18.336 Homework 3 :: Spring 2011 :: Due April 14

1. (20 pts) Consider the PDE
$$u_t = \pm u_{xxxx}$$
   with initial condition $u(x, 0) = u_0(x)$. Disregard boundary conditions in what follows.

   (a) What sign should you choose in $\pm$ to obtain stable solutions? Justify.

   (b) Assume the proper sign is taken to obtain a stable evolution. What is the stability condition on Δt for the explicit Euler method?

   (c) Your answer to (b) should indicate that the explicit Euler method is highly inefficient for this problem. In one sentence, how would remedy this situation?

   Biharmonic diffusion is sometimes used as a physical dissipation mechanism in fluid mechanics, to damp high frequencies in higher proportion to lower frequencies than regular diffusion does.

2. (80 pts) Consider the same 2D wave equation as in homework 2, with the same initial and boundary conditions. In this question we benchmark spectral vs. FD methods.

   (a) (15 pts) Consider the PDE discretized in time only, using the second order in time method of homework 2. Write the corresponding modified equation. The leading-order correction to the PDE should be a term proportional to $(\Delta t)^2 \Delta^2 u$, where Δ^2 is the square of the Laplacian. Explain how this observation can be used to formulate a fourth order in time method. [Hint: this trick is not unlike the Lax-Wendroff trick. In more generality, it involves calculus of FD operators and should be credited to Tal-Ezer (1986).]

   (b) (15 pts) Explain how you would initialize your scheme in (a) for $u(\Delta t)$, while maintaining fourth order accuracy.

   (c) (20 pts) Combine your method in (a), (b) with a N-by-N tensor Chebyshev grid for the spatial derivatives. Choose the right numbers of points such that the different Chebyshev grids are nested. Show a log-log plot of the error vs. N, and check from this plot that your method is (at least) fourth order accurate. [Hint: programs 19 and 20 in Trefethen's book solve the problem, but with the basic second order in time method. His programs can be downloaded.]

   (d) (15 pts) What is the CFL condition for your scheme? Justify by an analysis of the eigenvalues of the matrix that implements the spatial derivatives. [Hint: you have already found the region of stability for the 3-point timestepping rule in homework 2. For the eigenvalues of a spectral differentiation matrix, see the discussion around p.108 in Trefethen's book.]

   (e) (15 pts) The better comparison between spectral and FD methods is as follows. Fix N, the number of grid points per dimension, to be a reasonable power of 2 like 64 or 128. Consider the initial condition
$$u_t(x, y, 0) = \sin(B\pi x)\sin(B\pi y), \qquad u(x, y, 0) = 0,$$
   for various integer values of B, so that the explicit solution is a known standing wave. Produce a plot of the log of the error of the solution after some time (e.g. $T = 0.75$), as a function of B, both for the spectral method and a finite difference method. Draw a conclusion concerning the number of points per wavelength that each method requires in order to get a fixed error level like 10^{-3} (3 digits).

3. (Bonus, 10 pts). Show that if a function f is integrable and of bounded variation (possibly discontinuous), then its Fourier transform obeys

$$|\hat{f}(\xi)| \le \|f\|_{TV} |\xi|^{-1}.$$