Fast Methods for Partial Differential and Integral Equations
Problem Set 1

High-Frequency Magnetic Resonance Imaging

Problem Statement

Magnetic resonance imaging (MRI) is a non-invasive medical diagnostic technique, in which the atomic nuclei in the body are excited by a radio-frequency magnetic field. Older MRI systems are built with relatively weak permanent magnetics (e.g. 0.5 Tesla) with corresponding resonant frequencies of around 10-50 MHz. The wavelengths associated with these frequencies far exceed the dimensions of the human body, and the resultant imaging resolutions are insufficient for many advanced medical applications.

In search of higher imaging resolutions, many research institutions are now developing MRI with much higher magnetic fields and resonant frequencies. For example, modern MRI systems built with 7 Tesla superconducting magnets operate at 298.3 MHz. At these high frequencies, the magnetic fields strongly interact with the human body, causing degradations to imaging quality as well as raising concerns about safety. Some parts of the body, such as the brain and the eyes, can be particularly sensitive to changes in temperature caused by the magnetic fields.

Understanding the effects that high-frequency MRI have on the human body begins with a model of the underlying physics. To demonstrate these principles, we have made several simplifying assumptions and approximations that reduce the governing equations to the Helmholtz equation in non-conductive media[1]:

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = v(\mathbf{r}) \tag{1}$$

where $u(\mathbf{r})$ is the electric field strength in volts per meter, $v(\mathbf{r})$ is the electric excitation, k is the wave number, and $\mathbf{r} = (x, y, z)$ is the vector defining the location of the observation point. Note that k has the following relationship with the other constants of electromagnetics:

$$k^2 = \omega^2 \mu \epsilon$$

where $\omega = 2\pi f$ is the angular frequency in radians per second, μ is the magnetic permeability in henries per meter, and ϵ is the electric permittivity in farads per meter.

Question 1 - Constructing a Finite Differences Simulation

As a first attempt, let us construct a finite differences method simulation tool for the general electromagnetic scattering problem in free-space. By definition, the freespace permeability is exactly $\mu_0 = 4\pi \times 10^{-7}$ H/m, and the freespace permittivity ϵ_0 can be calculated by the well-known expression:

$$c_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

where the free-space speed of light is exactly $c_0 = 299792458$ m/s.

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[1]In fact, the Helmholtz expression on the left-hand side is the resultant of the curl-curl operation $\frac{1}{\epsilon} \nabla \times (\frac{1}{\mu} \nabla \times u(\mathbf{r}))$, and produces two other terms which are ignored for the purposes of this problem set. To see the derivation of this relation and exactly what terms were omitted, please refer to page 16 of the 18.325 course notes, available online at http://math.mit.edu/icg/resources/teaching/18.325/notes325-dec4.pdf
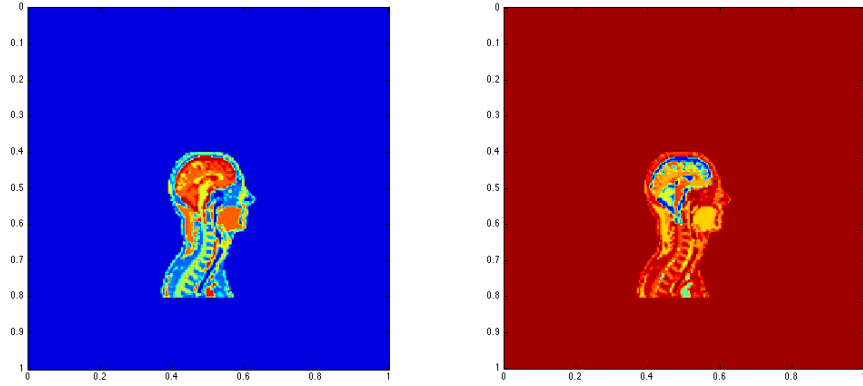
Figure 1: A plot of the $\epsilon_r = \epsilon/\epsilon_0$ provided, generated using the MATLAB code provided. (left) $\mathfrak{Re}\{\epsilon_r\}$; (right) $\mathfrak{Im}\{\epsilon_r\}$.

1. Set up a second-order accurate, 2D finite differences approximation to Equation (1), setting $k^2 = k_0^2 = \omega^2\mu_0\epsilon_0$. Contain the simulation within a square box of $[0,1]^2$ meters, with $n = 256$ divisions along x and y. Set all four boundaries of the box to the homogenous Dirichlet boundary condition: $u_{0,j} = u_{i,0} = u_{256,j} = u_{i,256} = 0$ for $i, j \in [0, 256]$. Let $v(\mathbf{r})$ be an impulse, located at $x = 0.5$ m and $y = 0.5$ m, discretized to a vector of zeros with a single element of $1/h^2$ at the appropriate row.

   (a) Solve the scattering problem using the backslash operator for $f = 21.3$ MHz and $f = 298.3$ MHz, corresponding to the older 0.5 T permanent magnet MRI and the modern 7 T superconductor magnet MRI respectively. Show your results as 2D images. (Hint: you may find the MATLAB commands `spdiags`, `speye`, `kron`, `reshape`, and `imagesc` helpful for this problem.) (4 pts)

2. Using the discrete sine transform (DST-I), implement two fast solvers for the Dirichlet problem: a *second-order accurate* solver, and a *spectrally accurate* solver. Using your two fast solvers and your backslash solver from Problem 1.1, simulate scattering for $f = 298.3$ MHz.

