X_1 \underset{\sim}{\xleftarrow{(d_0,d_2)}} X_2 \xrightarrow{d_1} X_1.$$

Saying that this weak multiplication has a *homotopy inverse* is the same as saying that all the diagrams

$$\begin{array}{ccc} X_p & \longrightarrow & X_1 \\ {\scriptstyle d_p}\downarrow & & \downarrow \\ X_{p-1} & \longrightarrow & * \end{array}$$

are homotopy cartesian, where the top map is induced by $[1] \cong \{0, p\} \subseteq [p]$ (this formulation essentially claims that you must have inverses in a monoid to be able to uniquely produce g and h such that $gh = m$, if the only thing you know is g and m).

The following proposition is proved in [109, page 296] and is used several times in the text. The natural map in question is gotten as follows: you always have a map of simplicial spaces $\Delta[1] \times X_1 \to X$, but if $X_0 = *$ you may collaps the endpoints and get a pointed map $S^1 \wedge X_1 \to X$. Take the diagonal, and consider the adjoint map $X_1 \to \underline{\mathcal{S}_*}(S^1, diag^* X)$, which we map map further to $\Omega X = \underline{\mathcal{S}_*}(S^1, \sin|X|)$.

{A1Segal1

Proposition 1.5.0.11 *Let X be a linear simplicial space with $X_0 = *$. Then the natural map*

$$X_1 \to \Omega X$$

is a weak equivalence if and only if the induced weak multiplication on X_1 has a homotopy inverse.

Proof: Since X is linear, we have that for each q the square

$$\begin{array}{ccc} X_1 & \longrightarrow & P_q X \\ \downarrow & & \downarrow \\ X_0 = * & \longrightarrow & X_q \end{array}$$

is homotopy cartesian. That X_1 has a homotopy inverse implies that X and PX satisfy the π_*-Kan condition, and that $\{[q] \mapsto \pi_0(P_q X)\} \to \{[q] \mapsto \pi_0(X_q)\}$ is isomorphic to the classifying fibration $E(\pi_0 X_1) = B(\pi_0 X_1, \pi_0 X_1, *) \twoheadrightarrow B(\pi_0 X_1)$ (which is a fibration since $\pi_0 X_1$ is a group). Hence theorem 1.5.0.4 gives that

$$\begin{array}{ccc} X_1 & \longrightarrow & diag^* PX \\ \downarrow & & \downarrow \\ * & \longrightarrow & diag^* X \end{array}$$

is homotopy cartesian, and the result follows by the contractibility of PX. ∎

Applying this proposition to the bar construction of a group-like simplicial monoid, we get:

of gplike}

Corollary 1.5.0.12 *Let M be a group-like simplicial monoid. Then the natural map $M \to \Omega BM$ is a weak equivalence.*

1.6 The plus construction

sec:A1plus}

1.6.1 Acyclic maps

Recall from I.1.6.2 that a map of pointed connected spaces is called *acyclic* if the integral homology of the homotopy fiber vanishes. We need some facts about acyclic maps.

If Y is a connected space, we may form it's *universal cover* $\tilde{Y}$ as follows. From $\sin|Y|$, form the space B by identifying two simplices $u, v \in \sin|Y|_q$ whenever, considered as maps $\Delta[q] \to \sin|Y|$, they agree on the one-skeleton of $\Delta[q]$. Then $\sin|Y| \twoheadrightarrow B$ is a fibration of fibrant spaces [82, 8.2], and $\tilde{Y}$ is defined by the pullback diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & B^{\Delta[1]} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

and we note that $\tilde{Y} \to Y$ is a fibration with fiber equivalent to the discrete set $\pi_1 Y$.

Lemma 1.6.1.1 *Let $f\colon X \to Y$ be a map of connected spaces, and $\tilde{Y}$ the universal cover of Y. Then f is acyclic if and only if*

$$H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y})$$

is an isomorphism.

Proof: We may assume that $X \to Y$ is a fibration with fiber F. Then $X \times_Y \tilde{Y} \to \tilde{Y}$ also is a fibration with fiber F, and the Serre spectral sequence

$$H_p(\tilde{Y}; H_q(F)) \Rightarrow H_{p+q}(X \times_Y \tilde{Y})$$

gives that if $\tilde{H}_*(F) = 0$, then the edge homomorphism (which is induced by $X \times_Y \tilde{Y} \to \tilde{Y}$) is an isomorphism as claimed.

Conversely, if $H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y})$ is an isomorphism. Then it is easy to check directly that $\tilde{H}_q(F) = 0$ for $q \leq 1$. Assume we have shown that $\tilde{H}_q(F) = 0$ for $q < k$ for a $k \geq 2$. Then the spectral sequence gives an exact sequence

$$H_{k+1}(X \times_Y \tilde{Y}) \xrightarrow{\simeq} H_{k+1}(\tilde{Y}) \longrightarrow H_k(F) \longrightarrow H_k(X \times_Y \tilde{Y}) \xrightarrow{\simeq} H_k(\tilde{Y}) \longrightarrow 0$$

which implies that $H_k(F) = 0$ as well. ∎

The lemma can be reformulated using homology with local coefficients: $H_*(\tilde{Y}) = H_*(Y; \mathbf{Z}[\pi_1 Y])$ and $H_*(X \times_Y \tilde{Y}) \cong H_*(\mathbf{Z}[\tilde{X}] \otimes_{\mathbf{Z}[\pi_1 X]} \mathbf{Z}[\pi_1 Y]) = H_*(X; f^*\mathbf{Z}[\pi_1 Y])$, so f is acyclic if and only if it induces an isomorphism

$$H_*(X; f^*\mathbf{Z}[\pi_1 Y]) \cong H_*(Y; \mathbf{Z}[\pi_1 Y])$$

This can be stated in more general coefficients:

{cor:acyl

Corollary 1.6.1.2 *A map $f\colon X \to Y$ of connected spaces is acyclic if and only if it for any local coefficient system $\mathcal{G}$ on Y, f induces an isomorphism*

$$H_*(X; f^*\mathcal{G}) \cong H_*(Y; \mathcal{G})$$

Proof: By the lemma we only need to verify one implication. If $i\colon F \to Y$ is the fiber of f, the Serre spectral sequence give

$$H_p(Y; H_q(F; i^* f^* \mathcal{G})) \Rightarrow H_{p+q}(X; f^*\mathcal{G})$$

However, $i^* f^* \mathcal{G}$ is a trivial coefficient system, so if $\tilde{H}_*(F) = 0$, the edge homomorphism must be an isomorphism. ∎

This reformulation of acyclicity is useful, for instance when proving the following lemma.

{lem:push

Lemma 1.6.1.3 *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow{f'} & S \end{array}$$

be a pushout cube of connected spaces with f acyclic, and either f or g a cofibration. Then f' is acyclic.

Proof: Let $\mathcal{G}$ be a local coefficient system in S. Using the characterization of acyclic maps as maps inducing isomorphism in homology with arbitrary coefficients, we get by excision that

$$H_*(S, Z; \mathcal{G}) \cong H_*(Y, X; (g')^*\mathcal{G}) = 0$$

implying that f' is acyclic. ∎

{lem:weac

Lemma 1.6.1.4 *Let $f\colon X \to Y$ be a map of connected spaces. Then f is a weak equivalence if and only if it is acyclic and induces an isomorphism of the fundamental groups.*

Proof: Let F be the homotopy fiber of f. If f induces an isomorphism $\pi_1 X \cong \pi_1 Y$ on fundamental groups, then $\pi_1 F$ is abelian. If f is acyclic, then $\pi_1 F$ is perfect. Only the trivial group is both abelian and perfect, so $\pi_1 F = 0$. As $\tilde{H}_* F = 0$ the Whitehead theorem tells us that $|F|$ is contractible. ∎

1.6.2 The construction

We now give a functorial construction of the plus construction, following the approach of [14, p. 218].

If X is any set, we may consider the free abelian group generated by X, and call it $\mathbf{Z}[X]$. If X is pointed we let $\tilde{\mathbf{Z}}[X] = \mathbf{Z}[X]/\mathbf{Z}[*]$. This defines a functor $\mathcal{E}ns_* \to \mathcal{A}b$ which is adjoint to the forgetful functor $U\colon \mathcal{A}b \to \mathcal{E}ns_*$, and extends degreewise to all spaces. The transformation given by the inclusion of the generators $X \to \tilde{\mathbf{Z}}[X]$ (where we symptomatically have forgotten to write the forgetful functor) induces the Hurewicz homomorphism $\pi_*(X) \to \pi_*(\tilde{\mathbf{Z}}[X]) = \tilde{H}_*(X)$.

As $\tilde{\mathbf{Z}}$ is a left adjoint functor, as explained in section 0.3 it gives rise to a cosimplicial space $\underset{\sim}{\mathbf{Z}}$ via

$$\underset{\sim}{\mathbf{Z}}[X] = \{[n] \mapsto \tilde{\mathbf{Z}}^{n+1}[X]\}$$

where the superscript $n+1$ means that we have used the functor $\tilde{\mathbf{Z}}$ $n+1$ times. The total (see section 1.8) of this cosimplicial space is called the *integral completion* of X and is denoted $\mathbf{Z}_\infty X$.

If Y is a pointed set, consider the category U/Y of functions $A \to Y$ from abelian groups to Y. The forgetful functor $U/Y \to \mathcal{E}ns_*/Y$, has a left adjoint $\dot{\mathbf{Z}}$ given by sending $f\colon X \to Y$ to $\dot{\mathbf{Z}}[X] \to Y$ where

$$\dot{\mathbf{Z}}[X] = \{\sum_{1 \le i \le n} n_i x_i \in \tilde{\mathbf{Z}}[X] | f(x_1) = \cdots = f(x_n)\}$$

and where we map $\sum n_i x_i$ to the common $f(x_i)$. Again we extend to simplicial sets. So, if $X \to Y$ is any map of spaces, there is a cosimplicial subspace of $\underset{\sim}{\mathbf{Z}}[X]$, whose total is called the *fiberwise integral completion* of X. The construction is natural in f.

If X is a space, there is a natural fibration $\sin|X| \to \sin|X|/P$ given by "killing, in each component, $\pi_i(X)$ for $i > 1$ and the maximal perfect subgroup $P\pi_1(X) \subseteq \pi_1(X)$". More precisely, let $\sin|X|/P$ be the space obtained from $\sin|X|$ by identifying two simplices $u, v \in \sin|X|_q$ whenever, for every injective map $\phi \in \Delta([1],[q])$, we have $d_i\phi^* u = d_i \phi^* v$ for $i = 0, 1$, and

$$[\phi^* u]^{-1} * [\phi^* v] = 0 \in \pi_1(X, d_0\phi^* u)/P\pi_1(X, d_0 \phi^* u)$$

The projection $\sin|Y| \to \sin|X|/P$ is a fibration.

Definition 1.6.2.1 The plus construction $X \mapsto X^+$ is the functor given by the fiberwise integral completion of $\sin|X| \to \sin|X|/P$, and $q_X\colon X \to X^+$ is the natural transformation coming from the inclusion $X \subseteq \dot{\mathbf{Z}}[\sin|X|]$.

That this is the desired definition follows from [14, p. 219], where they use the alternative description of corollary 1.6.1.2 for an acyclic map:

Proposition 1.6.2.2 *If X is a pointed connected space, then*

$$q_X\colon X \to X^+$$

is an acyclic map killing the maximal perfect subgroup of the fundamental group.

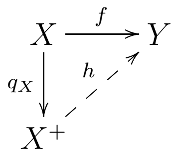
We note that q_X is always a cofibration (=inclusion).

1.6.3 Uniqueness of the plus construction

Now, Quillen provides the theorem we need: X^+ is characterized up to homotopy under X by this property

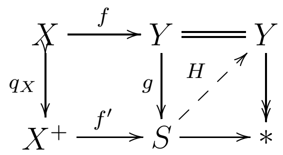
{theo:A1p

Theorem 1.6.3.1 *Consider the (solid) diagram of connected spaces*



If Y is fibrant and $P\pi_1 X \subseteq ker\{\pi_1 X \to \pi_1 Y\}$ then there exists a dotted map h making the resulting diagram commutative. Furthermore, the map is unique up to homotopy, and is a weak equivalence if f is acyclic.

Proof: Let $S = X^+ \coprod_X Y$ and consider the solid diagram



By lemma 1.6.1.3, we know that g is acyclic. The van Kampen theorem tells us that $\pi_1 S$ is the "free product" $\pi_1 X^+ *_{\pi_1 X} \pi_1 Y$, and the hypothesis imply that $\pi_1 Y \to \pi_1 S$ must be an isomorphism.

By lemma 1.6.1.4, this means that g is a weak equivalence. Furthermore, as q_X is a cofibration, so is g. Thus, as Y is fibrant, there exist a dotted H making the diagram commutative, and we may choose $h = Hf'$. By the universal property of S, any h must factor through f', and the uniqueness follows by the uniqueness of H.

If f is acyclic, then both $f = hq_X$ and q_X are acyclic, and so h must be acyclic. Furthermore, as f is acyclic $ker\{\pi_1 X \to \pi_1 Y\}$ must be perfect, but as $P\pi_1 X \subseteq ker\{\pi_1 X \to \pi_1 Y\}$ we must have $P\pi_1 X = ker\{\pi_1 X \to \pi_1 Y\}$. So, h is acyclic and induces an isomorphism on the fundamental group, and by 1.6.1.4 h is an equivalence. ∎

Recall that a space X is 0-connected if $\pi_0 X$ is a point, and if it is connected it is *k-connected* for a $k > 0$ if for all vertices $x \in X_0$ we have that $\pi_q(X, x) = 0$ for $0 \leq q \leq k$. A space is -1-connected by definition if it is nonempty. A map $X \to Y$ is *k-connected* if its homotopy fiber is $(k-1)$-connected.

{lemma:A1

Lemma 1.6.3.2 *Let $X \to Y$ be a k-connected map of connected spaces. Then $X^+ \to Y^+$ is also k-connected*

Proof: Either one uses the characterization of acyclic maps by homology with local coefficients, and check by hand that the lemma is right in low dimensions, or one can use our choice of construction and refer it away: [14, p. 113 and p. 42]. ∎

1.6.4 Spaces under BA_5

A1underBA5}

Let A_n be the alternating group on n letters. For $n \geq 5$ this is a perfect group with no nontrivial subgroups. We give a description of Quillen's plus for BA_5 by adding cells. Since A_5 is perfect, it is enough to display a map $BA_5 \to Y$ inducing an isomorphism in integral homology, where Y is simply connected.

Let $\alpha \neq I \in A_5$. This can be thought of as a map $S^1 = \Delta[1]/\partial\Delta[1] \to BA_5$ (consider α as an element in B_1A_5, since $B_0A_5 = *$ this is a loop). Form the pushout

$$\begin{array}{ccc} |S^1| & \longrightarrow & D^2 \\ {\scriptstyle |\alpha|}\downarrow & & \downarrow \\ |BA_5| & \longrightarrow & X_1 \end{array}$$

Since A_5 has no nontrivial normal subgroups, the van Kampen theorem tells us that X_1 is simply connected. The homology sequence of the pushout splits up into

$$0 \to H_2(A_5) \to H_2(X_1) \to H_1(S^1) \to 0, \qquad \text{and} \qquad H_q(A_5) \cong H_q(X_1), \text{ for } q \neq 1$$

Since $H_1(S^1) \cong \mathbf{Z}$, we may choose a splitting $\mathbf{Z} \to H_2(X_1) \cong \pi_2(X_1)$, and we let $\beta\colon |S^2| \to X_1$ represent the image of a generator of $\mathbf{Z}$. Form the pushout

$$\begin{array}{ccc} |S^2| & \longrightarrow & D^3 \\ {\scriptstyle \beta}\downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

We get isomorphisms $H_q(X_1) \cong H_q(X_2)$ for $q \neq 2, 3$, and an exact diagram

$$\begin{array}{ccccccccccc} & & & & & & H_2(A_5) & & & & \\ & & & & & & \downarrow & & & & \\ 0 & \longrightarrow & H_3(X_1) & \longrightarrow & H_3(X_2) & \longrightarrow & H_2(S^2) \longrightarrow H_2(X_1) & \longrightarrow & H_2(X_2) & \longrightarrow & 0 \\ & & & & & & \downarrow & & & & \\ & & & & & & H_1(S^1) & & & & \end{array}$$

But by the definition of β, the composite $H_2(S^2) \to H_2(X_1) \to H_1(S^1)$ is an isomorphism. Hence $H_3(X_1) \cong H_3(X_2)$ and $H_2(A_5) \cong H_2(X_2)$. Collecting what we have gotten, we get that the map $|BA_5| \to X_2$ is an isomorphism in homology and $\pi_1 X_2 = 0$, and $BA_5 \to$ "BA_5^+" $= \sin X_2$ is a model for the plus construction.

Proposition 1.6.4.1 *Let $\mathcal{C}$ be the category of spaces under BA_5 with the property that if $BA_5 \to Y \in ob\mathcal{C}$ then the image of A_5 normally generates $P\pi_1 Y$. Then the bottom arrow in the pushout diagram*

$$\begin{array}{ccc} BA_5 & \longrightarrow & \text{“}BA_5^+\text{”} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \text{“}Y^+\text{”} \end{array}$$

is a functorial model for the plus construction in $\mathcal{C}$.

Proof: As it is clearly functorial, we only have to check the homotopy properties of $Y \to$ “Y^+”. By lemma 1.6.1.3, it is acyclic, and by van Kampen $\pi_1(\text{“}Y^+\text{”}) = \pi_1 Y *_{A_5} \{1\}$. Using that the image of A_5 normally generate $P\pi_1 Y$ we get that $\pi_1(\text{“}Y^+\text{”}) = \pi_1 Y / P\pi_1 Y$, and we are done. ∎

Example 1.6.4.2 If R is some ring, we get a map

$$A_5 \subseteq \Sigma_5 \subseteq \Sigma_\infty \subseteq GL(\mathbf{Z}) \to GL(R)$$

We will show that $E(R)$ is normally generated by

$$\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in A_3 \subseteq A_5$$

In view of the proof in lemma IIINBNBref (I.1.6.2) that e_{41}^1 normally generate $E(R)$, this follows from the identity

$$e_{41}^1 = [[\alpha, e_{43}^{-1}], e_{21}^{-1}]$$

Hence, all spaces under BA_5, where the map on fundamental groups is the above map $A_5 \to GL(R)$ lie in $\mathcal{C}$. In particular in the language of chapter II $\widehat{GL}(A)$ is in this class for any $\mathbf{S}$-algebra A.

1.7 Simplicial abelian groups and chain complexes

1.7.0.3 Simplicial abelian groups, chain complexes and loop groups

The correspondence between chain complexes and simplicial abelian groups is well known, but we need some details pertaining to the various models for loops not usually described in the standard texts.

1.7.0.4 Chain complexes in general

As usual, a chain complex is a sequence

$$C_* = \{\cdots \leftarrow C_{q-1} \leftarrow C_q \leftarrow C_{q+1} \leftarrow \dots\}$$

such that any composite is zero, and a map of chain complexes $f_*\colon C_* \to D_*$ is a collection of maps $f_q\colon C_q \to D_q$ such that the diagrams

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ \downarrow & & \downarrow \\ C_q & \xrightarrow{f_{q-1}} & D_q \end{array}$$

commute. We let Ch be the category of chain complexes, and $Ch^{\geq 0}$ be the full subcategory of chain complexes C_* such that $C_q = 0$ if $q < 0$.

If C_* is a chain complex, we let $Z_qC = \ker\{C_q \to C_{q-1}\}$ (cycles), $B_qC = \operatorname{im}\{C_{q+1} \to C_q\}$ (boundaries) and $H_qC_* = Z_qC/B_qC$ (homology).

1.7.0.5 Truncations and shifts

Let C_* be a chain complex and k an integer. Then $C_*[k]$ is the shifted chain complex, i.e. $C_q[k] = C_{q+k}$ and the maps are moved accordingly. There are two functors $Ch \to Ch^{\geq 0}$. The first simply truncates: if C_* is a chain complex, then $t_+(C_*)$ is the chain complex you get by setting the groups in negative dimension to zero

$$t_+(C_*) = \{\cdots = 0 = 0 \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \dots\}$$

Then we get that

$$H_q(t_+(C_*)) = \begin{cases} H_q(C_*) & \text{if } q > 0 \\ C_0/B_0 & \text{if } q = 0 \\ 0 & \text{if } q < 0 \end{cases}$$

To remove the noise in dimension zero we have the gentler truncation $C_*[0,\infty)$ given by

$$C_*[0,\infty) = \{\cdots = 0 = 0 \leftarrow Z_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \dots\}$$

so that

$$H_q(C_*[0,\infty)) = \begin{cases} H_q(C_*) & \text{if } q \geq 0 \\ 0 & \text{if } q < 0 \end{cases}$$

Note that $C_*[0,\infty) \subseteq C_* \twoheadrightarrow t_+(C_*)$.

