# 18.336 Homework 1 :: Spring 2011 :: Due February 24 (in class or in the box outside my office, 2-392, before 5PM)

1. (20 pts) In this question we generalize the centered difference scheme for the first derivative to higher orders. Assume that the samples of a function are given on a Cartesian grid with spacing $h$, and assume that the point $x$ is on the grid. As in class, define the operator

$$\delta u(x) = u(x+h/2) - u(x-h/2).$$

Also define the averaging operator

$$\mu u(x) = \frac{u(x+h/2) + u(x-h/2)}{2}.$$

Notice that $\mu$ is a handy way to fall back on the grid right after $\delta$ is taken. In fact, $\mu\delta u(x) = hD_c u(x)$, where $D_c$ is the centered difference scheme.

   (a) Show that

$$\mu \left(1 + \frac{1}{4}\delta^2\right)^{-1/2} = 1.$$

   (b) Recall that we saw in class that $hD$ and $\delta$ are related by

$$hD = \delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + \ldots$$

   Explain in one sentence why this equation, in itself, does not give proper FD schemes after truncation.

   (c) Using the results in parts (a) and (b), derive a series for $hD$ as a function of $\mu$ and $\delta$, which gives the desired FD schemes upon truncation. It is fine to only write down the first two terms of this series. [Hint: Pre-multiply the whole equation in part (b) by $\mu\left(1 + \frac{1}{4}\delta^2\right)^{-1/2}$. To treat the right-hand-side, expand the parenthesis in a (Taylor) series, and conclude by multiplying the two series.]

2. (20 pts) In this question we study the approximation properties of finite differences for functions of limited differentiability. Among those, piecewise polynomials are perhaps the most interesting for numerical computations. After subtracting the smooth component, any piecewise polynomials is for some $n \geq 0$ locally of the form

$$u(x) = x_+^n = \begin{cases} x^n & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

Consider the generic situation in which the stencil of $D_c$ straddles the origin in an asymmetric manner, i.e., the two contentious differences are

$$D_c u(-\epsilon) = \frac{u(-\epsilon + h) - u(-\epsilon - h)}{2h}, \qquad D_c u(-\epsilon + h) = \frac{u(-\epsilon + 2h) - u(-\epsilon)}{2h},$$

for some $0 < \epsilon < h$. Take $u(x)$ to be defined on an interval like $[-1, 1]$, so we also consider $D_c u(-\epsilon + mh)$ for all relevant values of the integer $m$.

   (a) For which $n \geq 0$ is $D_c$ consistent for $u(x) = x_+^n$ in the maximum (uniform, $\ell_\infty$) norm? What is the order of accuracy of $D_c$ as a function of $n \geq 0$?

   (b) Same question in the $\ell_1$ norm, defined as $\|E\|_1 = h\sum_j |E_j|$.

   (c) For $n = 3$ we have the important example of cubic splines. Is the Taylor expansion argument for the error of $D_c$ (in the maximum norm) for cubic splines too optimistic / too pessimistic / just right?

3. (60 pts) In this question we solve the 2D Poisson equation of electrostatics using finite differences. Consider that the square $[0,1]^2$ is made of a dielectric material with permittivity 1. It is subjected to (1) a certain pattern of electric potential $f(y)$ on its left side, (2) it is grounded on its right side, and (3) it is insulated on its top and bottom sides. The resulting equation for the potential $u(x,y)$ inside the square, as a function of the excitation $f(y)$, is

$$-\Delta u(x,y) = 0, \qquad x \in [0,1]^2,$$

$$u(0,y) = f(y), \qquad u(1,y) = 0, \qquad 0 \le y \le 1,$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0, \qquad 0 \le x \le 1.$$

Note that the electric current is $\nabla u$. As usual, $-\Delta$ is minus the Laplacian. Unless otherwise stated, assume that

$$f(y) = \cos(2\pi y).$$

(a) Propose a second-order finite difference discretization for minus the Laplacian, which takes into account the boundary conditions. Detail your choice for the number of grid points that you use in $x$ vs. $y$, and whether your cells are square or rectangular. [Hint: written as a block matrix, the result should be of the form

$$-\tilde{\Delta} = A \otimes I + I \otimes B,$$

where $A$ is a matrix for minus the 1D Dirichlet Laplacian, $B$ is a matrix for minus the 1D Neumann Laplacian, and $\otimes$ is the Kronecker product. You already know $A$; this exercise is about finding $B$ and specifying the sizes of $A$, $B$, and the two identities.]

(b) Using this FD scheme, implement a solver for $u(x,y)$. Solve the linear system by an iterative method of your choice. Do not explicitly form a block matrix for the Laplacian, and do not use a direct method such as Matlab's backslash. Illustrate the convergence of your numerical scheme in a log-log plot of the maximum norm of the error vs. the grid spacing h. Check that the slope is approximately 2 in this graph. [Hint: either find the exact solution as a basis for comparison, or use a numerical solution on a very fine grid for that purpose. You may find it useful to use nested grids for the different values of $h$, so that the points on a coarse grid are a subset of the points on a finer grid.]

(c) Argue (very briefly; in one sentence perhaps) the consistency of your scheme. It is fine to assume that the solution $u$ is infinitely differentiable.

(d) What are the eigenvalues and eigenvectors of $A \otimes I + I \otimes B$ as a function of those of $A$ and $B$?

(e) There is a good chance that if the top and bottom rows of your matrix $B$ are scaled by an adequately small number (which does not change the original equation), then the eigenvalues of $B$ are real and positive, except for a zero eigenvalue corresponding to the constant eigenvector. Feel free to check this numerically. If you take this fact for granted, and use the result in (d), show that there is an eigenvalue gap:

$$\lambda_{\min}(-\tilde{\Delta}) \ge C > 0,$$

with $C$ independent of $h$. (This is a good step toward proving stability, but not the end of the story. There are complications due to the fact that $B$ is not symmetric. We won't pursue this further.)

(f) How many steps does the iterative method of your choice require for convergence to within a given error tolerance? Does the theory match the numerics? [Hint: see pages 74 and 75 of LeVeque 2007.]

(g) Replace $f(y)$ given above by

$$\tilde{f}(y) = \text{sgn}(cos(2\pi y)),$$

for the left boundary condition, where sgn($x$) $= 1$ if $x \ge 0$, and $-1$ if $x < 0$. Run your code again. Empirically, what does the order of convergence become? How do you explain this behavior?

4. (Bonus, 10 pts.) Implement a multigrid method of your choice to improve the convergence of the iterative method for question 3. Illustrate the gain in convergence speed in the way you find most adequate.

5. (Bonus, 20 pts). Make your explanation in point (f) quantitative, i.e., repeat the error analysis with $\tilde{f}(y)$ in place of $f(y)$.

Concerning bonus questions: totals will be rounded down to 100.