   (a) Produce a single log-log plot of the run-times for all three solvers, with the total number of grid points N on the horizontal axis. (4 pts)

   (b) Take the $n = 256$ results of your spectrally accurate solver as the true solution $\bar{u}$. Produce a single log-log plot of the l^∞-norm relative errors $\|u - \bar{u}\|_\infty / \|\bar{u}\|_\infty$ for all three solvers, with the total number of grid points N on the horizontal axis. (Hint: to obtain the right results, you must use *nested* grids, i.e. a grid at some scale should contain all the points of a grid at some coarser scale. You must also update the value of v at each new grid scale.) (4 pts)

Question 2 - Simulating the EM Scattering of a head

Now that we have an FD model, we can go ahead and simulate the scattering of EM waves by the presence of a human head. Human tissue is slightly capacitive, but non-magnetic. We can model it as a material with magnetic permeability $\mu = \mu_0$, and a location-dependent electric permittivity $\epsilon(\mathbf{r})$ being a function of the observation point $\mathbf{r}$. The presence of the head modifieds Equation (1) to the following:

$$\Delta u(\mathbf{r}) + k^2(\mathbf{r})u(\mathbf{r}) = v(\mathbf{r}) \tag{2}$$

where the wave number k is now too a function of $\mathbf{r}$.

To model the head, we have provided some real MRI data for $\epsilon_r(\mathbf{r}) = \epsilon(\mathbf{r})/\epsilon_0$ in the file `MRI_DATA.mat`. MATLAB code used to generate the reference x and y vectors and the plots shown in Figure 1 are provided below for your reference.

```
load('MRI_DATA');
x = linspace(0,1,257); y = linspace(0,1,257);
subplot(121); imagesc(x,y,real(e_r)); axis image;
subplot(122); imagesc(x,y,imag(e_r)); axis image;
```

1. Set the solution domain to $[0,1]^2$, discretized with $n = 256$ subintervals in the x and y directions. Set all four boundaries of the box to the homogenous Dirichlet boundary condition: $u_{0,j} = u_{i,0} = u_{256,j} = u_{i,256} = 0$ for $i, j \in [0, 256]$. Let us model the MRI current excitation $v(\mathbf{r})$ as an impulse, located at $x = 0.6$ m and $y = 0.7$ m, discretized as a vector of zeros with a single element of $1/h^2$ set at the appropriate row.

   (a) Solve the scattering problem using a second-order accurate scheme with the backslash operator for $f = 21.3$ MHz and $f = 298.3$ MHz, corresponding to the 0.5 T and the 7 T MRI systems respectively. Show your results as 2D images cropped to the head using the code below. (4 pts)

   ```
   sol = A\b; % solve matrix equation.
   sol(real(e_r) <2)=0; % crop solution to the head.
   new_x = linspace(0.35,0.65,100); new_y = linspace(0.4,0.8,100);
   [gx,gy]=meshgrid(new_x,new_y); sol = interp2(x,y,sol,gx,gy,'cubic');
   figure; imagesc(new_x,new_y,abs(sol)); axis image;
   ```

   (b) Is your new A matrix diagonalizable by the DST-I? Justify your answer. (2 pts)

2. The backslash operator becomes very computationally taxing at high discretization values of n, and iterative methods are generally used instead. The recipe for a simple iterative scheme that makes use of our DST solver is the following equation:

$$\Delta u^{(l+1)} + k_0^2 u^{(l+1)} = v(\mathbf{r}) - (k^2(\mathbf{r}) - k_0^2)u^{(l)},$$

   where the notation $u^{(l)}$ denotes the value of the unknown vector u at the l-th iteration of the solver, and k_0 is as defined from Problem 1.1. At each iteration, the right-hand side is evaluated for a given $u^{(l)}$, and solved using the fast DST-I solver to compute $u^{(l+1)}$. Implement this iterative method with the starting vector $u^{(0)}$ set to a vector of zeros, and with the inversion performed using your second-order accurate DST solver from Problem 1.2.

   (a) Keeping $n = 256$, take your results from Problem 2.1 as the true solution $\bar{u}$. For $f = 21.3$ MHz and $f = 298.3$ MHz, produce a log-log plot of the l^∞-norm relative errors $\|u^{(l)} - \bar{u}\|_\infty / \|\bar{u}\|_\infty$, with the iteration count l on the horizontal axis. Does the rate of converge depend on the frequency of excitation? (4 pts)

   (b) Denoting $\delta u^{(l+1)} \equiv u^{(l+1)} - \bar{u}$, show that $\delta u^{(l+1)} = (\Delta + k_0^2)^{-1}(k_0^2 - k^2(\mathbf{r}))\delta u^{(l)}$. Can you give a condition on $k^2(\mathbf{r})$, given a discretization spacing h, that guarantees the iterations will converge? (4 pts)

   (c) Solve the $f = 21.3$ MHz problem for $n = 1024$ using your iterative method, and record the run-time to convergence. Estimate the time it would take the backslash solver of Problem 2.1 to solve the problem at this higher discretization. (2 pts)

We will continue this last problem with a fast integral equation method in the next problem set. The concept of preconditioning an equation in a hetereogenous media by the equation in a nearby homogenous medium will be revisted. Stay tuned!

Bonus Questions - Extra Credit

These questions will give you extra credit for this problem set, but will not take your grade beyond 100%.

1. Prove the mixed-product property of the kronecker product, i.e. for matrices A, B and vectors v, w, show that $(A \otimes B)(v \otimes w) = Av \otimes Bw$ where $\otimes$ denotes the kronecker product. (2 pts)

2. Prove the discrete cyclic convolution theorem, i.e. for vectors f, g, show that $\widehat{f * g} = \hat{f}\hat{g}$, where $*$ denotes the cyclic convolution and $\hat{\ }$ denotes the discrete Fourier transform. (2 pts)