1.7.0.6 The algebraic mapping cone and homotopy pullback

Let $f\colon A_* \to C_*$ be a map of chain complexes. Then the mapping cone $C(f)_*$ is the chain complex with $C(f)_q = A_q \oplus C_{q+1}$ and boundary map $A_q \oplus C_{q+1} \to A_{q-1} \oplus C_q$ given by sending (a,c) to $(da + (-1)^q fc, dc)$. This gives a short exact sequence

$$0 \to C_*[1] \to C(f)_* \to A_* \to 0$$

and the boundary map $H_q(A_*) \to H_{q-1}(C_*[1]) \cong H_q(C_*)$ is f.

Note that $C(id_A)_*$ is contractible, and that $C(f)_*$ can be described as the pushout of chain complexes

$$\begin{array}{ccc} A_*[1] & \longrightarrow & C(id_A)_* \\ {\scriptstyle f[1]}\downarrow & & \downarrow \\ C_*[1] & \xrightarrow{\ i\ } & C(f)_* \end{array}$$

More generally, given a diagram $B_*@>f>>A_*@<g<<C_*$ we give a model for the homotopy pullback: $C(f,g)_*$ is the chain complex with

$$C(f,g)_q = B_q \oplus A_{q+1} \oplus C_q$$

and boundary map $B_q \oplus A_{q+1} \oplus C_q \to B_{q+1} \oplus A_q \oplus C_{q+1}$ given by sending (b,a,c) to $(db, (-1)^q fb + da + (-1)^q gc, gc)$. Then we have a short exact sequence

$$0 \to A_*[1] \to C(f,g)_* \to B_* \oplus C_* \to 0$$

1.7.0.7 The normalized chain complex

{normalize

The isomorphism between simplicial abelian groups and chain complexes concentrated in non-negative degrees is given by the *normalized chain complex*: If M is a simplicial abelian group, then $C_*^{\mathrm{norm}}(M)$ (which is usually called N_*M, an option unpalatable to us since this notation is already occupied by the nerve) is the chain complex given by

$$C_q^{\mathrm{norm}}(M) = \bigcap_{i=0}^{q-1} \ker\{d_i \colon M_q \to M_{q-1}\}$$

and boundary map $C_q^{\mathrm{norm}}M \to C_{q-1}^{\mathrm{norm}}M$ given by the remaining face map d_q. As commented earlier, this defines an isomorphism of categories.

1.7.0.8 The Moore complex

Associated to a simplicial abelian group M there is another chain complex, the Moore complex C_*M defined by $C_qM = M_q$ with boundary map given by the alternating sum $\delta = \sum_{j=0}^{q} (-1)^j d_j \colon M_q \to M_{q-1}$. The inclusion of the normalized complex into the Moore complex $C_*^{\mathrm{norm}}(M) \subseteq C_*(M)$ is a homotopy equivalence (see e.g., [**?**, 22.1]).

It should also be mentioned that the Moore complex is the direct sum of the normalized complex and the subcomplex generated by the images of the degeneracy maps. hence you will often see the normalized complex defined as the quotient of the Moore complex by the degenerate chains.

1.7.0.9 Various loop constructions

There are three natural loop constructions we need to consider, and we need to study their relations.

If M is a simplicial abelian group we may consider the simplicial abelian group

$$\mathcal{S}_*(S^1, M) \cong s\mathcal{A}b(\tilde{\mathbf{Z}}[S^1], M)$$

This fits into the short exact sequence

$$0 \to \mathcal{S}_*(S^1, M) \to \mathcal{S}_*(\Delta[1], M) \to M \to 0$$

and the middle group is contractible. We describe this group in a slightly different fashion using

Lemma 1.7.0.10 *There is a natural isomorphism of functors $\Delta^o \times \Delta \to \mathcal{E}ns$ between $\Delta([p],[1]) \wedge \Delta([p],[q])_+$ and $\Delta([p],[0] \sqcup [q])$.*

Proof: The isomorphism is gotten by sending $(\psi_1 \colon [p] \to [1], \psi_2 \colon [p] \to [q])$ to $\psi_1(\psi_2 + 1) \colon [p] \to [q+1]$. The inverse is gotten by sending $\phi \colon [p] \to [q+1]$ to $(\phi_1 \colon [p] \to [1], \phi_2 \colon [p] \to [q])$ where

$$\phi_1(j) = \begin{cases} 0, & \text{if } \psi(j) = 0 \\ 1, & \text{if } \psi(j) \neq 0 \end{cases} \qquad \text{and} \qquad \phi_2(j) = \begin{cases} 0, & \text{if } \psi(j) = 0 \\ \phi(j) - 1, & \text{if } \psi(j) \neq 0 \end{cases}$$

This is natural in $[p]$ and $[q]$ as one may check. ∎

If X is a simplicial object, we let PX be the simplicial object gotten by precomposition with

$$[0] \sqcup - \colon \Delta^o \to \Delta^o$$

(so that $(PX)_q = X_{q+1}$ and the face and degeneracy maps are shifted), and consider the map $PX \to X$ induced from the obvious inclusion $d^0 \colon [q] \to [0] \sqcup [q]$. If X is pointed we define LX as the pullback

$$\begin{array}{ccc} LX & \longrightarrow & PX \\ \downarrow & & \downarrow \\ & \longrightarrow & X \end{array}$$

Note that PX contains X_0 as a retract, and the nontrivial composition is homotopic to the identity. In our case this can be viewed as a corollary to the lemma above:

Corollary 1.7.0.11 *Let M be a simplicial abelian group, then the projection $\Delta[q+1]_+ \to \Delta[q+1]$ induces a (split) short exact sequence*

$$0 \to \mathcal{S}_*(\Delta[1], M) \to PM \to M_0 \to 0$$

If M is a simplicial abelian group, then

$$0 \to LM \to PM \to M \to 0$$

is a short exact sequence, and so we obtain yet a new short exact sequence

$$0 \to \mathcal{S}_*(S^1, M) \to LM \to M_0 \to 0$$

On the level of homotopy groups, the first exact sequences gives rise to the isomorphisms $\pi_q LM \cong \pi_{q+1} M$ for $q > 0$ and the exact sequence $0 \to \pi_1 M \to \pi_0 LM \to M_0 \to \pi_0 M \to 0$.

Lemma 1.7.0.12 *Let M be a simplicial abelian group. There are natural isomorphisms*

$$C_*^{norm}(LM) \cong t_+((C_*^{norm} M)[1])$$

and

$$C_*^{norm}(\mathcal{S}_*(S^1, M)) \cong ((C_*^{norm} M)[1])[0, \infty).$$

Proof: the first isomorphism follows from the definitions, and the second by the relationship between the loop spaces. ∎

Notice that these isomorphisms extend to maps of Moore complexes

$$\begin{array}{ccc} C_*\mathcal{S}_*(S^1, M) & \xrightarrow{\sim} & (C_*(M)[1])[0, \infty) \\ {\scriptstyle\subseteq}\big\downarrow & & {\scriptstyle\subseteq}\big\downarrow \\ C_* LM & \longrightarrow & t_+(C_*(M)[1]) \end{array}$$

and the upper map is an equivalence.

1.8 Cosimplicial spaces.

{A17}

Recall that a cosimplicial space is a functor

$$X : \Delta \to \mathcal{S}$$

The category of cosimplicial spaces is a simplical category: if Z is a space and X is a cosimplicial space, then $Z \times X$ is the cosimplicial space whose value on $[q] \in \Delta$ is $Z \times X^q$. The function space

$$c\mathcal{S}(X, Y) \in \mathcal{S}$$

of maps from the cosimplicial space X to the cosimplicial space Y has q-simplices the set of maps (natural transformation of functors $\Delta^o \times \Delta \to \mathcal{S}$)

$$\Delta[q] \times X \to Y$$

The *total space* of a cosimplicial space X is the space

$$\operatorname{Tot} X = c\mathcal{S}(\Delta[-], X)$$

(where $\Delta[-]$ is the cosimplicial space whose value on $[q] \in \Delta$ is $\Delta[q] = \Delta(-,[q])$). The q-simplices are cosimplicial maps $\Delta[q] \times \Delta[-] \to X$.

It helps me to keep my tongue straight in my mouth (but may simply confuse almost everybody else, so disregard this if it is not in your taste) that if $X, Y : I^o \times I \to \mathcal{E}ns$, then the "function space" is the functor $I^o \to \mathcal{E}ns$ sending $\alpha \in I^o$ to

$$\int_{i,j\in I} \mathcal{E}ns(I(i,\alpha) \times X(i,j), Y(i,j))$$

and the "total space" is the functor $I^o \to \mathcal{E}ns$ sending $\alpha \in I^o$ to

$$\int_{i,j\in I} \mathcal{E}ns(I(i,\alpha) \times I(i,j), Y(i,j))$$

1.8.1 The pointed case

In the pointed case we make the usual modifications: A pointed cosimplicial space is a functor $X : \Delta \to \mathcal{S}_*$, the function space $c\mathcal{S}_*(X,Y)$ has q-simplices the set of maps $\Delta[q]_+ \wedge X \to Y$, and the total space $\operatorname{Tot} X = c\mathcal{S}_*(\Delta[-]_+, X)$ is isomorphic (a an unbased space) to what you get if you forget the basepoint before taking Tot.

1.9 Homotopy limits and colimits.

{A18}

Let I be a small category, and $[I, \mathcal{S}_*]$ the category of functors from I to $\mathcal{S}_*$. This is a simplicial category in the sense that we have function spaces and "tensors" with pointed simplicial sets satisfying the usual properties. If $F, G \in [I, \mathcal{S}_*]$ we define the function space to be the simplicial set $\underline{I\mathcal{S}_*}(F,G)$ whose q simplices are

$$\underline{I\mathcal{S}_*}(F,G)_q = [I,\mathcal{S}_*](F \wedge \Delta[q]_+, G)$$

i.e. the set of all pointed natural transformations $F(i) \wedge \Delta[q]_+ \to G(i)$, and whose simplicial structure comes from regarding $[q] \mapsto \Delta[q]$ as a cosimplicial object. If $F \in [I^o, \mathcal{S}_*]$ and $G \in [I, \mathcal{S}_*]$ we define

$$F \wedge G \in \mathcal{S}_*$$

to be the colimit of

$$\bigvee_{\gamma : i \to j \in I} F(j) \wedge G(i) \rightrightarrows \bigvee_{i \in I} F(i) \wedge G(i)$$

where the upper map sends the γ summand to the j summand via $1 \wedge G\gamma$, and the lower map sends the γ summand to the i summand via $F\gamma \wedge 1$ (in other words: it is the coend $\int^I F \wedge G$).

If $F \in [I^o, \mathcal{S}_*]$, $G \in [I, \mathcal{S}_*]$ and $X \in \mathcal{S}_*$, we get that

$$\underline{\mathcal{S}_*}(F \wedge G, X) \cong \underline{I^o\mathcal{S}_*}(F, \underline{\mathcal{S}_*}(G,X)) \cong \underline{I\mathcal{S}_*}(G, \underline{\mathcal{S}_*}(F,X))$$

Recall the nerve and over construction. Let $N(I/-)_+ \in [I, \mathcal{S}_*]$, be the functor which sends $i \in obI$ to $N(I/i)_+$.

Definition 1.9.0.1 If $F \in [I, \mathcal{S}_*]$, then the *homotopy limit* is defined by

$$\operatorname{holim}_{\overleftarrow{I}} F = \underline{I\mathcal{S}_*}(N(I/-)_+, \sin|F|)$$

and the *homotopy colimit* is defined by

$$\operatorname{holim}_{\overrightarrow{I}} F = N(I^o/-)_+ \wedge F$$

Note that according to the definitions, we get that

$$\underline{\mathcal{S}_*}(N(I^o/-)_+ \wedge F, X) \cong \underline{I^o\mathcal{S}_*}(N(I^o/-)_+, \underline{\mathcal{S}_*}(F, X))$$

so many statements dualize. Most authors do not include the "$\sin|-|$" construction into their definition of the homotopy limit. This certainly has categorical advantages (i.e., the above duality becomes an on the nose duality between homotopy limits and colimits: $\underline{\mathcal{S}_*}(\operatorname{holim}_{\overrightarrow{I}} F, X) \cong \operatorname{holim}_{\overleftarrow{I^o}} \underline{\mathcal{S}_*}(F, X))$, but has the disadvantage that whenever they encounter a problem in homotopy theory they have to assume that their functor has "fibrant values".

1.9.1 Connection to categorical notions

We can express the categorical notions in the same language using the constant functor $*\colon I \to \mathcal{S}_*$ with value the one-point space:

$$\lim_{\overleftarrow{I}} F = \underline{I\mathcal{S}_*}(*_+, F)$$

and

$$\lim_{\overrightarrow{I}} F = *_+ \wedge F$$

The canonical maps $N(I/-) \to *$ and $N(I^o/-) \to *$ give natural maps (use in addition $F \to \sin|F|$ in the first map)

$$\lim_{\overleftarrow{I}} F \to \operatorname{holim}_{\overleftarrow{I}} F, \text{ and } \operatorname{holim}_{\overrightarrow{I}} F \to \lim_{\overrightarrow{I}} F$$

1.9.2 Functoriality

Let

$$I \xrightarrow{f} J \xrightarrow{F} \mathcal{S}_*$$

be functors between small categories. Then there are natural maps

$$f^*\colon \operatorname{holim}_{\overleftarrow{J}} F \to \operatorname{holim}_{\overleftarrow{I}} Ff$$

and

$$f_*\colon \operatorname{holim}_{\overrightarrow{I}} Ff \to \operatorname{holim}_{\overrightarrow{J}} F$$

Under certain conditions these maps are equivalences.

{A1lemma:

Lemma 1.9.2.1 (Cofinality lemma, cf. [14, XI.9.2]) *Let I and J be small categories and let*

$$I \xrightarrow{f} J \xrightarrow{F} \mathcal{S}_*$$

be functors. Then

$$\underset{\overrightarrow{I}}{\operatorname{holim}} Ff \xrightarrow{f_*} \underset{\overrightarrow{J}}{\operatorname{holim}} F$$

is an equivalence if the under categories j/f are contractible for all $j \in obJ$ (f is "right cofinal"); and dually

$$\underset{\overleftarrow{J}}{\operatorname{holim}} F \xrightarrow{f^*} \underset{\overleftarrow{I}}{\operatorname{holim}} Ff$$

is an equivalence if the over categories f/j are contractible for all $j \in obJ$ (f is "left cofinal").

For a sketch of the proof, see the simplicial version.

The corresponding categorical statement to the cofinality lemma only uses the path components of I, and we list it here for comparison:

Lemma 1.9.2.2 (Categorical cofinality lemma, cf. [79, p. 217]) *Let I and J be small categories and let*

$$I \xrightarrow{f} J \xrightarrow{F} \mathcal{S}_*$$

be functors. Then

$$\underset{\overrightarrow{I}}{\lim} Ff \xrightarrow{f_*} \underset{\overrightarrow{J}}{\lim} F$$

is an isomorphism if and only if the under categories j/f are connected for all $j \in obJ$; and dually

$$\underset{\overleftarrow{J}}{\lim} F \xrightarrow{f^*} \underset{\overleftarrow{I}}{\lim} Ff$$

is an isomorphism if and only if the over categories f/j are connected for all $j \in obJ$.

Homotopy colimits are functors of "natural modules" (really of $\mathcal{S}_*$-natural modules, see enriched section), that is the category of pairs (I, F) where I is a small category and $F\colon I \to \mathcal{S}_*$ is a functor. A morphism $(I, F) \to (J, G)$ is a functor $f\colon I \to J$ together with a natural transformation $F \to f^*G = G \circ f$ and induces the map

$$\underset{\overrightarrow{I}}{\operatorname{holim}}\, F \to \underset{\overrightarrow{I}}{\operatorname{holim}}\, f^*G \to \underset{\overrightarrow{J}}{\operatorname{holim}}\, G$$

Homotopy limits should be thought of as a kind of cohomology. It is a functor of "natural comodules" (I, F) (really $\mathcal{S}_*$-natural comodules), that is, the category of pairs as above, but where a map $(I, F) \to (J, G)$ now is a functor $f\colon J \to I$ and a natural transformation $f^*F \to G$. Such a morphism induces the map

$$\underset{\overleftarrow{I}}{\operatorname{holim}}\, F \to \underset{\overleftarrow{J}}{\operatorname{holim}}\, f^*F \to \underset{\overleftarrow{J}}{\operatorname{holim}}\, G$$

Lemma 1.9.2.3 (Homotopy lemma, cf. [14, XI.5.6 and XII.4.2]) *Let* $\eta\colon F \to G \in [I, \mathcal{S}_*]$ *be an equivalence (i.e.* $\eta_i\colon F(i) \to G(i)$ *is a weak equivalence for all* $i \in obI$*). Then*

$$\underset{\overleftarrow{I}}{\operatorname{holim}} F \xrightarrow{\sim} \underset{\overleftarrow{I}}{\operatorname{holim}} G$$

and

$$\underset{\overrightarrow{I}}{\operatorname{holim}} F \xrightarrow{\sim} \underset{\overrightarrow{I}}{\operatorname{holim}} G$$

are equivalences.

Proof: The first statement follows from the fact that $N(I/-)$ is cofibrant and $\sin|F|$ and $\sin|G|$ are fibrant in the closed simplicial model category of $[I, \mathcal{S}_*]$ of 1.3.4, and the second statement follows from duality. ∎

Lastly we have the following very useful observation. We do not know of any reference, but the first part is fairly obvious, and the second part follows by some work from the definition (remember that we take a functorial fibrant replacement when applying the homotopy limit):

{lemma:hol

Lemma 1.9.2.4 *Let* $f\colon I \subseteq J$ *be an inclusion of small categories and* $F\colon J \to \mathcal{S}_*$*. Then the natural map*

$$f_*\colon \underset{\overrightarrow{I}}{\operatorname{holim}} Ff \to \underset{\overrightarrow{J}}{\operatorname{holim}} F$$

is an cofibration (i.e., an injection) and

$$f^*\colon \underset{\overleftarrow{J}}{\operatorname{holim}} F \to \underset{\overleftarrow{I}}{\operatorname{holim}} Ff$$

is a fibration.

1.9.3 (Co)simplicial replacements

{subsec:A

There is another way of writing out the definition of the homotopy (co)limit of a functor $F\colon I \to \mathcal{S}_*$. Note that

$$\underline{I\mathcal{S}_*}(N_q(I/-), F) = \prod_{i_0 \leftarrow \cdots \leftarrow i_q \in N_q(I)} F(i_0)$$

Using the simplicial structure of $N_q(I/-)$ this defines a cosimplicial space. This gives a functor

$$[I, \mathcal{S}_*] \xrightarrow{\prod^*} [\Delta, \mathcal{S}_*],$$

the so-called *cosimplicial replacement*, and the homotopy limit is exactly the composite

$$[I, \mathcal{S}_*] \xrightarrow{\sin|\ \ |} [I, \mathcal{S}_*] \xrightarrow{\prod^*} [\Delta, \mathcal{S}_*] \xrightarrow{Tot} \mathcal{S}_*.$$

Likewise, we note that

$$N_q(I^o/-)_+ \wedge F = \bigvee_{i_0 \leftarrow \cdots \leftarrow i_q \in N_q(I)} F(i_q)$$

defining a functor $\bigvee_* \colon [I, \mathcal{S}_*] \to [\Delta^o, \mathcal{S}_*]$, the so-called *simplicial replacement*, and the homotopy colimit is the composite

$$[I, \mathcal{S}_*] \xrightarrow{\bigvee_*} [\Delta^o, \mathcal{S}_*] \xrightarrow{\text{diag}} \mathcal{S}_*.$$

There is a strengthening of the homotopy lemma for colimits which does not dualize:

ocolimconn}

Lemma 1.9.3.1 *Let* $\eta \colon F \to G \in [I, \mathcal{S}_*]$ *be a natural transformation such that*

$$\eta_i \colon F(i) \to G(i)$$

is n*-connected for all* $i \in obI$*. Then*

$$\underset{\overrightarrow{I}}{\operatorname{holim}} F \to \underset{\overrightarrow{I}}{\operatorname{holim}} G$$

is n*-connected.*

Proof: Notice that, by the description above, the map $N_q(I^o/-)_+ \wedge F \to N_q(I^o/-)_+ \wedge G$ is n-connected for each q. The result then follows upon taking the diagonal. ∎

toweroffib}

Lemma 1.9.3.2 *Let* $\ldots \twoheadrightarrow X_n \twoheadrightarrow X_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow X_0 \twoheadrightarrow *$ *be a tower of fibrations. Then the canonical map*

$$\underset{\overleftarrow{n}}{\lim} X_n \to \underset{\overleftarrow{n}}{\operatorname{holim}} X_n$$

is an equivalence.

