18.336J/6.335J :: Homework 4 :: Spring 2014 :: Due May 8, 2014

Consider the Helmholtz equation problem in the MRI 255-by-255 medium, with zero Dirichlet boundary conditions as described in Problem Set 1. The linear operator L is defined as

$$L = \Delta + \omega^2 \mu_0 \epsilon(\mathbf{r}), \qquad \text{zero Dirichlet b.c.}$$

In the following problem set, we will explore solving the $Lu = f$ problem using the *complex-shifted Laplace preconditioner* M^{-1}, where M is the differential operator

$$M = \Delta + i\omega^2 \mu_0 \epsilon(\mathbf{r}), \qquad i = \sqrt{-1}, \qquad \text{zero Dirichlet b.c.}$$

1. Implement a simple FD ("backslash") solver for the auxiliary problem $Mu = f$, then consider solving the preconditioned problem $M^{-1}Lu = M^{-1}f$ with a GMRES iteration.

   (a) Document how the number of GMRES iterations to reach some fixed accuracy level depends on (a representative set of) ω in the range $[0, \pi n c_0]$ with $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$ and n = 256. (The upper bound on ω is the one that corresponds to the largest wavenumber supported by the grid.) Is the preconditioner bad, good, or excellent at high frequencies? (5pts)

2. Implement a multigrid V-cycle of your choice for solving $Mu = f$. [Hint: we provide a Matlab function `solvepoissonmultigrid.m` to illustrate the implementation of simple a multigrid V-cycle for the 1D Poisson equation with Dirichlet boundary conditions.]

   (a) Compare the overall performance of your multigrid solver to the backslash solver, for the same set of frequencies as you chose earlier. (5 pts)

3. (Bonus) In `solvepoissonmultigrid.m`, the residual smoothing command `res = [0; 1/4*res(I+1) + 1/2*res(I) + 1/4*res(I-1); 0];` is commented out in the code, because downsampling the residual directly is more accurate than downsampling it after smoothing. (You can try uncommenting the line and see the error go up.) How do you explain this behavior? (5pts)