

Fast Methods for Partial Differential and Integral Equations Problem Set 3

1. Here are four functions:

$$f_1(x) = \cos(200\pi x) + \sin(200\pi x), \quad f_2(x) = \sqrt{1 - (x - 1/10)^2},$$

$$f_3(x) = \frac{1}{1 + 700(x + 1/20)^2}, \quad f_4(x) = |x - 1/2|^3.$$

For each function, compute the 1000 degree Chebyshev interpolant on $[-1, 1]$ and plot the corresponding Chebyshev coefficients of the interpolants. Explain the observed convergence rates of the coefficients. Repeat for $f_5(x) = f_1(x)f_3(x)$ and briefly explain the convergence behavior.

2. In this question we investigate the number of degrees of freedom required to resolve solutions using finite difference compared to spectral methods. Solve the second-order ordinary differential equation

$$u''(x) = -\omega^2 \pi^2 \sin(\omega\pi x), \quad x \in [-1, 1]$$

with $u(\pm 1) = 0$ for $\omega = 10, 100, 1000$ using the finite difference (FD) scheme and the spectral collocation method (colloc). What is the smallest n_ω^{FD} and n_ω^{colloc} required to resolve the solution to an absolute accuracy of approximately 10^{-2} ? How does n_ω^{colloc} increase if the accuracy is changed to 10^{-10} ? Explain why the spectral collocation method is advantageous if an accuracy of 10^{-10} in the solution is desired.

Use the explicit expansion,

$$\sin(\omega\pi x) \sim 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(\omega\pi) T_{2k+1}(x),$$

where J_k is the Bessel function of the first kind, to estimate the asymptotic behavior of n_ω^{colloc} for large ω .

3. Solve the nonlinear ordinary differential equation

$$0.01u''(x) + 2(1 - x^2)u(x) + u(x)^2 = 1, \quad x \in [-1, 1],$$

with $u(-1) = u(1) = 0$ using Newton's method in function space and spectral collocation method. There are many solutions to this equation

(perhaps, eight) so supply Newton's method with the initial guess

$$u^{init}(x) = 2x(x^2 - 1)(1 - 2/(1 + 20x^2))$$

and plot the computed solution and 2-norm of the update for each iteration.

4. Solve (using backslash)

$$u''(x) = -\omega^2 \pi^2 \sin(\omega \pi x), \quad x \in [-1, 1]$$

with $u(\pm 1) = 0$ for $\omega = 10, 100, 1000$ using the ultraspherical spectral method (ultra). Compute n_ω^{ultra} from question 2 and compare to n_ω^{colloc} . If $n_\omega^{ultra} \approx n_\omega^{colloc}$, explain why; if $n_\omega^{ultra} \not\approx n_\omega^{colloc}$, explain why.

Bonus Part: The ultraspherical spectral discretization is, in this question, banded-plus-rank-2. Derive your own backslash based on the Sherman-Morrison formula:

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}.$$

How does your own backslash compare to the inbuilt method?