1.9.4 Homotopy (co)limits in other categories

Note that, when defining the homotopy (co)limit we only used the simplicial structure of $[I, \mathcal{S}_*]$, plus the possibility of functorially replacing any object by an equivalent (co)fibrant object. If $\mathcal{C}$ is any category with all (co)products (at least all those indexed by the various $N_q(I/i)$s etc.), we can define the (co)simplicial replacement functors for any $F \in [I, \mathcal{C}]$:

$$\prod{}^* F = \{[q] \mapsto \prod_{i_0 \leftarrow \cdots \leftarrow i_q \in N_q(I)} F(i_0)\}$$

and

$$\coprod{}_* F = \{[q] \mapsto \coprod_{i_0 \leftarrow \cdots \leftarrow i_q \in N_q(I)} F(i_q)\}$$

In the special case of a closed simplicial model category, we can always precompose $\prod^*$ (resp. $\coprod_*$) with a functor assuring that $F(i)$ is (co)fibrant to get the right homotopy properties.

As an easy example, we could consider unbased spaces. For $F \in [I, \mathcal{S}]$ we let $\operatorname{holim}_{\overleftarrow{I}} F = Tot(\prod^* \sin |F|)$ and $\operatorname{holim}_{\overrightarrow{I}} F = diag^* \coprod_* F$. Recall the adjoint functor pair

$$\mathcal{S} \underset{U}{\overset{X \mapsto X_+}{\rightleftarrows}} \mathcal{S}_*$$

We get that if $F \in [I, \mathcal{S}]$ and $G \in [I, \mathcal{S}_*]$, then $\prod^* UG = U \prod^* G$, and $(\coprod_* F)_+ = \bigvee_* (F_+)$, so

$$U \operatorname{holim}_{\overleftarrow{I}} G \cong \operatorname{holim}_{\overleftarrow{I}} UG, \text{ and } (\operatorname{holim}_{\overrightarrow{I}} F)_+ \cong \operatorname{holim}_{\overrightarrow{I}} (F_+)$$

More generally, the (co)simplicial replacement will respect left (right) adjoint functors.

1.9.5 Simplicial abelian groups

{subsec:A

In abelian groups, the product is the product of the underlying sets, whereas the coproduct is the direct sum. All simplicial abelian groups are fibrant, and we choose a functorial factorization $0 \rightarrowtail C(M) \overset{\sim}{\twoheadrightarrow} M$, for instance the one coming from the free/forgetful adjoint functor pair to sets. Note that the diagonal (total) of a (co)simplicial simplicial abelian group is a simplicial abelian group, and we define

$$\operatorname{holim}_{\overleftarrow{I}} F = Tot \prod{}^* F$$

and

$$\operatorname{holim}_{\overrightarrow{I}} F = diag^* \coprod{}_* F$$

Note that this last definition is "wrong" in that we have not replaced $F(i)$ by a cofibrant object. But this does not matter since it is an easy excercise to show that

$$\operatorname{holim}_{\overrightarrow{I}} F \overset{\simeq}{\longleftarrow} \operatorname{holim}_{\overrightarrow{I}} CF$$

(forget down to simplicial spaces, use that homotopy groups commute with filtered colimits 1.1.7.3 and finite products, and use that a degreewise equivalence of bisimplicial sets induces an equivalence on the diagonal1.5.0.2).

If F is a functor to abelian groups, the (co)homotopy groups of the (co)simplicial replacement functors above are known to algebraists as the derived functors of the (co)limit, i.e.

$$\lim_{\overleftarrow{I}}{}^{(s)} F = H^s(I, F) = \pi^s \prod{}^* F, \text{ and } \lim_{\overrightarrow{I}}{}_{(s)} F = H_s(I, F) = \pi_s \coprod{}_* F$$

This is used by the following statements for the general case. We say that a category J has finite cohomological dimension if there is some n such that $\lim_{\overleftarrow{J}}{}^{(s)} F = 0$ for all F and $s > n$. For instance, $\mathbf{N}$ has finite cohomological dimension ($n = 1$).

{theo:lim

Theorem 1.9.5.1 *Let $X\colon J \to \mathcal{S}_*$ be a functor. If $\pi_q X$ take values in abelian groups for all $q \geq 0$, then there is a spectral sequence with E^2 term*

$$E^2_{s,t} = \varprojlim_J{}^{(-s)} \pi_t X, \qquad 0 \leq -s \leq t$$

which under favourable conditions converges to $\pi_{s+t} \operatorname{holim}_{\overleftarrow{J}} X$. Especially, if J has finite cohomological dimension, the spectral sequence converges. If $J = \mathbf{N}$ it collapses to the exact sequence

$$0 \to \varprojlim_{\mathbf{N}}{}^{(1)} \pi_{t+1} X \to \pi_t \operatorname*{holim}_{\overleftarrow{\mathbf{N}}} X \to \varprojlim_{\mathbf{N}} \pi_t X \to 0$$

If h is some connected reduced homology theory satisfying the wedge axiom then there is a convergent spectral sequence

$$E^2_{s,t} = \varinjlim_J{}_{(s)} h_t X \to h_{s+t} \operatorname*{holim}_{\overrightarrow{J}} X$$

The homotopy limits in abelian groups coincide with what we get if we forget down to $\mathcal{S}_*$, but generally the homotopy colimit will differ. However, if $F\colon I \to \mathcal{A}$, and $U\colon \mathcal{A}b \to \mathcal{E}ns_*$ is the forgetful functor, there is a natural map

$$\operatorname*{holim}_{\overrightarrow{I}} UF \longrightarrow U \operatorname*{holim}_{\overrightarrow{I}} F = U diag^* \{[q] \mapsto \bigoplus_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_q} X(i_q)\}$$

given by sending wedges to sums. We leave the proof of the following lemma as an excercise (use that homotopy groups commute with filtered direct limit, and the Blakers–Massey theorem 1.10.0.8)

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Lemma 1.9.5.2 *Let $F\colon I \to \mathcal{A}$ be a functor such that $F(i)$ is n-connected for all $i \in obI$. Then*

$$\operatorname*{holim}_{\overrightarrow{I}} UF \to U \operatorname*{holim}_{\overrightarrow{I}} F$$

is $(2n+1)$-connected.

1.9.6 Spectra

The category of spectra has two useful notions of fibrations and weak equivalences, the stable and the pointwise. For the pointwise case there is no difference from the space case, and so we concentrate on the stable structure. Any spectrum is pointwise equivalent to a cofibrant spectrum (i.e., on for which all the structure maps $S^1 \wedge X^k \to X^{k+1}$ are cofibrations, see 1.2.0.3), so it is no surprise that the pointwise homotopy colimit has good properties also with respect to the stable structure. For homotopy limits we need as usual a bit of preparations. We choose a fibrant replacement functor $X \mapsto QX$ as in 1.2. Let $X\colon J \to \mathcal{S}pt$ be a functor from a small category to spectra. Then

$$\operatorname*{holim}_{\overrightarrow{J}} X = \{k \mapsto diag^* \coprod{}_* X^k\}$$

which is just $\text{holim}_{\rightarrow} J$ applied pointwise, and

$$\underset{\overleftarrow{J}}{\text{holim}}\, X = \{k \mapsto Tot^* \prod{}^* Q^k X\}$$

which is equivalent to $k \mapsto \text{holim}_{\overleftarrow{J}} Q^k X$ (we just have skipped the extra application of $\sin|-|$).

Lemma 1.9.6.1 *Pointwise homotopy limits and colimits preserve stable equivalences of spectra.*

Proof: For the homotopy limit this is immediate from the construction since all stable equivalences are transformed into pointwise equivalences of pointwise fibrant spectra by Q. For the homotopy colimit, notice that we just have to prove that for a diagram of specta X, the canonical map $X \to QX$ induces an stable equivalence of homotopy colimits. By lemma 1.2.0.3 we may assume that all spectra in X are $-n$ connected, and then Freudenthal's suspension theorem 1.10.0.9 gives that the maps in $X^k \to Q^k X$ are $2k - n$ connected. Since homotopy colimits preserve connectivity (lemma 1.9.3.1) this means that the map of pointwise homotopy colimits is a weak equivalence. ∎

1.9.7 Enriched categories

(This presupposes MA2) Let $V = (V, \otimes, e)$ be a symmetric monoidal closed category with all coproducts. The most important examples beside the case $(\mathcal{E}ns, \times, *)$ treated above, is the case $(\mathcal{S}_*, \wedge, S^0)$, i.e. the case of (pointed) simplicial categories.

We model the simplicial and cosimplicial replacement functors as follows. Let I be a small V-category, and $i_{-1} \in obI$. The nerve of I/i_{-1} can in this setting be reinterpreted as a simplicial V-object with q-simplices

$$N_q^V(I/i_{-1}) = \coprod_{i_0,\dots,i_q} \bigotimes_{0\le k\le q} \underline{I}(i_k, i_{k-1}) \in obV$$

Let $\mathcal{C}$ be a (co)tensored V-category. Let $X \colon I \to \mathcal{C}$ be a V-functor, then we define

$$\underset{\overleftarrow{I}}{\text{holim}}^V X = \{[q] \mapsto \int_I \underline{V}(N_q^V(I/-), X) \cong \prod_{i_0,\dots,i_q} \underline{V}(\bigotimes_{1\le k\le q} \underline{I}(i_k, i_{k-1}), X(i_0))\}$$

and

$$\underset{\overrightarrow{I}}{\text{holim}}^V X = \{[q] \mapsto \int^I N_q^V(I^o/-) \otimes X \cong \coprod_{i_0,\dots,i_q} \bigotimes_{1\le k\le q} \underline{I}(i_k, i_{k-1}) \otimes X(i_q)\}$$

The homotopy V-limit is a cosimplicial $\mathcal{C}$-object, and the homotopy V-colimit is a simplicial $\mathcal{C}$-object.

We see that the homotopy limit is a functor of V-natural comodules, and homotopy colimits are functors of V-natural modules (ref MA2).

1.9.8 Example

Let I be an $\mathcal{S}_*$-category, and let $X\colon I \to \mathcal{S}_*$ be a $\mathcal{S}_*$-functor. Then the homotopy (co)limit of X is defined by

$$\underset{\overleftarrow{I}}{\operatorname{holim}} X = Tot\, \underset{\overleftarrow{I}}{\operatorname{holim}}^{\mathcal{S}_*} \sin|X| = Tot\{[q] \mapsto \prod_{i_0,\dots,i_q} \underline{\mathcal{S}_*}(\bigwedge_{1\le k\le q} \underline{I}(i_k, i_{k-1}), \sin|X(i_0)|)\}$$

and the homotopy colimit as

$$\underset{\overrightarrow{I}}{\operatorname{holim}} X = diag^* \underset{\overrightarrow{I}}{\operatorname{holim}}^{\mathcal{S}_*} X = diag^*\{[q] \mapsto \bigvee_{i_0,\dots,i_q} \bigwedge_{1\le k\le q} \underline{I}(i_k, i_{k-1}) \wedge X(i_q)\}$$

If I is a constant simplicial category (i.e. an ordinary category), then this definition agrees with the usual one (of course you have to adjoin basepoints to all morphism sets).

As a particularly important example, let G be a simplicial monoid, and X a G-space (i.e. an $\mathcal{S}_*$-functor $X\colon G_+ \to \mathcal{S}_*$), then the homotopy fixed point and orbit spaces are just $\operatorname{holim}_{\overleftarrow{G}} X$ and $\operatorname{holim}_{\overrightarrow{G}} X$. See appendix C for further details.

All the usual results for homotopy (co)limits generalize, for instance

Lemma 1.9.8.1 (Homotopy lemma) *Let $X, Y\colon I \to \mathcal{S}_*$ be $\mathcal{S}_*$-functor, and let $\eta\colon X \to Y$ be a $\mathcal{S}_*$-natural equivalence. Then η induces a weak equivalence on* $\operatorname{holim}_{\overleftarrow{I}}$ *and* $\operatorname{holim}_{\overrightarrow{I}}$.

Proof: [Note on proof]The homotopy colimit statement is clear since a map of simplicial spaces which induce an equivalence in each degree induce an equivalence on the diagonal. For the homotopy limit case, the proof proceeds just as the one sketched in [14, page 303]: first one shows that

$$\{[q] \mapsto \prod_{i_0,\dots,i_q} \underline{\mathcal{S}_*}(\bigwedge_{1\le k\le q} \underline{I}(i_k, i_{k-1}), \sin|X(i_0)|)\}$$

is a fibrant cosimplicial space (this uses the "matching spaces" of [14, page 274], essentially you fix an i_0 and use that the degeneracy map

$$\sum_j s_j\colon \bigvee_{0\le j\le q} \bigvee_{i_1,\dots,i_{q-1}} \bigwedge_{1\le k\le q-1} \underline{I}(i_k, i_{k-1}) \to \bigvee_{i_1,\dots,i_q} \bigwedge_{1\le k\le q} \underline{I}(i_k, i_{k-1})$$

is an inclusion). Then one uses that a map of fibrant cosimplicial spaces that is a pointwise equivalence, induces an equivalence on Tot. ∎

Do we actually need the following?

Lemma 1.9.8.2 (Cofinality lemma) *Let $f\colon I \to J$ be $\mathcal{S}_*$-functors. Then*

$$\underset{\overleftarrow{J}}{\operatorname{holim}} F \xrightarrow{f^*} \operatorname{holim}_{\overleftarrow{I}} Ff$$

is an equivalence for all simplicial $F\colon J \to \mathcal{S}_*$ *if and only if* f *is "left cofinal" in the sense that for all* $j \in obJ$

$$S^0 \simeq N(f/j) = \{[q] \mapsto \bigvee_{i_0,\dots,i_q} \underline{J}(f(i_0), j) \bigwedge_{1\leq k\leq q} \underline{I}(i_k, i_{k-1})\}$$

with the obvious face and degeneracy maps .

Proof: Assume $S^0 \simeq N(f/j)$, and let $X = \sin|F|$. Consider the bicosimplicial space C which in bidegree p, q is given by

$$C^{pq} = \prod_{\substack{i_0,\dots i_p \in I \\ j_0,\dots j_q \in J}} \underline{\mathcal{S}_*}(\underline{J}(f(i_0), j_q) \wedge \bigwedge_{1\leq k\leq p} \underline{I}(i_k, i_{k-1}) \wedge \bigwedge_{1\leq l\leq q} \underline{J}(j_l, j_{l-1}), X(j_0))$$

Fixing q, we get a cosimplicial space

$$\prod_{j_0,\dots j_q\in J} \underline{\mathcal{S}_*}(\bigwedge_{1\leq l\leq q} \underline{J}(j_l, j_{l-1}), \underline{\mathcal{S}_*}(N(f/j_q), X(j_0)))$$

which by hypothesis is equivalent to

$$\prod_{j_0,\dots j_q\in J} \underline{\mathcal{S}_*}(\bigwedge_{1\leq l\leq q} \underline{J}(j_l, j_{l-1}), X(j_0))$$

which, when varying q, is $\operatorname{holim}_{\overleftarrow{J}} X$. Fixing p we get a cosimplicial space

$$[q] \mapsto \prod_{i_0,\dots i_p\in I} \underline{\mathcal{S}_*}(\bigwedge_{1\leq k\leq p} \underline{I}(i_k, i_{k-1}), \prod_{j_0,\dots j_q\in J} \underline{\mathcal{S}_*}(\underline{J}(f(i_0), j_q) \wedge \bigwedge_{1\leq l\leq q} \underline{J}(j_l, j_{l-1}), X(j_0)))$$

Note that $X(f(i_0)) \to \{[q] \mapsto \prod_{j_0,\dots j_q\in J} \underline{\mathcal{S}_*}(\underline{J}(f(i_0), j_q) \wedge \bigwedge_{1\leq l\leq q} \underline{J}(j_l, j_{l-1}), X(j_0))\}$ is an equivalence (it has an extra codegeneracy), and so, when varying p again, we get $\operatorname{holim}_{\overleftarrow{I}} Ff$. One also has to show compatibility with the map in the statement.

In the opposite direction, let $F(j) = \underline{\mathcal{S}_*}(\underline{J}(j, j'), Z)$ for some $j' \in obJ$ and fibrant space Z. Writing out the cosimplicial replacements for $\operatorname{holim}_{\overleftarrow{J}} F$ and $\operatorname{holim}_{\overleftarrow{J}} Ff$ we get that the first is contractible, whereas the latter is $\underline{\mathcal{S}_*}(N(f/j'), Z)$, and so $N(f/j')$ must be contractible. ∎

Note that in the proof, for a given $F\colon J \to \mathcal{S}_*$, the crucial point was that for all $j, j' \in obJ$ $\underline{\mathcal{S}_*}(N(f/j'), X(j)) \simeq X(j)$. This gives the corollary

Corollary 1.9.8.3 *Given simplicial functors*

$$I \xrightarrow{f} J \xrightarrow{F} \mathcal{S}_*,$$

then

$$\operatorname{holim}_{\overleftarrow{J}} F \xrightarrow{f^*} \operatorname{holim}_{\overleftarrow{I}} Ff$$

is an equivalence if for all $j, j' \in obJ$ $Map_*(N(f/j'), F(j)) \simeq F(j)$.

This corollary is often very useful, for instance in the form

Corollary 1.9.8.4 *Given simplicial functors*

$$I \xrightarrow{f} J \xrightarrow{F} \mathcal{S}_*$$

such that F has p-complete values and such that $(N(f/j)\hat{_p} \simeq S^0$ (resp. such that $N(j/f)\hat{_p} \simeq S^0$) for every $j \in obJ$, then

$$\operatorname{holim}_{\overleftarrow{J}} F \xrightarrow{f^*} \operatorname{holim}_{\overleftarrow{I}} Ff$$

is an equivalence.

Proof: We have to show that for all $j, j' \in obJ$ $Map_*(N(f/j'), F(j)) \simeq F(j))$. If we can show that for any spaces Y and Z such that $Y\hat{_p} \simeq S^0$ and $Z \simeq Z\hat{_p}$, then $Z \simeq Map_*(Y, Z)$, we are done. This follows from the string of isomorphisms $[S^n, Map_*(Y, Z)] \cong [S^n \wedge Y, Z] \cong [S^n \wedge Y, Z\hat{_p}] \cong [(S^n \wedge Y)\hat{_p}, Z\hat{_p}] \cong [((S^n)\hat{_p} \wedge Y\hat{_p})\hat{_p} \cong [(S^n)\hat{_p}, Z\hat{_p}] \cong [S^n, Z]$. The third and the last isomorphism comes from the fact that if $Z \simeq Z\hat{_p}$, then $[V\hat{_p}, Z] \cong [V, Z]$. This follows by examining the diagram

$$\begin{array}{ccc} [V, Z] & \longleftarrow & [V\hat{_p}, Z] \\ \downarrow \cong & \searrow & \downarrow \cong \\ [V, Z\hat{_p}] & \longleftarrow & [V\hat{_p}, Z\hat{_p}] \end{array}$$

∎

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Example 1.9.8.5 The inclusion $C_{p^\infty} = \lim_{r\to\infty} C_{p^r} \subseteq S^1$ induces an equivalence $BC_{p^\infty}\hat{_p} \to BS^1\hat{_p}$ (the S^1s should be sin $|S^1|$s, but we allow ourself this lapse of consistency for once). Thus we get that for any p-complete space X with S^1 action, the map

$$X^{hS^1} \to X^{hC_{p^\infty}}$$

is an equivalence. (Proof that $BC_{p^\infty}\hat{_p} \xrightarrow{\sim} BS^1\hat{_p}$: We have a short exact sequence $C_{p^\infty} \subseteq S^1 \to \lim_{\to} pS^1$, and so it is enough to show that $B(\lim_{\overrightarrow{p}} S^1)\hat{_p} \simeq *$. But this is clear, since the homotopy groups of $B(\lim_{\overrightarrow{p}} S^1) \cong H(\mathbf{Z}[1/p], 2)$ are uniquely p-divisible.)

1.10 Cubical diagrams

{A1Cubical}

(rel to simp/cosimp. and B-M results.)

Introduce the categories $\mathcal{P}$ and $\mathcal{P}n$,

Definition 1.10.0.6 An n-cube is a functor $\mathcal{X}$ from the category $\mathcal{P}n$ of subsets of $\{1, \ldots, n\}$ to any of the categories where we have defined homotopy (co)limits (ref). We say that $\mathcal{X}$ is k-cartesian if $\mathcal{X}_\emptyset \to \operatorname{holim}_{\overleftarrow{S \neq \emptyset}} \mathcal{X}_S$ is k-connected, and k-cocartesian if $\operatorname{holim}_{\overrightarrow{S \neq \{1,\ldots,n\}}} \mathcal{X}_S \to \mathcal{X}_{\{1,\ldots,n\}}$ is k-connected. It is homotopy cartesian if it is k-cartesian for all k, and homotopy cocartesian if it is k-cocartesian for all k.

When there is no possibility of confusing with the categorical notions, we write just cartesian and cocartesian. Homotopy (co)cartesian cubes are also called homotopy pullback cubes (resp. homotopy pushout cubes), and the initial (resp. final) vertex is then called the homotopy pullback (resp. homotopy pushout) of the rest of the diagram.

As a convention we shall say that a 0 cube is k-cartesian (resp. k-cocartesian) if $\mathcal{X}_\emptyset$ is $(k-1)$-connected (resp. k-connected).

So, a 0 cube is an object $\mathcal{X}_\emptyset$, a 1 cube is a map $\mathcal{X}_\emptyset \to \mathcal{X}_{\{1\}}$, and a 1 cube is k-(co)cartesian if it is k-connected as a map. A 2 cube is a square

$$\begin{array}{ccc} \mathcal{X}_\emptyset & \longrightarrow & \mathcal{X}_{\{1\}} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\{2\}} & \longrightarrow & \mathcal{X}_{\{1,2\}} \end{array}$$

and so on. We will regard a natural transformation of n cubes $\mathcal{X} \to \mathcal{Y}$ as an $n+1$ cube. In particular, if $F \to G$ is some natural transformation of functors of simplicial sets, and $\mathcal{X}$ is an n cube of simplicial sets, then we get an $n+1$ cube $F\mathcal{X} \to G\mathcal{X}$.

The equivalence $\mathcal{P} - \emptyset @ >\sim>> \Delta$....

This means that for any functor from a small category $J\colon X \to \mathcal{S}_*$

$$\underset{\overleftarrow{J}}{\operatorname{holim}} X \simeq \underset{\overleftarrow{S \in \mathcal{P}-\emptyset}}{\operatorname{holim}} \prod_{j_0 \leftarrow \cdots \leftarrow j_{|S|} \in N_{|S|} J} X(j_0)$$

and

$$\underset{\rightarrow}{\operatorname{holim}} JX \simeq \underset{\overrightarrow{S \in \mathcal{P}^o - \emptyset}}{\operatorname{holim}} \bigvee_{j_0 \leftarrow \cdots \leftarrow j_{|S|}} X(j_{|S|}).$$

This is especially interesting if J is finite (which is equivalent to saying that $N(J)$ is a finite space), for then the homotopy limit is a homotopy pullback of a (finite) cube, and the homotopy colimit is the homotopy pushout of a (finite) cube. Explicitly, if $N(J) = sk_k N(J)$ (that is, as a functor from Δ^o, it factors through the homotopy equivalent subcategory Δ_k of objects $[q]$ for $q \le k$), then $\operatorname{holim}_{\overleftarrow{J}} X$ is equivalent to the homotopy pullback of the punctured k-cube which sends $S \in \mathcal{P}k - \emptyset$ to $\prod_{j_0 \leftarrow \cdots \leftarrow j_{|S|} \in N_{|S|} J} X(j_0)$, and dually for the homotopy colimit.

This means that statements for homotopy pullbacks and pushouts are especially worth while listening to. The Blakers–Massey theorem is an instance of such a statement. It relates homotopy limits and homotopy colimits in a certain range. The ultimate Blakers–Massey theorem is the following.

Theorem 1.10.0.7 *Let S be a finite set with $|S| = n \ge 1$, and let $k\colon \mathcal{P}S \to \mathbf{Z}$ be a monotone function. Set $M(k)$ to be the minimum of $\sum_\alpha k(T_\alpha)$ over all partitions $\{T_\alpha\}$ of S by nonempty sets. Let $\mathcal{X}$ be an S cube.*

1. *If $\mathcal{X}|_T$ is $k(T)$-cocartesian for each nonempty $T \subseteq S$, then $\mathcal{X}$ is $1-n+M(k)$ cartesian.*
2. *If $\mathcal{X}(-\cup(S-T))|_T$ is $k(T)$ cartesian for each nonempty $T \subseteq S$, then $\mathcal{X}$ is $n-1+M(k)$ cocartesian.*

∎

See [40, 2.5 and 2.6] for a proof.

The usual Blakers–Massey theorem is a direct corollary of this. We say that a cube is strongly (co)cartesian if all subcubes of dimension strictly greater than one are homotopy (co)cartesian (demanding this also for dimension one would be the same as demanding that all maps were equivalences, and would lead to a rather uninteresting theory!).

ers-Massey}

Corollary 1.10.0.8 (Blakers–Massey) *Let $\mathcal{X}$ be a strongly cocartesian n cube, and suppose that $\mathcal{X}_\emptyset \to \mathcal{X}_{\{s\}}$ is k_s-connected for all $1 \leq s \leq n$. Then $\mathcal{X}$ is $1-n+\sum_s k_s$ cartesian. Dually, if $\mathcal{X}$ is strongly cartesian, and $\mathcal{X}_{\{1,\dots,n\}-\{s\}} \to \mathcal{X}_{\{1,\dots,n\}}$ is k_s connected for $1 \leq s \leq n$, then $\mathcal{X}$ is $n-1+\sum_s k_s$ cocartesian.*

By applying the Blakers-Massey theorem to the cocartesian square

$$\begin{array}{ccc} X & \longrightarrow & \Delta[1]\wedge X \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1\wedge X \end{array}$$

you get

reudenthal}

Corollary 1.10.0.9 (Freudenthal) *If X is $(n-1)$-connected, then the natural map $X \to \Omega^1(S^1\wedge X)$ is $(2n-1)$-connected.*

For reference we list the following useful corollary which is the unstable forerunner of the fact that stably products are sums.

um is prod}

Corollary 1.10.0.10 *Let X and Y be pointed spaces and X be m-connected and Y be n-connected. Then $X \vee Y \to X \times Y$ is $m+n$-connected.*

Proof: This is much easier by using the Whitehead and Künneth theorems, but here goes. Assume for simplicity that $m \geq n$ Consider the cocartesian square

$$\begin{array}{ccc} X\vee Y & \longrightarrow & X\times Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X\wedge Y \end{array} .$$

Now, $X\wedge Y$ is $m+n+1$-connected (by e.g., considering the spectral sequence 1.5.0.6 of the associated bisimplicial set), the left vertical map is $n+1$-connected and the top horizontal map is – for trivial reasons – n-connected. Hence the diagram $2n$-cartesian and so the top horizontal map must be at least $2n$-connected (since $m+n \geq 2n$). With this improved connectivity, we can use Blakers-Massey again. Repeating this procedure until we get cartesianness that exceeds $m+n$ we get that the top map is $m+n$-connected (and finally, the diagram is $m+2n$-cartesian). ∎

The Blakers–Massey theorem has the usual consequence for spectra:

Corollary 1.10.0.11 *Let $\mathcal{X}$ be an n-cube of bounded below spectra. Then $\mathcal{X}$ is homotopy cartesian if and only if it is homotopy cocartesian.*

Lemma 1.10.0.12 *Let $X\colon I \times J \to \mathcal{S}pt$ be a functor where I is finite. Then the canonical maps*

$$\underset{\overrightarrow{I}}{\text{holim}}\, \underset{\overleftarrow{J}}{\text{holim}}\, X \to \underset{\overleftarrow{J}}{\text{holim}}\, \underset{\overrightarrow{I}}{\text{holim}}\, X$$

and

$$\underset{\overrightarrow{J}}{\text{holim}}\, \underset{\overleftarrow{I}}{\text{holim}}\, X \to \underset{\overleftarrow{I}}{\text{holim}}\, \underset{\overrightarrow{J}}{\text{holim}}\, X$$

are equivalences.

Proof: The homotopy colimit of X over I is equivalent to the homotopy pushout of a punctured cube with finite wedges of copies of $X(i)$'s on each vertex. But in spectra finite wedges are equivalent to products, and homotopy pushout cubes are homotopy pullback cubes, and homotopy pullbacs commute with homotopy limits. This proves the first equivalence, the other is dual. ∎

Corollary 1.10.0.13 *Let $X.\colon \Delta^o \times J \to \mathcal{S}pt$ be a functor regarded as a functor from J to simplicial spectra. Assume J has finite cohomological dimension and $diag^*X$ is bounded below. Then*

$$diag^* \underset{\overleftarrow{J}}{\text{holim}}\, X. \to \underset{\overleftarrow{J}}{\text{holim}}\, diag^* X.$$

is an equivalence.

Proof: Assume $\lim_{\overleftarrow{J}}{}^{(s)} \equiv 0$ for $s > n$, and $\pi_s diag^* X. = 0$ for $s < m$. Let

$$sk_k X. = \underset{[q]\in\Delta_k}{\underrightarrow{\text{holim}}}\, X_q$$

This maps by a $k-m$-connected map to $diag^*X.$, and let F be the homotopy fiber of this map. Then $E^2_{s,t} = \lim_{\overleftarrow{J}}{}^{(-s)}\pi_t F = 0$ if $s < -n$ or $t < k-m$, so $\pi_q \operatorname{holim}_{\overleftarrow{J}} F = 0$ for $q < k-m-n$. All in all, this means that the last map in

$$sk_k \underset{\overleftarrow{J}}{\text{holim}}\, X. = \underset{[q]\in\Delta_k}{\underrightarrow{\text{holim}}}\, \underset{\overleftarrow{J}}{\text{holim}}\, X_q \xrightarrow{\sim} \underset{\overleftarrow{J}}{\text{holim}}\, \underset{[q]\in\Delta_k}{\underrightarrow{\text{holim}}}\, X_q = \underset{\overleftarrow{J}}{\text{holim}}\, sk_k X \to \underset{\overleftarrow{J}}{\text{holim}}\, diag^* X.$$

is $k-n-m$-connected. Letting k go to infinity we have the desired result. ∎

Even in the nonstable case there is a shadow of these nice properties.

Definition 1.10.0.14 If f is some integral function, we say that an S cube $\mathcal{X}$ is f cartesian if each d subcube (face) of $\mathcal{X}$ is $f(d)$ cartesian. Likewise for f cocartesian.

Corollary 1.10.0.15 *Let $k > 0$. An S cube of spaces is $id + k$ cartesian if and only if it is $2 \cdot id + k - 1$ cocartesian. The implication cartesian to cocartesian holds even if $k = 0$.*

Proof: Note that it is trivially true if $|S| \leq 1$. Assume it is proven for all d cubes with $d < n$.

To prove one implication, let $\mathcal{X}$ be an $id+k$ cartesian $n = |S|$ cube. All strict subcubes are also $id + k$ cartesian, and so $2 \cdot id + k - 1$ cocartesian, and the only thing we need to show is that $\mathcal{X}$ itself is $2n + k - 1$ cocartesian. This follows from 2.2.2: $\mathcal{X}$ is K cocartesian where

$$K = n - 1 + \min(\sum_\alpha(|T_\alpha| + k))$$

where the minimum is taken over all partitions $\{T_\alpha\}$ of S by nonempty sets. But this minimum is clearly attained by the trivial partition, for if we subdivide T into T_1 and T_2 then $|T|+k = |T_1|+|T_2|+k \leq |T_1|+k+|T_2|+k$, and so $K = (n-1)+(n+k) = 2n+k-1$.

In the opposite direction, let $\mathcal{X}$ be a $2 \cdot id + k - 1$ cocartesian $n = |S|$ cube. This time, all strict subcubes are by assumption $id+k$ cartesian, and so we are left with showing that $\mathcal{X}$ is $n + k$ cartesian. Again this follows from 2.2.1: $\mathcal{X}$ is K cartesian where

$$K = (1 - n) + \min(\sum_\alpha(2|T_\alpha| + k - 1))$$

where the minimum is taken over all partitions $\{T_\alpha\}$ of S by nonempty sets. But this minimum is clearly attained by the trivial partition, for if we subdivide T into T_1 and T_2 then $2|T| + k - 1 = 2|T_1| + 2|T_2| + k - 1 \leq 2|T_1| + k - 1 + 2|T_2| + k - 1$, and so $K = (1 - n) + (2n + k - 1) = n + k$. ∎

Homology takes cofiber sequences to long exact sequences. This is a reflection of the well-known statement

Lemma 1.10.0.16 *If $\mathcal{X}$ is a cocartesian cube, then $\tilde{\mathbf{Z}}\mathcal{X}$ is cartesian.*

Proof: This follows by induction on the dimension of $\mathcal{X}$. If $dim\mathcal{X} = 1$ it is true by definition, and if $\mathcal{X}$ has dimension $d > 1$ split $\mathcal{X}$ into two $d-1$ dimensional cubes $\mathcal{X}^i \to \mathcal{X}^f$ and take the cofiber $\mathcal{X}^i \to \mathcal{X}^f \to \mathcal{X}^c$. As $\mathcal{X}$ was cocartesian, so is $\mathcal{X}^c$, and by assumption $\tilde{\mathbf{Z}}\mathcal{X}^c$ is cartesian, and $\tilde{\mathbf{Z}}\mathcal{X}^i \to \tilde{\mathbf{Z}}\mathcal{X}^f \to \tilde{\mathbf{Z}}\mathcal{X}^c$ is a fiber sequence, and so $\tilde{\mathbf{Z}}\mathcal{X}$ must be cartesian. ∎

We will need a generalization of the Hurewicz theorem. Recall that the Hurewicz theorem states that if X is $k - 1 > 0$ connected, then $\pi_k X \to H_k(X)$ is an isomorphism and $\pi_{k+1}X \to H_{k+1}X$ is a surjection, or in other words that

$$X \xrightarrow{h_X} \tilde{\mathbf{Z}}X$$

is $k + 1$-connected.

Using the transformation $h\colon 1 \to \tilde{\mathbf{Z}}$ on $h_X\colon X \to \tilde{\mathbf{Z}}X$ we get a square

$$\begin{array}{ccc} X & \xrightarrow{h_X} & \tilde{\mathbf{Z}}X \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_{\tilde{\mathbf{Z}}X}} \\ \tilde{\mathbf{Z}}X & \xrightarrow{\tilde{\mathbf{Z}}h_X} & \tilde{\mathbf{Z}}\tilde{\mathbf{Z}}X \end{array}$$

One may check by brute force that this square is $k+2$ cartesian if X is $k-1>0$ connected. We may continue this process to obtain arbitrarily high dimensional cubes by repeatedly applying h and the generalized Hurewicz theorem states that the result gets linearly closer to being cartesian with the dimension.

{theo:Hur

Theorem 1.10.0.17 *[The Hurewicz theorem (generalized form)] Let $k>1$. If $\mathcal{X}$ is an $id+k$ cartesian cube of simplicial sets, then so is $\mathcal{X}\to\tilde{\mathbf{Z}}\mathcal{X}$.*

Proof: To fix notation, let $\mathcal{X}$ be an $n=|S|$ cube with iterated fiber F and iterated cofiber C. Let $\mathcal{C}$ be the S cube which sends S to C, and all strict subsets to $*$. Then the $|S|+1$ cube $\mathcal{X}\to\mathcal{C}$ is cocartesian.

As $\mathcal{X}$ is $id+k$ cartesian, it is $2\cdot id+k-1$ cocartesian, and in particular C is $2n+k-1$ connected. Furthermore, if $\mathcal{X}|T$ is some d subcube of $\mathcal{X}$ where $\{S\}\notin T$, then $\mathcal{X}|T$ is $2d+k-1$ cocartesian, and so $\mathcal{X}|T\to\mathcal{C}|T=*$ is $2d+k$ cocartesian. Also, if $\mathcal{X}|T$ is some strict subcube with $\{S\}\in T$, then $\mathcal{X}|T\to\mathcal{C}|T$ is still $2d+k$ cocartesian because C is $2n+k-1$ connected, and $d<n$. Thus $\mathcal{X}\to\mathcal{C}$ is $2\cdot id+k-2$ cocartesian, and cocartesian. Using 2.2.1 again, we see that $\mathcal{X}\to\mathcal{C}$ is $1-n+2(n+1+k-2)=n+2k-1$ cartesian as the minimal partition is obtained by partitioning $S\cup\{n+1\}$ in two.

This implies that the map of iterated fibers $F\to\Omega^nC$ is $n+2k-1$ connected. We note that $n+2k-1\geq n+k+1$ as $k>1$.

Furthermore, as C is $2n+k-1$ connected, $\Omega^nC\to\Omega^n\tilde{\mathbf{Z}}C$ is $n+k+1$ connected.

But lemma 2.5 implies that

$$\tilde{\mathbf{Z}}\mathcal{X}\to\tilde{\mathbf{Z}}\mathcal{C}$$

is cartesian. Hence the iterated fiber of $\tilde{\mathbf{Z}}\mathcal{X}$ is $\Omega^n\tilde{\mathbf{Z}}C$, and we have shown that the map from the iterated fiber of $\mathcal{X}$ is $n+1+k$ connected. Doing this also on all subcubes gives the result. ∎

In particular

Corollary 1.10.0.18 *Let X be a $k-1>0$-connected space. Then the cube you get by applying h n times to X is $id+k$-cartesian.*

1.11 Completions and localizations

Let R be either, $\mathbf{F}_p$ for a prime p or a subring of $\mathbf{Q}$. The free/forgetful adjoint pair

$$sR-mod\overset{\tilde{R}}{\leftrightarrows}\mathcal{S}_*$$

give rise to a cosimplicial functor on spaces wich in dimension q takes $X\in\mathcal{S}_*$ to the simplicial R-module $\tilde{R}^{q+1}(X)$ considered as a space. In favourable circumstances the total, or homotopy limit $R_\infty X$, of this cosimplicial space has the right properties of an "R-completion". More precisely, we say that X is *good* (with respect to R) if $X\to R_\infty X$ induces an isomorphism in R-homology, and $R_\infty X\to R_\infty R_\infty X$ is an equivalence.

Especially, simply connected spaces and loop spaces are good, and so R-completion of spectra is well behaved (this is a homotopy limit construction, so we should be prepared to make our spectra Ω-spectra before applying R_∞ to each space).

Explictly, for a spectrum X, let J be a set of primes, I the set of primes not in J and p any prime, we let

$$X_{(J)} = \{k \mapsto X^k_{(J)} = (\mathbf{Z}[I^{-1}])_\infty X^k\}$$

and

$$X\hat{_p} = \{k \mapsto Q^k X\hat{_p} = (\mathbf{Z}/p\mathbf{Z})_\infty (Q^k X)\}$$

We write $X_{\mathbf{Q}}$ or $X_{(0)}$ for the rationalization $X_\emptyset$, and we say that X is *rational* if $X \to X_{\mathbf{Q}}$ is an equivalence, which is equivalent to asserting that $\pi_* X$ is a rational vector space. Generally, $X_{(J)}$ is a localization, in the sense that $X \to X_{(J)}$ induces an equivalence in spectrum homology with coefficients in $\mathbf{Z}[I^{-1}]$, and $\pi_* X_{(J)} \cong \pi_* X \otimes \mathbf{Z}[I^{-1}]$.

Also, $X\hat{_p}$ is a p-completion in the sense that $X \to X\hat{_p}$ induces an equivalence in spectrum homology with coefficients in $\mathbf{Z}/p\mathbf{Z}$, and there is a natural short exact (nonnaturally splittable) sequence

$$0 \to \mathrm{Ext}^1(C_{p^\infty}, \pi_* X) \to \pi_* X\hat{_p} \to Hom(C_{p^\infty}, \pi_{*-1} X) \to 0$$

where $C_{p^\infty} = \mathbf{Z}[1/p]/\mathbf{Z}$. One says that an abelian group M is "Ext $-p$-complete" if $M \to \mathrm{Ext}(C_{p^\infty}, M)$ is an isomorphism and $Hom(C_{p^\infty}, M) = 0$. A spectrum X is p-complete (i.e. $X \to X\hat{_p}$ is an equivalence) if and only if $\pi_* X$ is Ext $-p$-complete.

Lemma 1.11.0.19 *Any simplicial space satisfying the π_*-Kan condition and which is "good" in every degree (and in particular, any simplicial spectrum) may be p-completed or localized degreewise*

Proof: We prove the less obvious completion part. Let Y be the simplicial space $\{q \mapsto (X_q)\hat{_p}\}$. We must show that the map $diag^* X \to diag^* Y$ is a p-completion. Use the spectral sequence for the simplicial space Y, and that Ext$-p$ completeness is closed under extension to see that $diag^*(Y)$ is p-complete. Then use the spectral sequence for the simplicial space $\mathbf{F}_p Y$ to see that $H_*(diag^* X, \mathbf{F}_p) \to H_*(diag^* Y, \mathbf{F}_p)$ is an isomorphism. ∎

Theorem 1.11.0.20 *[12] Let X be any spectrum, then*

$$\begin{array}{ccc} X & \longrightarrow & X_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ \prod_{p\ prime} X\hat{_p} & \longrightarrow & (\prod_{p\ prime} X\hat{_p})_{\mathbf{Q}} \end{array}$$

is homotopy cartesian.

Also from the description of Bousfield we get that p-completion commutes with arbitrary homotopy limits and J-localization with arbitrary homotopy colimits.

1.11.1 Completions and localizations of simplicial abelian groups

If M is a simplicial abelian group, then we can complete or localize the Eilenberg-MacLane spectrum HM. The point here is that this gives new Eilenberg-MacLane spectra which can be described explicitly. The proofs of the statements below follow from the fact that Eilenberg-MacLane spectra and completion and localization are determined by their homotopy groups.

Let $M \in ob\mathcal{A} = s\mathcal{A}b$ be a simplicial abelian group. Then $H(M \otimes_{\mathbf{Z}} \mathbf{Q})$ is clearly a model for $HM_{(0)}$. The map $HM \to HM_{(0)}$ is given by $M@ > m \mapsto m \otimes 1 >> M \otimes_{\mathbf{Z}} \mathbf{Q}$.

Choose a free resolution $R \overset{\sim}{\twoheadrightarrow} \mathbf{Z}[1/p]/\mathbf{Z}$. Then we may define the p-completion as

$$M\hat{_p} = \mathcal{A}(R, \tilde{\mathbf{Z}}[S^1] \otimes_{\mathbf{Z}} M)$$

(internal function object in $\mathcal{A}$, see ?) which is a simplicial abelian group whose Eilenberg-MacLane spectrum $H(M\hat{_p})$ is equivalent to $(HM)\hat{_p}$ (note the similarity with the up to homotopy definition commonly used for spectra). The homotopy groups are given by considering the second quadrant spectral sequence (ref?)

$$E^2_{s,t} = \operatorname{Ext}^{-s}_{\mathbf{Z}}(\mathbf{Z}[1/p]/\mathbf{Z}, \pi_{t-1}M) \Rightarrow \pi_{s+t}M\hat{_p}$$

whose only nonvanishing colums are in degree 0 and -1. The map $M \to M\hat{_p}$ is given as follows. Let $Q = R\prod_{\mathbf{Z}[1/p]/\mathbf{Z}} \mathbf{Z}[1/p]$, and consider the short exact sequence

$$0 \to \mathbf{Z} \to Q \to R \to 0$$

giving rise to the exact sequence

$$0 \to \mathcal{A}(R, \tilde{\mathbf{Z}}[S^1] \otimes_{\mathbf{Z}} M) \to \mathcal{A}(Q, \tilde{\mathbf{Z}}[S^1] \otimes_{\mathbf{Z}} M) \to \tilde{\mathbf{Z}}[S^1] \otimes_{\mathbf{Z}} M \to 0$$

which gives the desired map $M \to M\hat{_p}$.

If M is a cyclic abelian group then, $M \to M \otimes_{\mathbf{Z}} \mathbf{Q}$ and $M \to M\hat{_p}$ are cyclic maps.

Chapter B

Some language

{A2}

B.1 A quick review on enriched categories

To remind the reader, and set notation, we give a short presentation of enriched categories (see e.g., [23], [64] or [**?**]), together with some relevant examples. Our guiding example will be $\mathcal{A}b$-categories, also known as linear categories. These are categories where the morphism sets are actually Abelian groups, and composition is bilinear. That is: in the definition of "category", sets are replaced by Abelian groups, Cartesian product by tensor product and the one point set by the group of integers. Besides $\mathcal{A}b$-categories, the most important example will be the $\Gamma\mathcal{S}_*$-categories, which are used frequently from chapter II on, and we go out of our way to point out some relevant details for this case. Note however, that scary things like limits and ends are after all not that scary since limits (and colimits for that matter) are calculated pointwise.

B.1.1 Closed categories

Recall the definition of a symmetric monoidal closed category $(V, \Box, e)$, see e.g., [79]. For convenience we repeat the definition below, but the important thing to remember is that it behaves as $(\mathcal{A}b, \otimes_{\mathbf{Z}}, \mathbf{Z})$.

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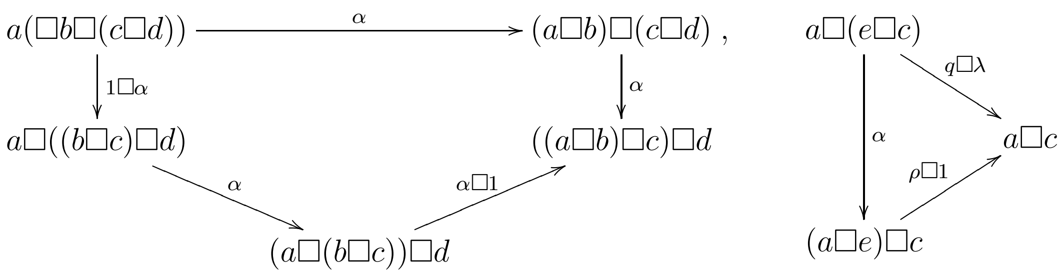
Definition B.1.1.1 A *monoidal category* is a tuple $(\mathcal{C}, \Box, e, \alpha, \lambda, \rho)$ where $\mathcal{C}$ is a category, $\Box$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and α, λ and γ are natural isomorphisms

$$\alpha_{a,b,c} \colon a\Box(b\Box c) \xrightarrow{\simeq} (a\Box b)\Box c, \qquad \lambda_a \colon e\Box a \xrightarrow{\simeq} a, \quad \text{and } \rho_a \colon a\Box e \xrightarrow{\simeq} a$$

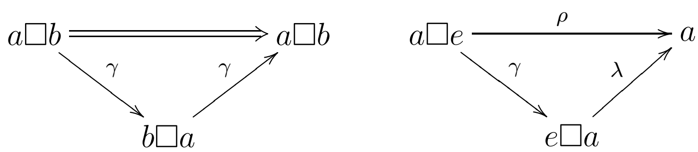
with $\lambda_e = \rho_e \colon e\Box e \to e$, satisfying the coherence laws given by requiring that the following diagrams commute:

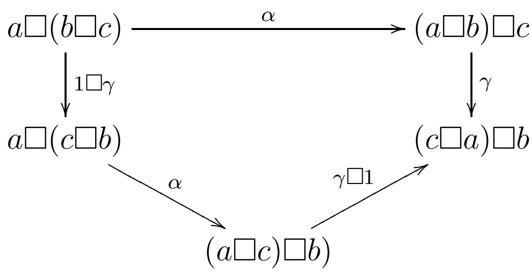


A monoidal category is *symmetric* when it is equipped with a natural isomorphism

$$\gamma_{a,b}\colon a\Box b \xrightarrow{\simeq} b\Box a$$

such that the following diagrams commute





A *symmetric monoidal closed category* (often just called a *closed category*) is a symmetric monoidal category such that

$$-\Box b\colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint $\underline{\mathcal{C}}(b,-)\colon \mathcal{C} \to \mathcal{C}$ (which is considered to be part of the data).

If $\mathcal{C}$ is a closed category, we will refer to $\underline{\mathcal{C}}(b,c)$ as *the internal morphism objects*.

B.1.2 Enriched categories

Let $(V,\Box,e)$ be any closed symmetric monoidal category.

Definition B.1.2.1 A V-category $\mathcal{C}$ is a class of objects, $ob\mathcal{C}$, and for objects $c_0, c_1, c_2 \in ob\mathcal{C}$ objects in V, $\underline{\mathcal{C}}(c_i, c_j)$, and a "composition"

$$\underline{\mathcal{C}}(c_1,c_0)\Box\underline{\mathcal{C}}(c_2,c_1) \to \underline{\mathcal{C}}(c_2,c_0)$$

and a "unit"

$$e \to \underline{\mathcal{C}}(c,c)$$

in V subject to the usual unit and associativity axioms: given objects $a, b, c, d \in ob\mathcal{C}$ then the following diagrams in V commute

$$\begin{array}{ccc}
\underline{\mathcal{C}}(c,d)\Box(\underline{\mathcal{C}}(b,c)\Box\underline{\mathcal{C}}(a,b)) & \xrightarrow{\cong} & (\underline{\mathcal{C}}(c,d)\Box\underline{\mathcal{C}}(b,c))\Box\underline{\mathcal{C}}(a,b) \\
\downarrow & & \downarrow \\
\underline{\mathcal{C}}(c,d)\Box\underline{\mathcal{C}}(a,c) & & \underline{\mathcal{C}}(b,d)\Box\underline{\mathcal{C}}(a,b) \\
& \searrow \quad \swarrow & \\
& \underline{\mathcal{C}}(a,d) &
\end{array}$$

$$\begin{array}{ccccc}
\underline{\mathcal{C}}(a,b)\Box e & \xrightarrow{\cong} & \underline{\mathcal{C}}(a,b) & \xleftarrow{\cong} & e\Box\underline{\mathcal{C}}(a,b) \\
\downarrow & & \downarrow = & & \downarrow \\
\underline{\mathcal{C}}(a,b)\Box\underline{\mathcal{C}}(a,a) & \longrightarrow & \underline{\mathcal{C}}(a,b) & \longleftarrow & \underline{\mathcal{C}}(b,b)\Box\underline{\mathcal{C}}(a,b)
\end{array}.$$

We see that $\mathcal{C}$ is an ordinary category (an "$\mathcal{E}ns$-category") too, which we will call $\mathcal{C}$ too, or $U_0\mathcal{C}$ if we need to be precise, with the same objects and with morphism sets $U_0\mathcal{C}(c,d) = V(e, \underline{\mathcal{C}}(c,d))$.

We see that $\underline{\mathcal{C}}$ can be viewed as a functor $U\mathcal{C}^o \times U\mathcal{C} \to V$: if $f \in \mathcal{C}(c',c) = V(e, \underline{\mathcal{C}}(c',c))$ and $g \in \mathcal{C}(d,d') = V(e, \underline{\mathcal{C}}(d,d'))$ then $f^*g_* = g_*f^* = \underline{\mathcal{C}}(f,g)\colon \underline{\mathcal{C}}(c,d) \to \underline{\mathcal{C}}(c',d') \in V$ is defined as the composite

$$\underline{\mathcal{C}}(c,d) \cong e\Box\underline{\mathcal{C}}(c,d)\Box e \xrightarrow{g\Box id\Box f} \underline{\mathcal{C}}(d,d')\Box\underline{\mathcal{C}}(c,d)\Box\underline{\mathcal{C}}(c,c') \to \underline{\mathcal{C}}(c',d')\cdot$$

B.1.2.2 Some further definitions

If $\mathcal{C}$ and $\mathcal{D}$ are two V-categories, we define their *tensor product* (or whatever the operator in V is called) $\mathcal{C}\Box\mathcal{D}$ to be the V-category given by $ob(\mathcal{C}\Box\mathcal{D}) = ob\mathcal{C} \times ob\mathcal{D}$, and $\underline{\mathcal{C}\Box\mathcal{D}}((c,d),(c',d')) = \underline{\mathcal{C}}(c,c')\Box\underline{\mathcal{D}}(d,d')$.

Let $\mathcal{C}$ be a V-category where V has finite products. If $U\mathcal{C}$ is a category with sum (i.e. it has an initial object $*$, and categorical coproducts), then we say that $\mathcal{C}$ is a *V-category with sum* if the canonical map $\underline{\mathcal{C}}(c \vee c', d) \to \underline{\mathcal{C}}(c,d) \times \underline{\mathcal{C}}(c',d)$ is an isomorphism.

A *V-functor* F from $\mathcal{C}$ to $\mathcal{D}$ is an assignment $ob\mathcal{C} \to ob\mathcal{D}$ together with maps $\underline{\mathcal{C}}(c,c') \to \underline{\mathcal{D}}(F(c),F(c'))$ preserving unit and composition. A V-functor $F\colon \mathcal{C} \to \mathcal{D}$ is *V-full* (resp. *V-faithful*) if $\underline{\mathcal{C}}(c,d) \to \underline{\mathcal{D}}(F(c),F(d))$ is epic (resp. monic). A *V-natural transformation* between two V-functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is a map $\eta_c\colon F(c) \to G(c) \in U\mathcal{D}$ for every $c \in ob\mathcal{C}$ such that all the diagrams

$$\begin{array}{ccc}
\underline{\mathcal{C}}(c,c') & \longrightarrow & \underline{\mathcal{D}}(F(c),F(c')) \\
\downarrow & & \downarrow (\eta_{c'})_* \\
\underline{\mathcal{D}}(G(c),G(c')) & \xrightarrow{(\eta_c)^*} & \underline{\mathcal{D}}(F(c),G(c'))
\end{array}$$

commute.

If

$$\mathcal{D} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

is a pair of V-functors, we say that F is V*-left adjoint* to U (and U is V*-right adjoint* to F) if there are V-natural transformations $FU \xrightarrow{\epsilon} 1_{\mathcal{C}}$ (the *counit*) and $UF \xleftarrow{\eta} 1_{\mathcal{D}}$ (the *unit*) such that the following diagrams commute:

$$\begin{array}{ccc} U & \xrightarrow{\eta U} & UFU \\ & \searrow^{=} & \downarrow U\epsilon \\ & & U \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{F\eta} & FUF \\ & \searrow^{=} & \downarrow \epsilon F \\ & & U \end{array}$$

B.1.2.3 Examples of enriched categories

1. Any symmetric monoidal closed category $(V, \square, e)$ is enriched in itself due to the internal morphism objects.

2. A *linear* category is nothing but an $\mathcal{A}b$-category, that is a category enriched in $(\mathcal{A}b, \otimes, \mathbf{Z})$. Note that an additive category is *something else* (it is a linear category with a zero object and all finite sums). “Linear functor” is another name for $\mathcal{A}b$-functor.

3. Just as a ring is an $\mathcal{A}b$-category with one object, or a k-algebra is a $(k-\text{mod})$-category with only one object, an $\mathbf{S}$-algebra is a $\Gamma\mathcal{S}_*$-category with only one object. This is equivalent to saying that it is a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$, which is another way of saying that an $\mathbf{S}$-algebra is something which satisfies all the axioms of a ring, if you replace every of $\mathcal{A}b$, $\otimes$ or $\mathbf{Z}$ by $\Gamma\mathcal{S}_*$, $\wedge$ or $\mathbf{S}$ (in that order).

4. Let $\mathcal{C}$ be a category with sum (and so is “tensored over Γ^0” by the formula $c\square k_+ = \bigvee_k c$). This defines a (discrete) $\Gamma\mathcal{S}_*$-category $\mathcal{C}^\vee$ by setting $\underline{\mathcal{C}}^\vee(c,c')(X) = \mathcal{C}(c, c'\square X)$ for $X \in ob\Gamma^o$ and $c, c' \in ob\mathcal{C}$, and with composition given by

$$\begin{array}{rcc} \underline{\mathcal{C}}^\vee(c,d)(X)\wedge\underline{\mathcal{C}}^\vee(b,c)(Y) & = & \mathcal{C}(c,d\square X)\wedge\mathcal{C}(b,c\square Y) \\ & \xrightarrow{(-\square Y)\wedge id} & \mathcal{C}(c\square Y,(d\square X)\square Y)\wedge\mathcal{C}(b,c\square Y) \\ & \longrightarrow & \mathcal{C}(b,(d\square X)\square Y) \\ \cong \mathcal{C}(b,d\square(X\square Y)) & = & \underline{\mathcal{C}}^\vee(b,d)(X\wedge Y) \end{array}.$$

Slightly more general, we could have allowed $\mathcal{C}$ to be an $\mathcal{S}_*$-category with sum.

B.1.3 Monoidal V-categories

There is nothing hindering us from adding a second layer of complexity to this. Given a closed category $(V, \boxtimes, \epsilon)$, a *(symmetric) monoidal (closed) V-category* is a (symmetric) monoidal (closed) category $(\mathcal{C}, \square, e)$ in the sense that you use definition B.1.1.1, but do it in the V-enriched world (i.e., $\mathcal{C}$ is a V-category, $\square\colon \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ a V-functor, the required natural transformations are V-natural (and $\underline{\mathcal{C}}(b, -)$ is V-right adjoint to $-\square b$)).

B.1.3.1 Important convention

All categories are considered to be enriched over $(\mathcal{S}_*, \wedge, S^0)$ without further mention. In particular, $(V, \square, e)$ is a closed $\mathcal{S}_*$-category, and any V-category $\mathcal{C}$ is also an $\mathcal{S}_*$-category which is sometimes also called $\mathcal{C}$, but when accuracy is important $U\mathcal{C}$, with morphism spaces $\mathcal{C}(b, c) = V(e, \underline{\mathcal{C}}(b, c)) \in ob\mathcal{S}_*$. This fits with the convention of not underlining function spaces. Of course, it also defines a set-based category $U_0\mathcal{C}$ too by considering zero-simplices only.

B.1.4 Modules

A left $\mathcal{C}$-module P is an assignment $ob\mathcal{C} \to obV$, and a morphism $P(c)\square\underline{\mathcal{C}}(c, b) \to P(b)$ in V such that the obvious diagrams commute; or in other words, a $\mathcal{C}$-module is a V-functor $P\colon \mathcal{C} \to V$. Right modules and bimodules are defined similarly as V-functors $\mathcal{C}^o \to V$ and $\mathcal{C}^o\square\mathcal{C} \to V$. If V has finite products and $\mathcal{C}$ is a V-category with sum, a $\mathcal{C}^o$-module M is said to be additive if the canonical map $M(c \vee c') \to M(c) \times M(c')$ is an isomorphism, and a bimodule is additive if $P(c \vee c', d) \to P(c, d) \times P(c', d)$ is an isomorphism.

Example B.1.4.1 If a ring A is considered to be an $\mathcal{A}b$-category with just one object, one sees that a left A-module M in the ordinary sense is nothing but a left A-module in the sense above: consider the functor $A \to \mathcal{A}b$ with M as value, and sending the morphism $a \in A$ to multiplication on $M \xrightarrow{m \mapsto am} M$. Likewise for right modules and bimodules.

Likewise, if A is an $\mathbf{S}$-algebra, then an A-module is a $\Gamma\mathcal{S}_*$ functor $A \to \Gamma\mathcal{S}_*$. Again, this another way of saying that an A-module is an "$-\wedge A$"-algebra, which is to say that it satisfies all the usual axioms for a module, mutatis mutandem.

B.1.4.2 V-natural modules

A *V-natural bimodule* is a pair $(\mathcal{C}, P)$ where $\mathcal{C}$ is a V-category and P is a $\mathcal{C}$-bimodule. A {Def:natu
map of V-natural bimodules $(\mathcal{C}, P) \to (\mathcal{D}, Q)$ is a V-functor $F\colon \mathcal{C} \to \mathcal{D}$ and a V-natural transformation $P \to F^*Q$ where F^*Q is the $\mathcal{C}$-bimodule given by the composite

$$\mathcal{C}^o\square\mathcal{C} \xrightarrow{F \times F} \mathcal{D}^o\square\mathcal{D} \xrightarrow{Q} V\cdot$$

Similarly one defines V-natural modules as pairs $(\mathcal{C}, P)$ where $\mathcal{C}$ is a V-category and P a $\mathcal{C}$-module. A map of V-natural modules $(\mathcal{C}, P) \to (\mathcal{D}, Q)$ is a V-functor $F\colon \mathcal{C} \to \mathcal{D}$ and a

V-natural transformation $P \to F^*Q$ where F^*Q is the $\mathcal{C}$-bimodule given by the composite

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{Q} V.$$

The V-natural (bi)modules form a 2-category: the maps of V-natural (bi)modules are themselves objects of a category. The morphisms in this category are (naturally) called natural transformations; a natural transformation $\eta\colon F \to G$ where F, G are two maps of V-natural bimodules $(\mathcal{C}, P) \to (\mathcal{D}, Q)$ is a V-natural transformation $\eta\colon F \to G$ of V-functors $\mathcal{C} \to \mathcal{D}$ such that the diagram

$$\begin{array}{ccc} P(c, c') & \longrightarrow & Q(F(c), F(c')) \\ \downarrow & & \downarrow{\scriptstyle (\eta_{c'})_*} \\ Q(G(c), G(c')) & \xrightarrow{(\eta_c)^*} & Q(F(c), G(c')) \end{array}$$

commutes. A natural isomorphism is a natural transformation such that all the η_c are isomorphisms. Likewise one defines the notion of a natural transformation/isomorphism for maps of V-natural modules.

For cohomology considerations, the dual notion of V-natural co(bi)modules is useful. The objects are the same as above, but a morphism $f\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ is a functor $f\colon \mathcal{D} \to \mathcal{C}$ together with a natural transformation $f^*P \to Q$, and so on.

Example B.1.4.3 Let $\mathcal{C}$ be a category with sum, and let P be an additive $\mathcal{C}$-bimodule (i.e., $P(c \vee c', d) \xrightarrow{\cong} P(c, d) \times P(c', d)$). Recall the definition of $\mathcal{C}^\vee$. We define a $\mathcal{C}^\vee$-bimodule $P^\vee$ by the formula $P^\vee(c, d)(X) = P(c, d\Box X)$. Note that since P is additive we have a canonical map $P(c, d) \to P(c\Box X, d\Box X)$, and the right module action uses this. Then $(\mathcal{C}^\vee, P^\vee)$ is a natural module, and $(\mathcal{C}^\vee, P^\vee) \to ((-\Box X)^*\mathcal{C}^\vee, (-\Box X)^*P^\vee)$ is a map of natural modules.

B.1.5 Ends and coends

Ends and coends are universal concepts as good as limits and colimits, but in the set-based world you can always express them in terms of limits and colimits, and hence are less often used. The important thing to note is that this is the way we construct natural transformations: given two (set-based) functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation η from F to G is a collection of maps $\eta_c\colon F(c) \to G(c)$ satisfying the usual condition. Another way to say the same thing is that the set of natural transformations is a set $\mathcal{D}^{\mathcal{C}}(F, G)$ together with a family of functions

$$\mathcal{D}^{\mathcal{C}}(F, G) \xrightarrow{\eta \mapsto p_c(\eta) = \eta_c} \mathcal{D}(F(c), G(c))$$

such that for every $f\colon c_1 \to c_0$

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{C}}(F, G) & \xrightarrow{p_{c_1}} & \mathcal{D}(F(c_1), G(c_1)) \\ {\scriptstyle p_{c_0}}\downarrow & & \downarrow{\scriptstyle G(f)_*} \\ \mathcal{D}(F(c_0), G(c_0)) & \xrightarrow{F(f)^*} & \mathcal{D}(F(c_1), G(c_0)) \end{array}$$

Furthermore, $\mathcal{D}^{\mathcal{C}}(F, G)$ is universal among sets with this property: It is "the end of the functor $\mathcal{D}(F(-), G(-)) \colon \mathcal{C}^o \times \mathcal{C} \to \mathcal{D}$". This example is the only important thing to remember about ends. What follows is just for reference.

Definition B.1.5.1 Let $\mathcal{C}$ and $\mathcal{D}$ be V-categories and $T \colon \mathcal{C}^o \square \mathcal{C} \to \mathcal{D}$ a V-functor. A V-natural family is an object $d \in ob\mathcal{D}$, and for every object $c \in ob\mathcal{C}$ a map $f_c \colon d \to T(c, c)$ such that the following diagram commute

$$\begin{array}{ccc} \underline{\mathcal{C}}(c_1, c_0) & \xrightarrow{T(c_1,-)} & \underline{\mathcal{D}}(T(c_1, c_1), T(c_1, c_0)) \\ {\scriptstyle T(-,c_0)}\downarrow & & \downarrow{\scriptstyle f_{c_1}^*} \\ \underline{\mathcal{D}}(T(c_0, c_0), T(c_1, c_0)) & \xrightarrow{f_{c_0}^*} & \underline{\mathcal{D}}(d, T(c_1, c_0)) \end{array}$$

Definition B.1.5.2 Let $\mathcal{C}$ be a V-category. The *end* of a bimodule $T \colon \mathcal{C}^o \square \mathcal{C} \to V$ is a V-natural family

$$\int_c T(c, c) \xrightarrow{p_x} T(x, x)$$

such that for any other V-natural family $f_x \colon v \to T(x, x)$, there exists a unique morphism $v \to \int_c T(c, c)$ making the following diagram commute:

$$\begin{array}{ccc} v & \longrightarrow & \int_c T(c, c) \\ & {\scriptstyle f_x}\searrow \quad \swarrow{\scriptstyle p_x} & \\ & T(x, x) & \end{array}$$

Definition B.1.5.3 Let $T \colon \mathcal{C}^o \square \mathcal{C} \to \mathcal{D}$ be a V-functor. The *end* of T is a V-natural family

$$\int_c T(c, c) \xrightarrow{p_x} T(x, x)$$

such that for every $d \in ob\mathcal{D}$

$$\underline{\mathcal{D}}(d, \int_c T(c, c)) \xrightarrow{p_{x*}} \underline{\mathcal{D}}(d, T(x, x))$$

is the end of

$$\mathcal{C}^o \square \mathcal{C} \xrightarrow{\underline{\mathcal{D}}(d, T(-,-))} V$$

With mild assumptions, this can be expressed as a limit in $\mathcal{D}$ (see [**?**, page 39]). The dual of the end is the *coend*. The most basic is the tensor product: considering a ring A as an $\mathcal{A}b$-category with one object (called A), a left module $M \colon A \to \mathcal{A}b$ and a right module $N \colon A^o \to \mathcal{A}b$, the tensor product $N \otimes_A M$ is nothing but the coend $\int^A N \otimes M$.

B.1.6 Functor categories

Assume that V has all limits. If I is a small category, we define the V-category $\int_I \mathcal{C}$ of "functors from I to $\mathcal{C}$" as follows. The objects are just the functors from I to $U\mathcal{C}$, but the morphisms $\int_I \underline{\mathcal{C}}(F, G)$ is set to be the end $\int_I \underline{\mathcal{C}}(F, G) = \int_{i \in I} \mathcal{C}(F(i), G(i))$ of

$$I^o \times I \xrightarrow{(F,G)} U\mathcal{C}^o \times U\mathcal{C} \xrightarrow{\mathcal{C}} V \cdot$$

We check that this defines a functor $[I, U\mathcal{C}]^o \times [I, U\mathcal{C}] \to V$. The composition is defined by the map

$$\left(\int_I \underline{\mathcal{C}}(G,H)\right) \square \left(\int_I \underline{\mathcal{C}}(F,G)\right) \to \int_I \int_I \underline{\mathcal{C}}(G,H) \square \underline{\mathcal{C}}(F,G) \xrightarrow{diag^*} \int_I \underline{\mathcal{C}}(G,H) \square \underline{\mathcal{C}}(F,G)$$
$$\to \int_I \underline{\mathcal{C}}(F,H)$$

Note that I is here an ordinary category, and the end here is an end of set-based categories. In the case where the forgetful map $V \xrightarrow{N \mapsto V(e,N)} \mathcal{E}ns$ has a left adjoint, say $X \mapsto e \square X$, then there is a left adjoint functor from categories to V-categories, sending a category I to a "free" V-category $e\square I$, and the functor category we have defined is the usual V-category of V-functors from $e\square I$ to $\mathcal{C}$ (see [23], [?] or [?]).

Also, a $\mathcal{C}$-bimodule P gives rise to a $\int_I \mathcal{C}$-bimodule $\int_I P$ with $\int_I P(F,G)$ defined as the end. The bimodule structure is defined as

$$\int_I \underline{\mathcal{C}} \square \int_I P \square \int_I \underline{\mathcal{C}} \to \int_{I^{\times 3}} \underline{\mathcal{C}} \square P \square \underline{\mathcal{C}} \to \int_I \underline{\mathcal{C}} \square P \square \underline{\mathcal{C}} \to \int_I P.$$

As an example, one has the fact that if $\mathcal{C}$ is any category and $\mathcal{D}$ is an $\mathcal{A}b$-category, the free functor from sets to abelian groups $\mathbf{Z}\colon \mathcal{E}ns_* \to \mathcal{A}b$ induces an equivalence between the $\mathcal{A}b$-category of $\mathcal{A}b$-functors $\mathbf{Z}\mathcal{C} \to \mathcal{D}$ and the $\mathcal{A}b$-category of functors $\mathcal{C} \to \mathcal{D}$. See [?] for a discussion on the effect of change of base-category.

Example B.1.6.1 (Modules over an S-algebra) Let A be an $\mathbf{S}$-algebra. The category $\mathcal{M}_A$ of A-modules is again a $\Gamma\mathcal{S}_*$-category . Explicitly, if M and N are A-modules, then

$$\underline{\mathcal{M}}_A(M,N) = \int_A \underline{\Gamma\mathcal{S}_*}(M,N) \cong \lim_{\leftarrow}\{\underline{\Gamma\mathcal{S}_*}(M,N) \rightrightarrows \underline{\Gamma\mathcal{S}_*}(A \wedge M, N)\}$$

with the obvious maps.

We refer to [107] for a more thorough discussion of $\mathbf{S}$-algebras and A-modules and their homotopy properties. See also chapter II.

Chapter C

Group actions

{A3}

In this appendix we will collect some useful facts needed in chapter VI. We will not strive for the maximal generality, and there is nothing here which can not be found elsewhere in some form.

C.1 G-spaces

{sec:A3Gs
{Def:A3Gs

Let G be a simplicial monoid. A *G-space* X is a space X together with a pointed G action $\mu\colon G_+\wedge X \to X$ such that the expected diagrams commute. Or said otherwise: it is a simplicial functor

$$G \xrightarrow{X} \mathcal{S}_*$$

with G considered as a simplicial category with one object. We let X denote both the functor and the image of the object in G. That the functor is simplicial assures that the resulting map $G \to \underline{\mathcal{S}_*}(X,X)$ is simplicial, and by adjointness it gives rise to μ (the "plus" in $G_+\wedge X \to X$ comes from the fact that $G \to \underline{\mathcal{S}_*}(X,X)$ is not basepoint preserving as it must send the identity to the identity). Then the functoriality encodes the desired commuting diagrams.

According to our general convention of writing $\mathcal{C}\mathcal{S}_*$ for the category of functors from a category $\mathcal{C}$ to $\mathcal{S}_*$ (blatantly violated in our notation $\Gamma\mathcal{S}_*$ for functors from Γ^o to spaces), we write $G\mathcal{S}_*$ for the category of G-spaces. This is a pointed simplicial category with function spaces

$$\underline{G\mathcal{S}_*}(X,Y) = \{[q] \mapsto G\mathcal{S}_*(X\wedge\Delta[q]_+, Y)\}$$

If X is a G-space and Y is a G^o-space (a right G-space), we let their smash product be the space

$$Y\wedge_G X = Y\wedge X/(yg\wedge x \sim y\wedge gx)$$

The forgetful map $G\mathcal{S}_* \to \mathcal{S}_*$ has a left adjoint, namely $X \mapsto G_+\wedge X$, the *free G-space on the space X*.

Generally we say that a G-space X is *free* if for all non-base points $x \in X$ the *isotropy groups* $I_x = \{g \in G | gx = x\}$ are trivial, whereas $I_{\text{base point}} = G$ ("free away from the basepoint"). A *finite free G-space* a G-space Y with only finitely many non-degenerate *G-cells*. (you adjoin a "G-cell" of dimension n to Y_j by taking a pushout of maps of G spaces

$$\begin{array}{ccc} \partial\Delta(n)\wedge G_+ & \xrightarrow{\text{incl.}\wedge id} & \Delta(n)\wedge G_+ \\ \downarrow & & \downarrow \\ Y_j & \longrightarrow & Y_{j+1} \end{array}$$

where G acts trivially on $\partial\Delta(n)$ and $\Delta(n)$.)

C.1.1 The orbit and fixed point spaces

Let $f\colon M \to G$ be a map of monoids. Precomposition with f gives a functor $f^*\colon [G, \mathcal{E}ns_*] \to [M, \mathcal{E}ns_*]$ and since all (co)limits exists this functor has both a right and a left adjoint. If f is surjective and G a group, let $H \subset M$ be the submonoid of elements mapping to the identity. Then the right adjoint of f^* is

$$X \mapsto X^H = \lim_{\overleftarrow{H}} X = \{x \in X | h \cdot x = x \text{ for all } h \in H\}$$

the set of *fixed points*, and the left adjoint is

$$X \mapsto X_H = \lim_{\overrightarrow{H}} X = X/(h \cdot x \sim x)$$

the set of *orbits*. The same considerations and definitions holds in the simplicial case, and we even get simplicial adjoints:

$$M\mathcal{S}_* \underset{X \mapsto X_H}{\overset{X \mapsto X^H}{\rightleftarrows}} G\mathcal{S}_* \quad (f^* \text{ from } G\mathcal{S}_* \text{ to } M\mathcal{S}_*)$$

$$\underline{G\mathcal{S}_*}(X_H, Y) \cong \underline{M\mathcal{S}_*}(X, f^*Y), \text{ and } \underline{G\mathcal{S}_*}(Y, X^H) \cong \underline{M\mathcal{S}_*}(f^*Y, X)$$

If G is a simplicial group, the homomorphism $G \to G^o \times G$ sending g to (g^{-1}, g) makes it possible to describe $-\wedge_G-$ and $\underline{G\mathcal{S}_*}(-,-)$ in terms of orbit and fixed point spaces. If $X, Y \in obG\mathcal{S}_*$ and $Z \in obG^o\mathcal{S}_*$ then $Z\wedge X$ and $\underline{\mathcal{S}_*}(X, Y)$ are naturally $G^o \times G$-spaces, and since G is a group also G-spaces, and we get that

$$Z\wedge_G X \cong (Z\wedge X)_G, \qquad \text{and} \qquad \underline{G\mathcal{S}_*}(X, Y) \cong \underline{\mathcal{S}_*}(X, Y)^G.$$

C.1.2 The homotopy orbit and homotopy fixed point spaces

Let G be a simplicial monoid. When regarded as a simplicial category, with only one object $*$, we can form the over (resp. under) categories, and the nerve $N(G/*)_+$ (resp. $N(*/G)_+$) is a contractible free G-space (resp. contractible free G^o-space), and the G orbit space is $BG = NG$. For G a group, $N(G/*) \cong B(G, G, *)$ (resp. $N(*/G) \cong B(*, G, G)$) the one sided bar construction, and we note that in this case the left and right distinction is inessential. We write EG_+ for any contractible free G-space.

Definition C.1.2.1 Let G be a simplicial monoid and X a G-space. Then the *homotopy fixed point space* is

$$X^{hG} = \underset{\overleftarrow{G}}{\operatorname{holim}} X = \underline{G\mathcal{S}_*}(N(G/*)_+, \sin|X|)$$

and the *homotopy orbit space* is

$$X_{hG} = \underset{\overrightarrow{G}}{\operatorname{holim}} X = N(*/G)_+ \wedge_G X.$$

A nice thing about homotopy fixed point and orbit spaces is that they preserve weak equivalences (since homotopy (co)limits do). We have maps $X^G \to X^{hG}$ and $X_{hG} \to X_G$, and a central problem in homotopy theory is to know when they are equivalences.

Note that if G is a group, then

$$X^{hG} \simeq Map_*(EG_+, X)^G, \qquad \text{and} \qquad X_{hG} \simeq (EG_+ \wedge X)_G$$

Any free G-space EG_+ whose underlying space is contractible will do in the sense that they all give equivalent answers.

{lemma:A3

Lemma C.1.2.2 *Let U be a free G-space, and X any fibrant G-space (i.e., a G space which is fibrant as a space). Then*

$$\underline{G\mathcal{S}_*}(U, X) \xrightarrow{\sim} \underline{G\mathcal{S}_*}(U \wedge EG_+, X),$$

and so if G is a group $\underline{\mathcal{S}_*}(U, X)^G \simeq \underline{\mathcal{S}_*}(U, X)^{hG}$. Furthermore, if U is d-dimensional, then $\underline{G\mathcal{S}_*}(U, -)$ sends n-connected maps of fibrant spaces to $(n-d)$-connected maps.

Proof: By induction on the G-cells, it is enough to prove it for $U = S^k \wedge G_+$. But then the map is the composite from top left to top right in

$$\begin{array}{ccccc}
\underline{G\mathcal{S}_*}(S^k \wedge G_+, X) & \longrightarrow & \underline{G\mathcal{S}_*}(S^k \wedge G_+ \wedge EG_+, X) & \underset{\cong}{\xrightarrow{i^*}} & \underline{G\mathcal{S}_*}(S^k \wedge G_+ \wedge EG_+, X) \\
\cong\downarrow & & \cong\downarrow & & \\
\underline{\mathcal{S}_*}(S^k, X) & \xrightarrow{\sim} & \underline{\mathcal{S}_*}(S^k \wedge EG_+, X) & &
\end{array}$$

where i_* is the G-isomorphism from $S^k \wedge G_+ \wedge EG_+$ (no action on EG_+) to $S^k \wedge G_+ \wedge EG_+$ (diagonal action) given by the *shear map* $(s, g, e) \mapsto (s, g, ge)$. The last statement follows {shear map from induction on the skeleta, and the fact that $\underline{G\mathcal{S}_*}(S^k \wedge G_+, -) \cong \underline{\mathcal{S}_*}(S^k, -)$ sends n-connected maps of fibrant spaces to $(n-k)$-connected maps. ∎

C.2 (Naïve) G-spectra

Let G be a simplicial monoid. The category of G-*spectra*, $GSpt$ is the category of simplicial functors from G to the category of spectra. A map of G-spectra is called a pointwise (resp. stable) equivalence if the underlying map of spectra is.

This notion of G-spectra is much less rigid than what most people call G-spectra (see e.g., [71]), and they would prefer to call these spectra something like "naïve pre-G-spectra". To make it quite clear: our G-spectra are just sequences of G-spaces together with structure maps $S^1 \wedge X^n \to X^{n+1}$ that are G-maps. A map of G-spectra $X \to Y$ is simply a collection of G-maps $X^n \to Y^n$ commuting with the structure maps.

Again, G-spectra form a simplicial category, with function objects

$$\underline{G\mathcal{S}pt}^0(X, Y) = \{[q] \mapsto G\mathcal{S}pt(X \wedge \Delta[q]_+, Y)\}$$

Even better, it has function spectra

$$\underline{G\mathcal{S}pt}(X, Y) = \{k \mapsto \underline{G\mathcal{S}pt}^0(X, Y^{k+?})\}$$

If X is a G-spectrum we could define the homotopy orbit and fixed point spectra pointwise, i.e.

$$X_{hG} = \{k \mapsto (X^k)_{hG})\}$$

and

$$\text{``}X^{hG} = \{k \mapsto (X^k)^{hG}\}\text{''}$$

These construction obviously preserve pointwise equivalences, but just as the homotopy limit naïvely defined (without the $\sin|-|$, see appendix A.1.9) may not preserve weak equivalences, some care is needed in the stable case.

Pointwise homotopy orbits always preserve stable equivalences, but pointwise homotopy fixed points may not. (it is the old story: since all spectra are pointwise equivalent to cofibrant ones in the stable model category of spectra, pointwise homotopy colimits are well behaved with respect to stable equivalences, but homotopy limits are only well behaved on the fibrant spectra). However, if the spectrum X is an Ω-spectrum, stable and pointwise equivalences coincide, and this may always be assured by applying the construction $QX = \{k \mapsto Q^k X = \lim_{\overrightarrow{n}} \Omega^n X^{k+n}\}$ of appendix A.1.2. This is encoded in the real definition.

Definition C.2.0.3 Let G be a simplicial monoid and X a G-spectrum. Then the *homotopy orbit spectrum* is given by

$$X_{hG} = \{k \mapsto (X^k)_{hG})\}$$

whereas the *homotopy fixed point spectrum* is given by

$$X^{hG} = \{k \mapsto (Q^k X)^{hG}\}$$

Lemma C.2.0.4 *Let G be a simplicial monoid and $f\colon X \to Y$ a map of G-spectra. If f is a stable equivalence of spectra, then $f_{hG}\colon X_{hG} \to Y_{hG}$ and $f^{hG}\colon X^{hG} \to Y^{hG}$ are stable equivalences.*

For G-spectra X there are spectral sequences

$$E^2_{p,q} = H_p(BG; \pi_q X) \Rightarrow \pi_{p+q}(X_{hG})$$

and

$$E^2_{p,q} = H^{-p}(BG; \pi_q X) \Rightarrow \pi_{p+q}(X^{hG})$$

which can be obtained by filtering EG.

C.2.1 The norm map for finite groups

Except for an occasional S^1-homotopy fixed point, we will mostly concern ourselves with finite groups. In these cases the theory simplifies considerably, so although some of the considerations to follow have more general analogs (see e.g., [71] or [44]) we shall restrict our statements to this context. We have to define the *norm map* $X_{hG} \to X^{hG}$, and we use ideas close to [134].

Consider the weak equivalence

$$Q^0(\underline{\mathbf{S}} \wedge G_+) \to Q^0(\textstyle\prod_G \underline{\mathbf{S}}) \cong \prod_G Q^0\underline{\mathbf{S}}$$

(we should really have written $\underline{\mathcal{S}_*}(G_+, Q^0\underline{\mathbf{S}})$ instead of $\prod_G Q^0\underline{\mathbf{S}}$ since that makes some of the manipulations below easier to guess, but we chose to stay with the less abstract notion for now). Regard this as a $G\times G$ map under the multiplication $(a,b)g = bga^{-1}$, and we have an equivalence

$$\underline{\mathcal{S}_*}(E(G\times G)_+, Q^0(\underline{\mathbf{S}} \wedge G_+))^{G\times G} \xrightarrow{\ \sim\ } \underline{\mathcal{S}_*}(E(G\times G)_+, \textstyle\prod_G Q^0(\underline{\mathbf{S}})^{G\times G}$$

We have a preferred point in the latter space, namely the one defined by the diagonal

$$\Delta \in (\prod_G S^0)^{G\times G} \subseteq \prod_G Q^0\underline{\mathbf{S}})^{G\times G} \subseteq \underline{\mathcal{S}_*}(E(G\times G)_+, \prod_G Q^0\underline{\mathbf{S}})^{G\times G}.$$

Note that the homotopy class of Δ represents the "norm" of the finite group G in the usual sense:

$$[\Delta] = \sum_{g\in G} g \in \mathbf{Z}[G] = \pi_0\Omega^l \prod_G S^l.$$

Now, pick the point f in $\mathcal{S}_*(E(G\times G)_+, Q^0(\underline{\mathbf{S}} \wedge G_+))^{G\times G}$ in the component of

$$[\Delta] \in \pi_0\underline{\mathcal{S}_*}(E(G\times G)_+, Q^0(\underline{\mathbf{S}} \wedge G_+))^{G\times G}$$

of your choice (we believe in the right to choose freely, the so called axiom of choice, but now that you've chosen we ask you kindly to stick to your choice).

So we have a preferred $G\times G$-map

$$E(G\times G)_+ \xrightarrow{\ f\ } Q^0(\underline{\mathbf{S}} \wedge G_+)$$

such that the composite with

$$Q^0(\underline{\mathbf{S}} \wedge G_+) \to Q^0(\prod_G \underline{\mathbf{S}})$$

is homotopic (by a homotopy we may fix once and for all) to the projection onto S^0 followed by the diagonal. Otherwise said, if $f'\colon EG_+ \to \underline{\mathcal{S}_*}(EG_+, Q^0(G_+ \wedge \underline{\mathbf{S}}))$ is the adjoint of f, then

$$S^0 \xrightarrow{\ \sim\ } EG_+ \xrightarrow{\ f'\ } \underline{\mathcal{S}_*}(EG_+, Q^0(G_+ \wedge \underline{\mathbf{S}})) \longrightarrow Q^0(G_+ \wedge \underline{\mathbf{S}})$$

maps the nonbasepoint of S^0 into the component $\sum_{g\in G} g \in \pi_0(G_+\wedge\underline{\mathbf{S}})$. Using this map, we define

$$\tau\colon X \to QX$$

to be the composite $X = S^0\wedge X \to Q(G_+\wedge\underline{\mathbf{S}})\wedge X \to Q(G_+\wedge X) \to QX$. On homotopy groups it is simply the endomorphism of π_*X given by multiplication with $\sum_{g\in G} g \in \pi_0(G_+\wedge\underline{\mathbf{S}})$.

ite groups}

Proposition C.2.1.1 *Let G be a finite group and X a G-spectrum. Then there is a natural map, called the norm map*

$$N\colon X_{hG} \to X^{hG}$$

such that the composite $X \twoheadrightarrow X_{hG} \xrightarrow{N} X^{hG} \hookrightarrow QX$ *equals τ. If $X = G_+\wedge Y$ with trivial G action on Y, then the norm map is an equivalence.*

Proof: For each $k \in \mathbf{N}$ let ν be the name of the composite

$$\begin{array}{ccccc} EG_+\wedge X^k & \xrightarrow{f'\wedge 1} & \underline{\mathcal{S}_*}(EG_+, Q^0(G_+\wedge\underline{\mathbf{S}}))\wedge X^k & & \\ & & \downarrow & & \\ & & \underline{\mathcal{S}_*}(EG_+, Q^0(G_+\wedge\underline{\mathbf{S}}\wedge X^k)) & \xrightarrow{\mu} \underline{\mathcal{S}_*}(EG_+, Q^0(\underline{\mathbf{S}}\wedge X^k)) \xrightarrow{\lambda} & \underline{\mathcal{S}_*}(EG_+, Q^kX) \end{array}$$

where μ is induced by the G action of X and λ by the structure map

$$Q^0(\underline{\mathbf{S}}\wedge X^k) = \varinjlim_{n} \Omega^n(S^n\wedge X^k) \to \varinjlim_{n} \Omega^n(X^{n+k}) = Q^kX.$$

From the $G\times G$-structure on f, we get that ν actually factors through the orbits and fixed points:

$$\nu^G\colon EG_+\wedge_G X^k \to \underline{\mathcal{S}_*}(EG_+, Q^kX)^G$$

Varying k, this gives the norm map.

From the commutativity of the diagram

$$\begin{array}{ccccccc} X^k & \longrightarrow & Q^0(G_+\wedge\underline{\mathbf{S}})\wedge X^k & \longrightarrow & Q^0(G_+\wedge\underline{\mathbf{S}}\wedge X^k) & \longrightarrow & Q^kX \\ \downarrow & & \uparrow & & \uparrow & & \uparrow \\ EG_+\wedge X^k & \xrightarrow{f'\wedge 1} & \underline{\mathcal{S}_*}(EG_+, Q^0(G_+\wedge\underline{\mathbf{S}}))\wedge X^k & \longrightarrow & \underline{\mathcal{S}_*}(EG_+, Q^0(G_+\wedge\underline{\mathbf{S}}\wedge X^k)) & \longrightarrow & \underline{\mathcal{S}_*}(EG_+, Q^kX) \\ \downarrow & & & & & & \uparrow \\ (X_{hG})^k & & & \xrightarrow{\qquad N \qquad} & & & (X^{hG})^k \end{array}$$

where the top row is τ, the second claim follows.

The last statement may be proven as follows. If $X = G_+\wedge Y$ then consider the commutative diagram

$$\begin{array}{ccccccc} \pi_*(G_+\wedge Y) & \longrightarrow & \pi_*(G_+\wedge Y)_{hG} & \longrightarrow & \pi_*(G_+\wedge Y)^{hG} & \longrightarrow & \pi_*(G_+\wedge Y) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \bigoplus_G \pi_*Y & \xrightarrow{\nabla} & \pi_*Y & & \pi_*Y & \xrightarrow{\Delta} & \bigoplus_G \pi_*Y \end{array}$$

where $\nabla(g \mapsto y_g) = \sum_g y_g$, and $\Delta(y) = \{g \mapsto y\}$. The "missing" arrow can of course be filled in as the vertical maps are isomorphisms, but there is only one map $\pi_* Y \to \pi_* Y$ making the bottom composite the norm, namely the identity. ∎

{cor:A3 n

Corollary C.2.1.2 *Let U be a finite free G-space and Y a G-spectrum. Then the norm maps*

$$(U \wedge Y)_{hG} \to (U \wedge Y)^{hG}$$

and

$$\underline{\mathcal{S}_*}(U, Y)_{hG} \to \underline{\mathcal{S}_*}(U, Y)^{hG}$$

are both equivalences.

Proof: By induction on G-cells in U, reduce to the case $U = S^n \wedge G_+$. Use a shear map as in the proof C.1.2 to remove action from $S^n \wedge Y$ and $\underline{\mathcal{S}_*}(S^n, Y)$ in the resulting expressions. Note the stable product to sum shift in the last case. Use the proposition C.2.1.1. ∎

We have one very important application of this corollary:

{cor:A3 n

Corollary C.2.1.3 *Let U be a finite free G-space, and X any G-space. Then there is a chain of natural equivalences*

$$\lim_{\overrightarrow{k}} \Omega^k (Map_*(U, S^k \wedge X)_{hG}) \simeq Map_*(U, \lim_{\overrightarrow{k}} \Omega^k (S^k \wedge X))^{hG}.$$

If U is d-dimensional and X n-connected, then

$$Map_*(U, X)^G \to Map_*(U, \lim_{\overrightarrow{k}} \Omega^k (S^k \wedge X))^{hG}$$

is $2n - d + 1$ connected.

Proof: Recall that $Map_*(-,-) = \underline{\mathcal{S}_*}(-, \sin|-|)$. Corollary C.2.1.2 tells us that the norm map

$$\lim_{\overrightarrow{k}} \Omega^k (Map_*(U, S^k \wedge X)_{hG}) \xrightarrow{\sim} \lim_{\overrightarrow{k}} \Omega^k (\lim_{\overrightarrow{l}} \Omega^l Map_*(U, S^l \wedge S^k \wedge X))^{hG}$$

is an equivalence, and the latter space is equivalent to $Map_*(U, \lim_{\overrightarrow{k}} \Omega^k S^k \wedge X)^{hG}$ by lemma A.1.1.7.1 since U and EG_+ (and G) are finite. The last statement is just a reformulation of lemma C.1.2.2 since $X \to \lim_{\overrightarrow{k}} \Omega^k (S^k \wedge X)$ is $2n + 1$ connected by the Freudenthal suspension theorem A.1.10.0.9. ∎

C.3 Circle actions and cyclic homology

The theory for finite groups has a nice continuation to a theory for compact Lie groups. We will only need one case: $G = \mathbf{S^1} = \sin|S^1|$, and in an effort to be concrete, we cover that case in some detail. For the more general theory, please consult other and better sources.

If X is an $\mathbf{S^1}$-spectrum, we can also consider the homotopy fixed points under the finite subgroups $C \subset \mathbf{S^1}$. As $Map_*(E\mathbf{S^1}_+, X) \to Map_+(EC_+, X)$ is a C-equivariant homotopy we can calculate X^{hC} equally well as $Map_*(E\mathbf{S^1}_+, X)^C$. Thus, if $C' \subseteq C$ is a subgroup, we can think of $X^{hC} \to X^{hC'}$ most conveniently as the inclusion $Map_*(E\mathbf{S^1}_+, X)^C \subseteq Map_*(E\mathbf{S^1}_+, X)^{C'}$.

Lemma C.3.0.4 *If X is an $\mathbf{S^1}$-spectrum and p some prime, then the natural map*

$$X^{h\mathbf{S^1}} \to \underset{\overleftarrow{r}}{\operatorname{holim}}\, X^{hC_{p^r}}$$

is an equivalence after p-completion.

Proof: This is just a reformulation of A.1.9.8.5 ∎

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Lemma C.3.0.5 *Let Y be a spectrum, and let the functorial (in Y) $\mathbf{S^1}$-map of spectra*

$$f' \colon \mathbf{S^1}_+ \wedge Y \xrightarrow{\sim} Map_*(\mathbf{S^1}_+, \mathbf{S^1} \wedge Y)$$

be the adjoint of the composite

$$\mathbf{S^1}_+ \wedge \mathbf{S^1}_+ \xrightarrow{+} \mathbf{S^1}_+ \xrightarrow{pr} \mathbf{S^1}$$

smashed with Y. Then f' is an equivalence of spectra.

Proof: The diagram

$$\begin{array}{ccccc}
Y^l & \longrightarrow & \mathbf{S^1}_+ \wedge Y^l & \xrightarrow{pr} & \mathbf{S^1} \wedge Y^l \\
\downarrow & & \downarrow & & \downarrow \simeq \\
Map_*(\mathbf{S^1}, \mathbf{S^1} \wedge Y^l) & \xrightarrow{pr^*} & Map_*(\mathbf{S^1}_+, \mathbf{S^1} \wedge Y^l) & \longrightarrow & \sin|\mathbf{S^1} \wedge Y^l|
\end{array}$$

commutes, and both horizontal sequences are (stable) fiber sequences of spectra (when varying l). The outer vertical maps are both stable equivalences, and the so the middle map (which is the map in question) must also be a stable equivalence. ∎

Corollary C.3.0.6 *If Y is a spectrum, then there is a natural chain of stable equivalences $(Y \wedge \mathbf{S^1}_+)^{hS^1} \simeq S^1 \wedge Y$.*

Proof: The lemma gives us that

$$\begin{aligned}
(\mathbf{S^1}_+ \wedge Y)^{hS^1} &\xrightarrow{\sim} Map_*(\mathbf{S^1}_+, \mathbf{S^1} \wedge Y)^{hS^1} \simeq Map_*(E\mathbf{S^1}_+ \wedge \mathbf{S^1}_+, Q(\mathbf{S^1} \wedge Y))^{S^1} \\
&\cong Map_*(E\mathbf{S^1}_+, Q(\mathbf{S^1} \wedge Y)) \simeq Q(\mathbf{S^1} \wedge Y) \simeq S^1 \wedge Y
\end{aligned}$$

∎

C.3.1 The norm for $\mathbf{S^1}$-spectra

Nothing of what follows are new ideas, but since we have stubbornly insisted on giving explicit models for everything we do, we offer the following brief explanation of the $\mathbf{S^1}$-norm. See [44] for a fuller description of a theory containing the discussion below as a particular example.

First use the stable equivalence (and $\mathbf{S^1}\times\mathbf{S^1}$-map) from lemma C.3.0.5 with $Y=\underline{\mathbf{S}}^{-1}$:

$$\mathbf{S^1}_+\wedge\underline{\mathbf{S}}^{-1}\to\underline{\mathcal{S}_*}(\mathbf{S^1}_+,\underline{\mathbf{S}}),$$

inducing an equivalence

$$\underline{\mathcal{S}_*}(E(\mathbf{S^1}\times\mathbf{S^1})_+,Q^0(\mathbf{S^1}\wedge\mathbf{S}^{-1}))^{\mathbf{S^1}\times\mathbf{S^1}}\xrightarrow{\sim}\underline{\mathcal{S}_*}(E(\mathbf{S^1}\times\mathbf{S^1})_+,Q^0(\underline{\mathcal{S}_*}(\mathbf{S^1},\mathbf{S})))^{\mathbf{S^1}\times\mathbf{S^1}}.$$

The latter space has a preferred element, given by $()_+$ of the unbased $\mathbf{S^1}\times\mathbf{S^1}$-map $\mathbf{S^1}\to *$, under the map

$$\underline{\mathcal{S}_*}(\mathbf{S^1}_+,S^0)^{\mathbf{S^1}\times\mathbf{S^1}}\to Q^0\underline{\mathcal{S}_*}(\mathbf{S^1}_+,\mathbf{S})^{\mathbf{S^1}\times\mathbf{S^1}}\to\underline{\mathcal{S}_*}(E(\mathbf{S^1}\times\mathbf{S^1})_+,Q^0(\underline{\mathcal{S}_*}(\mathbf{S^1},\mathbf{S})))^{\mathbf{S^1}\times\mathbf{S^1}},$$

and we choose an element in $\mathcal{S}_*(E(\mathbf{S^1}\times\mathbf{S^1})_+,Q^0(\mathbf{S^1}\wedge\mathbf{S}^{-1}))^{\mathbf{S^1}\times\mathbf{S^1}}$ which is sent to this homotopy class.

Let X be an $\mathbf{S^1}$-spectrum and consider the following composite of $\mathbf{S^1}\times\mathbf{S^1}$-maps

$$\begin{array}{rl}E(\mathbf{S^1}\times\mathbf{S^1})_+\wedge X_{k+1}\xrightarrow{f\wedge 1} & Q^0(\mathbf{S^1}_+\wedge\mathbf{S}^{-1})\wedge X_{k+1}\\ & Q^0(\mathbf{S^1}_+\wedge\mathbf{S}^{-1}\wedge X_{k+1})\xrightarrow{\mu}Q^0(\mathbf{S}^{-1}\wedge X_{k+1})\longrightarrow Q^kX\end{array}.$$

The adjoint of this composite

$$E\mathbf{S^1}_+\wedge X_{k+1}\to\underline{\mathcal{S}_*}(E\mathbf{S^1}_+,Q^kX)$$

factors through orbits and fixed points

$$(E\mathbf{S^1}_+\wedge X_{k+1})_{\mathbf{S^1}}\to\underline{\mathcal{S}_*}(E\mathbf{S^1}_+,Q^kX)^{\mathbf{S^1}}$$

to define the norm map $S^1\wedge X_{h\mathbf{S^1}}\to X^{h\mathbf{S^1}}$. The norm map is obviously functorial in the $\mathbf{S^1}$-spectrum X.

C.3.2 Cyclic spaces

{Cyclic s

Recall the relevant notions: Let Λ be the category with the same objects as Δ, but with morphism sets given by

$$\Lambda([p],[q])=\Delta([p],[q])\times C_{p+1}$$

with composition subject to the extra relations (where t_n is the generator of C_{n+1})

$$\begin{aligned}t_nd^i&=d^{i-1}t_{n-1} && 1\le i\le n\\ t_nd^0&=d^n\\ t_ns^i&=s^{i-1}t_{n+1} && 1\le i\le n\\ t_ns^0&=s^nt^2_{n+1}\end{aligned}$$

A *cyclic object* in some category $\mathcal{C}$ is a functor $\Lambda^o \to \mathcal{C}$ and a *cyclic map* is a natural transformation between cyclic objects. Due to the inclusion $j\colon \Delta \subset \Lambda$, any cyclic object X gives rise to a simplicial object j^*X.

As noted by Connes [19], this is intimately related to objects with a circle action (see also [59], [28] and [6]). In analogy with the standard n-simplices $\Delta[n] = \{[q] \mapsto \Delta([q],[n])$, we define the cyclic sets $\Lambda[n] = \Lambda(-,[n])\colon \Lambda^o \to \mathcal{E}ns$.

Lemma C.3.2.1 ([28]) *For all n, $|j^*\Lambda[n]|$ is an $\mathbb{T}$-space, naturally (in $[n] \in ob\Lambda^o$) homeomorphic to $\mathbb{T} \times |\Delta[n]|$.*

Proof: ([28]). Consider the the "twisted product" $\Delta[1] \times_t \Delta[n]$ whose q-simplicies are $(q+1)$-tuples of pairs of integers

$$(0,i_0)\dots(0,i_a),(1,i_{a+1}),\dots(1,i_q)$$

where $0 \le i_{a+1} \le \dots \le i_q \le i_0 \le \dots \le i_a \le n$ and $0 \le a \le q$ and the obvious face and degeneracy maps . Note that $j^*\Lambda[n]$ is the quotient of $\Delta[1] \times_t \Delta[n]$ by identifying $((0,i_0),\dots,(0,i_q))$ and $((1,i_0),\dots,(1,i_q))$. Furthermore, write out......NBNB ∎

∎

There are lots of adjoint functors that are nice to have: if $\mathcal{C}$ is a category with finite sums we get an adjoint pair

$$\mathcal{C}^{\Lambda^o} \underset{j^*}{\overset{N^{cy}}{\leftrightarrows}} \mathcal{C}^{\Delta^o}$$

where the cyclic nerve N^{cy} is the left adjoint given in degree q by $N^{cy}X([q]) = \bigvee_{C_{q+1}} X_q$ and with a twist in the simplicial structure, just as in the proof of the lemma above (in fact, $\Lambda[n] \cong N^{cy}\Delta[n]$, see e.g., [74, 7.1.5] for more details where N^{cy} is called F).

More concretely, we also have an adjoint pair

$$\mathbb{T} - Top_* \underset{\sin_\Lambda}{\overset{|-|_\Lambda}{\leftrightarrows}} \mathcal{E}ns_*^{\Lambda^o}$$

given by

$$|X|_\Lambda = \int^{[q]\in\Lambda^o} |\Lambda[q]|_\Lambda\, {}_+\wedge X_q = \coprod_{[q]\in\Lambda^o} |\Lambda[q]|_\Lambda\, {}_+\wedge X_q/\sim$$

where X is a cyclic set and $|\Lambda[q]|_\Lambda$ is $|j^*\Lambda[q]| \cong \mathbb{T} \times |\Delta[n]|$ considered as a $\mathbb{T}$-space, and

$$\sin_\Lambda Z = \{[q] \mapsto \mathbb{T} - Top(|\Lambda[q]|_\Lambda, Z)\}$$

for Z a $\mathbb{T}$-space. Note that since $|\Lambda[q]|_\Lambda \cong \mathbb{T} \times |\Delta[n]|$ we have a natural isomorphism

$$j^* \sin_\Lambda Z \cong \sin(UZ)$$

where U denotes the forgetful functor from $\mathbb{T}$-spaces to (topological pointed) spaces (right adjoint to $\mathbb{T}_+\wedge -$). Furthermore, by formal nonsense, if X is a cyclic set and Y is a space (pointed simplicial set!) then we have natural homeomorphisms

$$U|X|_\Lambda \cong |j^*X| \qquad \text{and} \qquad |N^{cy}Y|_\Lambda \cong \mathbb{T}_+\wedge|Y|$$

where U denotes the forgetful functor from $\mathbb{T}$-spaces to (topological pointed) spaces (right adjoint to $\mathbb{T}_+\wedge-$). Occasionally it is more convenient to consider the adjoint pair

$$\mathbb{T}-Top_* \underset{\sin_\Lambda}{\overset{|-|_\Lambda}{\leftrightarrows}} \mathcal{S}_*{}^{\Lambda^o}$$

given by

$$|X|_\Lambda = \int^{[q]\in\Lambda^o} |\Lambda[q]|_{\Lambda\ +}\wedge|X_q| = \coprod_{[q]\in\Lambda^o,[p]\in\Delta^o} |\Lambda[q]|_{\Lambda\ +}\wedge|\Delta[p]|_+\wedge X_q/\sim$$

where X is a cyclic space, and

$$\sin_\Lambda Z = \{[q],[p]\mapsto \mathbb{T}-Top(|\Lambda[q]|_\Lambda\times|\Delta[p]|, Z)\}$$

for Z a $\mathbb{T}$-space. We record the relations in this case in the

Lemma C.3.2.2 *There are natural isomorphisms*

$$j^*\sin_\Lambda Z\cong \sin(UZ), \qquad U|X|_\Lambda\cong|j^*X| \qquad \textit{and} \qquad |N^{cy}Y|_\Lambda\cong\mathbb{T}_+\wedge|Y|$$

where X is a cyclic space, Y a simplicial space and Z a $\mathbb{T}$-space.

This makes it possible to write out the $\mathbb{T}$ action quite explicit: Let X be a cyclic space, then the $\mathbb{T}$-action on $|X|$ is given by

$$\mathbb{T}_+\wedge|X|\cong|N^{cy}X|\to|X|$$

where the last map is induced by the unit of adjunction $N^{cy}X\to X$ (we have suppressed some forgetful functors).

C.3.2.3 Connes' B-operator and the cyclic action

Let M be a cyclic abelian group and G a simplicial abelian group. Then the free cyclic functor takes the form $N^{cy}G=\{[q]\mapsto \mathbf{Z}[C_q]\otimes G\}$ (with the twist as before). Analogous to the stable equivalence $\mathbb{T}_+\wedge X\simeq S^1\wedge X\vee X$ for spectra, we get a natural equivalence $N^{cy}G\simeq(\tilde{\mathbf{Z}}[S^1]\otimes G)\oplus G$, and so a weak map $\tilde{\mathbf{Z}}[S^1]\otimes M\to\tilde{\mathbf{Z}}[S^1]\otimes M)\oplus M\simeq N^{cy}M\to M$, which on homotopy groups takes the form $\pi_{*-1}M\to\pi_*M$. This is the same map as was described for the associated Eilenberg-MacLane specrum in lemma 3.1.1.

Connes' defines the B-operator to be the map

$$M_q \xrightarrow{N=\sum(-1)^{qj}t^j} M_q \xrightarrow{(-1)^q s_q} M_{q+1} \xrightarrow{(1+(-1)^q t)} M_{q+1}.$$

One checks that B satisfies the relations $B\circ B=0$ and $B\circ b+b\circ B=0$ where $b=\sum(-1)^j d_j$. This latter relation implies that B defines a complex

$$\dots \xrightarrow{B} \pi_nM \xrightarrow{B} \pi_{n+1}M \xrightarrow{B} \dots.$$

Bibliography

[1] John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1978.

[2] M. F. Atiyah. *K-theory*. Advanced Book Classics. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989. Notes by D. W. Anderson.

[3] Christian Ausoni and John Rognes. Algebraic K-theory of topological K-theory. *Acta Math.*, 188(1):1–39, 2002.

[4] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[5] A. Jon Berrick. *An approach to algebraic K-theory*, volume 56 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[6] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[7] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):7–8, 57–143, 1994. K-theory (Strasbourg, 1992).

[8] M. Bökstedt and I. Madsen. Algebraic K-theory of local number fields: the unramified case. In *Prospects in topology (Princeton, NJ, 1994)*, volume 138 of *Ann. of Math. Stud.*, pages 28–57. Princeton Univ. Press, Princeton, NJ, 1995.

[9] Marcel Bökstedt. Topological hochschild homology. Preprint, Bielefeld, 1986.

[10] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[11] Armand Borel and Jean-Pierre Serre. Le théorème de Riemann-Roch. *Bull. Soc. Math. France*, 86:97–136, 1958.

[12] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[13] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978.

[14] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[15] Lawrence Breen. Extensions du groupe additif. *Inst. Hautes Études Sci. Publ. Math.*, (48):39–125, 1978.

[16] D. Burghelea, Z. Fiedorowicz, and W. Gajda. Power maps and epicyclic spaces. *J. Pure Appl. Algebra*, 96(1):1–14, 1994.

[17] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120:189–224, 1984.

[18] P. M. Cohn. Some remarks on the invariant basis property. *Topology*, 5:215–228, 1966.

[19] Alain Connes. Cohomologie cyclique et foncteurs Ext^n. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(23):953–958, 1983.

[20] Alain Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.

[21] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, Lecture Notes in Mathematics, Vol. 137, pages 1–38. Springer, Berlin, 1970.

[22] R. Keith Dennis and Michael R. Stein. K_2 of discrete valuation rings. *Advances in Math.*, 18(2):182–238, 1975.

[23] Eduardo J. Dubuc. *Kan extensions in enriched category theory.* Lecture Notes in Mathematics. 145. Berlin-Heidelberg-New York: Springer-Verlag. XVI, 173 p., 1970.

[24] Bjørn Ian Dundas. Relative K-theory and topological cyclic homology. *Acta Math.*, 179(2):223–242, 1997.

[25] Bjørn Ian Dundas. Localization of V-categories. *Theory Appl. Categ.*, 8:No. 10, 284–312 (electronic), 2001.

[26] Bjørn Ian Dundas and Randy McCarthy. Stable K-theory and topological Hochschild homology. *Ann. of Math. (2)*, 140(3):685–701, 1994.

[27] Bjørn Ian Dundas and Randy McCarthy. Topological Hochschild homology of ring functors and exact categories. *J. Pure Appl. Algebra*, 109(3):231–294, 1996.

[28] W. G. Dwyer, M. J. Hopkins, and D. M. Kan. The homotopy theory of cyclic sets. *Trans. Amer. Math. Soc.*, 291(1):281–289, 1985.

[29] W. G. Dwyer and D. M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.

[30] F. Thomas Farrell and Lowell E. Jones. Rigidity in geometry and topology. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 653–663, Tokyo, 1991. Math. Soc. Japan.

[31] Z. Fiedorowicz, C. Ogle, and R. M. Vogt. Volodin K-theory of A_∞-ring spaces. *Topology*, 32(2):329–352, 1993.

[32] Vincent Franjou, Jean Lannes, and Lionel Schwartz. Autour de la cohomologie de Mac Lane des corps finis. *Invent. Math.*, 115(3):513–538, 1994.

[33] Vincent Franjou and Teimuraz Pirashvili. On the Mac Lane cohomology for the ring of integers. *Topology*, 37(1):109–114, 1998.

[34] Ofer Gabber. K-theory of Henselian local rings and Henselian pairs. In *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, volume 126 of *Contemp. Math.*, pages 59–70. Amer. Math. Soc., Providence, RI, 1992.

[35] S. M. Gersten. K-theory of free rings. *Comm. Algebra*, 1:39–64, 1974.

[36] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[37] Thomas G. Goodwillie. Cyclic homology, derivations, and the free loopspace. *Topology*, 24(2):187–215, 1985.

[38] Thomas G. Goodwillie. On the general linear group and Hochschild homology. *Ann. of Math. (2)*, 121(2):383–407, 1985.

[39] Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math. (2)*, 124(2):347–402, 1986.

[40] Thomas G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.

[41] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[42] Daniel R. Grayson. The K-theory of endomorphisms. *J. Algebra*, 48(2):439–446, 1977.

[43] Daniel R. Grayson. On the K-theory of fields. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 31–55. Amer. Math. Soc., Providence, RI, 1989.

[44] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.

[45] T. Gunnarsson, R. Schwänzl, R. M. Vogt, and F. Waldhausen. An un-delooped version of algebraic K-theory. *J. Pure Appl. Algebra*, 79(3):255–270, 1992.

[46] Jean-Claude Hausmann and Dale Husemoller. Acyclic maps. *Enseign. Math. (2)*, 25(1-2):53–75, 1979.

[47] Lars Hesselholt. On the p-typical curves in Quillen's K-theory. *Acta Math.*, 177(1):1–53, 1996.

[48] Lars Hesselholt. Witt vectors of non-commutative rings and topological cyclic homology. *Acta Math.*, 178(1):109–141, 1997.

[49] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the K-theory of truncated polynomial algebras. *Invent. Math.*, 130(1):73–97, 1997.

[50] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[51] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[52] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

[53] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[54] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[55] Luc Illusie. *Complexe cotangent et déformations. II.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 283.

[56] Hvedri Inassaridze. *Algebraic K-theory*, volume 311 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.

[57] J. F. Jardine. The homotopical foundations of algebraic K-theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 57–82. Amer. Math. Soc., Providence, RI, 1989.

[58] Mamuka Jibladze and Teimuraz Pirashvili. Cohomology of algebraic theories. *J. Algebra*, 137(2):253–296, 1991.

[59] John D. S. Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87(2):403–423, 1987.

[60] D. M. Kan and W. P. Thurston. Every connected space has the homology of a $K(\pi, 1)$. *Topology*, 15(3):253–258, 1976.

[61] Daniel M. Kan. A combinatorial definition of homotopy groups. *Ann. of Math. (2)*, 67:282–312, 1958.

[62] Christian Kassel. K-théorie relative d'un idéal bilatère de carré nul: étude homologique en basse dimension. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 249–261. Springer, Berlin, 1981.

[63] Christian Kassel. La K-théorie stable. *Bull. Soc. Math. France*, 110(4):381–416, 1982.

[64] Gregory Maxwell Kelly. *Basic concepts of enriched category theory.* London Mathematical Society Lecture Note Series, 64. Cambridge etc.: Cambridge University Press. 245 p., 1982.

[65] Michel A. Kervaire. Smooth homology spheres and their fundamental groups. *Trans. Amer. Math. Soc.*, 144:67–72, 1969.

[66] Michel A. Kervaire. Multiplicateurs de Schur et K-théorie. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages pp 212–225. Springer, New York, 1970.

[67] Michael Larsen and Ayelet Lindenstrauss. Topological Hochschild homology of algebras in characteristic p. *J. Pure Appl. Algebra*, 145(1):45–58, 2000.

[68] Michael Larsen and Ayelet Lindenstrauss. Topological Hochschild homology and the condition of Hochschild-Kostant-Rosenberg. *Comm. Algebra*, 29(4):1627–1638, 2001.

[69] W. G. Leavitt. Modules without invariant basis number. *Proc. Amer. Math. Soc.*, 8:322–328, 1957.

[70] Ronnie Lee and R. H. Szczarba. The group $K_3(Z)$ is cyclic of order forty-eight. *Ann. of Math. (2)*, 104(1):31–60, 1976.

[71] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[72] Ayelet Lindenstrauss. A relative spectral sequence for topological Hochschild homology of spectra. *J. Pure Appl. Algebra*, 148(1):77–88, 2000.

[73] Ayelet Lindenstrauss and Ib Madsen. Topological Hochschild homology of number rings. *Trans. Amer. Math. Soc.*, 352(5):2179–2204, 2000.

[74] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[75] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59(4):569–591, 1984.

[76] Manos Lydakis. Smash products and Γ-spaces. *Math. Proc. Cambridge Philos. Soc.*, 126(2):311–328, 1999.

[77] Hendrik Maazen and Jan Stienstra. A presentation for K_2 of split radical pairs. *J. Pure Appl. Algebra*, 10(3):271–294, 1977/78.

[78] Saunders Mac Lane. Homologie des anneaux et des modules. In *Colloque de topologie algébrique, Louvain, 1956*, pages 55–80. Georges Thone, Liège, 1957.

[79] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[80] Ib Madsen. Algebraic K-theory and traces. In *Current developments in mathematics, 1995 (Cambridge, MA)*, pages 191–321. Internat. Press, Cambridge, MA, 1994.

[81] Bruce A. Magurn, editor. *Reviews in K-theory, 1940–84*. American Mathematical Society, Providence, RI, 1985. Reviews reprinted from Mathematical Reviews.

[82] J. Peter May. *Simplicial objects in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[83] Randy McCarthy. On fundamental theorems of algebraic K-theory. *Topology*, 32(2):325–328, 1993.

[84] D. McDuff and G. Segal. Homology fibrations and the "group-completion" theorem. *Invent. Math.*, 31(3):279–284, 1975/76.

[85] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[86] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[87] where?? MilnorK2=Quillen. . ., . .

[88] Barry Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.

[89] Stephen A. Mitchell. On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint. In *Algebraic topology and its applications*, volume 27 of *Math. Sci. Res. Inst. Publ.*, pages 163–240. Springer, New York, 1994.

[90] Robert Oliver. *Whitehead groups of finite groups*, volume 132 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[91] Hiller or Kratzer. Let a be a perfect ring of characteristic p (meaning that the frobenius homomorphism $a \mapsto a^p$ is an isomorphism), then $k_i(a)$ is uniquely p-divisible for $i > 0$. ., . .

[92] T. Pirashvili. On the topological Hochschild homology of $\mathbf{Z}/p^k\mathbf{Z}$. *Comm. Algebra*, 23(4):1545–1549, 1995.

[93] Teimuraz Pirashvili. Polynomial approximation of Ext and Tor groups in functor categories. *Comm. Algebra*, 21(5):1705–1719, 1993.

[94] Teimuraz Pirashvili. Spectral sequence for Mac Lane homology. *J. Algebra*, 170(2):422–428, 1994.

[95] Teimuraz Pirashvili and Friedhelm Waldhausen. Mac Lane homology and topological Hochschild homology. *J. Pure Appl. Algebra*, 82(1):81–98, 1992.

[96] Daniel Quillen. Letter to graeme Segal ., . .

[97] Daniel Quillen. Cohomology of groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 47–51. Gauthier-Villars, Paris, 1971.

[98] Daniel Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[99] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[100] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[101] John Rognes. Algebraic K-theory of the two-adic integers. *J. Pure Appl. Algebra*, 134(3):287–326, 1999.

[102] John Rognes. Two-primary algebraic K-theory of pointed spaces. *Topology*, 41(5):873–926, 2002.

[103] John Rognes. The smooth Whitehead spectrum of a point at odd regular primes. *Geom. Topol.*, 7:155–184 (electronic), 2003.

[104] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[105] R. Schwänzl, R. M. Vogt, and F. Waldhausen. Topological Hochschild homology. *J. London Math. Soc. (2)*, 62(2):345–356, 2000.

[106] Roland Schwänzl, Ross Staffeldt, and Friedhelm Waldhausen. Stable K-theory and topological Hochschild homology of A_∞ rings. In *Algebraic K-theory (Poznań, 1995)*, volume 199 of *Contemp. Math.*, pages 161–173. Amer. Math. Soc., Providence, RI, 1996.

[107] Stefan Schwede. Stable homotopical algebra and Γ-spaces. *Math. Proc. Cambridge Philos. Soc.*, 126(2):329–356, 1999.

[108] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[109] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[110] Jean-Pierre Serre. *Corps locaux*. Publications de l'Institut de Mathématique de l'Université de Nancago, VIII. Actualités Sci. Indust., No. 1296. Hermann, Paris, 1962.

[111] Nobuo Shimada and Kazuhisa Shimakawa. Delooping symmetric monoidal categories. *Hiroshima Math. J.*, 9(3):627–645, 1979.

[112] V. Srinivas. *Algebraic K-theory*, volume 90 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[113] Ross E. Staffeldt. On fundamental theorems of algebraic K-theory. *K-Theory*, 2(4):511–532, 1989.

[114] suitable ref for the Barratt-Priddy-Quillen theorem. . ., . .

[115] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, 73(2):241–245, 1983.

[116] A. A. Suslin. On the equivalence of K-theories. *Comm. Algebra*, 9(15):1559–1566, 1981.

[117] A. A. Suslin. Algebraic K-theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 222–244, Providence, RI, 1987. Amer. Math. Soc.

[118] Richard G. Swan. Excision in algebraic K-theory. *J. Pure Appl. Algebra*, 1(3):221–252, 1971.

[119] Thomas G. Goodwillie. Letter to F. Waldhausen. August 10 1987.

[120] Thomas G. Goodwillie. Notes on the cyclotomic trace. Lecture notes for a series of seminar talks at MSRI, spring 1990, December 1991.

[121] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85(1):91–109, 1979.

[122] Stavros Tsalidis. On the topological cyclic homology of the integers. *Amer. J. Math.*, 119(1):103–125, 1997.

[123] Stavros Tsalidis. Topological Hochschild homology and the homotopy descent problem. *Topology*, 37(4):913–934, 1998.

[124] B. L. Tsygan. Homology of matrix Lie algebras over rings and the Hochschild homology. *Uspekhi Mat. Nauk*, 38(2(230)):217–218, 1983.

[125] Wilberd van der Kallen, Hendrik Maazen, and Jan Stienstra. A presentation for some $K_2(n, R)$. *Bull. Amer. Math. Soc.*, 81(5):934–936, 1975.

[126] Friedhelm Waldhausen. The k-theory (in his sense) of the category of retractive spaces over a given space x, is equivalent to the product of the suspension spectrum of x and the differentiable whitehead spectrum of x. .

[127] Friedhelm Waldhausen. Algebraic K-theory of generalized free products. I, II. *Ann. of Math. (2)*, 108(1):135–204, 1978.

[128] Friedhelm Waldhausen. Algebraic K-theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.

[129] Friedhelm Waldhausen. Algebraic K-theory of topological spaces. II. In *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, volume 763 of *Lecture Notes in Math.*, pages 356–394. Springer, Berlin, 1979.

[130] Friedhelm Waldhausen. Algebraic K-theory of spaces, a manifold approach. In *Current trends in algebraic topology, Part 1 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 141–184. Amer. Math. Soc., Providence, R.I., 1982.

[131] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[132] Friedhelm Waldhausen. Algebraic K-theory of spaces, concordance, and stable homotopy theory. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 392–417. Princeton Univ. Press, Princeton, NJ, 1987.

[133] Charles Weibel. An introduction to algebraic k-theory. In progress, some of it is on the web, `http://math.rutgers.edu/w̃eibel/Kbook.html`, . .

[134] Michael Weiss and Bruce Williams. Automorphisms of manifolds and algebraic K-theory. II. *J. Pure Appl. Algebra*, 62(1):47–107, 1989.

Index