18.336 Spring 2010 02/02/10.

name, email, office, tel.
OH : TR 2:00-3:30
http://math.mit.edu/18.336

Textbooks:

- R. LeVeque, Finite diff. methods for ordinary & PDE, SIAM 2007
- N. Trefethen, Spectral methods in Matlab, SIAM 2000
- R. LeVeque, Finite volume methods for hyperbolic problems, C.U.P. 2002.

Prereqs:
- ODE, PDE
- Fourier analysis
- Linear algebra, G. Elim.
- Programming (language: you)

Eval: 50% Hw (4), drop lowest, | no late, write own
50% Project

Register: also if listener.

Who took 2.097J, 6.339J, 16.920J

Material (Review)

Hw1: Poisson eq. 2D / multigrid

1. FD: Basics, consistency.
   Elliptic problems.
   Iterative schemes (GS, CG, etc.)
   Multigrid preconditioning

Hw2: Wave eq. 2D, staggered

2. FDT: time-dep. problems.
   Explicit, implicit methods for ODE
   Stability: Von Neumann analysis.
   Lax Equivalence thm.
   Parabolic & Hyperbolic problems.
   Numerical dispersion, group velocity
   Nonreflecting boundary conditions
   Absorbing.

Hw3: Some spectral method.

3. SP: spectral methods.
   Interpolation, differentiation
   FFT.
   (Poisson summation formula)
   Chebyshev grids
   Spectral accuracy

Hw4: Burgers eq., shock capturing

4. FV: Finite volume.
   Conservation laws, shocks
   Riemann problems, Godunov method
   Limiters, TVD methods
   ENO

NOT: FE, BIE.

3

Projects: attached.

Timeline:

02	02	Intro	
	04	FD	
	09	FD	
	11	FD	
	16	FD	
	18	FD,	Choose project & submit
	23	FD	
	25	FDT	Hw1 due
03	02	FDT	
	04	FDT	
	09	FDT	
	11	FDT	Hw2 due
	16	FDT	
	18	FDT (ABC)	

– Break –

	30	SP	
04	01	SP	Project update
	06	SP	
	08	SP	
	13	SP	
	15	FV	Hw3 due
	20	Patriots day	
	22	FV	

04 | 27 FV
| 29 FV $\boxed{\text{Hw 4}}$

05 | 04 FV } or special
| 06 FV }

| 11 } presentations.
| 13 }

$\boxed{\text{Project}}$ report due.

<u>Why?</u> - Central techniques in sc. and tech. ⊛.
→ develop <u>proficiency</u>
- Accent on mathematics
→ develop con<u>fidence</u>
- Leads to research
→ fun, great testbed

⊛: Simulations for: • Inverse problems
• PDE-based control
• Model validation
• its own sake, want features of the solution
• Image processing?

<u>Review of PDE</u>

- PDE = relationship between the p.d. of a function, to be solved for the fn.
- Linear: $\sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u(x) = f(x)$

$\alpha = (\alpha_1, \ldots, \alpha_m)$

$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}$ $\quad |\alpha| = \sum \alpha_i$

(homogeneous if $f=0$)

- Semilinear:

$$\sum_{|\alpha|=m} a_\alpha(x)\,\partial^\alpha u(x) + F(\text{lower order der.}, u, x) = 0$$

- Quasilinear:

$$\sum_{|\alpha|=m} a_\alpha(\text{l.o.d.}, u, x)\,\partial^\alpha u(x) + F(\text{l.o.d.}, u, x) = 0.$$

- Else, fully nonlinear.

Ex. • Linear:

1) Elliptic: Poisson's eq.

$\Delta = \sum_i \partial^2/\partial x_i^2$

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

$f=0$: Laplace eq.
(u is harmonic)

Prop. u is smoother than f
(gain 2 derivatives)

Testbed problem! → FE

Useful: • $\Delta p = f(v, u)$ in fluids
• $\Delta V = \rho$, e-m.

Convection-diff. eq. (time indep.)

$$v \cdot \nabla u + \alpha \Delta u = f$$

$\frac{dx}{dt} = v$ are characteristics

Var. coeff.: $\nabla \cdot D(x) \nabla u = f$

Biharmonic: $\Delta\Delta u = f$

2) <u>Parabolic</u>: Heat equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha \Delta u(x,t) \qquad x \in \Omega, t > 0$$

IC $\quad u(x,0) = f(x)$

BC $\quad u(x,t) = g(x,t) \qquad x \in \partial\Omega, t > 0$

$(g(x,0) = f(x), \ x \in \partial\Omega)$

<u>Prop.</u> Damps out high frequencies in u, = Blur.

Testbed for time integrators (implicit)

Var. coeff. $\frac{\partial u}{\partial t} = \alpha \nabla \cdot \alpha(x) \nabla u$

Convection-diffusion

$$\frac{\partial u}{\partial t} = \alpha \Delta u + v \cdot \nabla u$$

Useful:
- heat transfer $(u = T)$
- derivative pricing, Black-Scholes $(u = \text{price})$
- diffusion of chemicals $(u = \text{conc.})$
- probability, Fokker-Planck

3) <u>Hyperbolic</u>: Wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \alpha \Delta u(x,t) \quad , x \in \Omega, t > 0$$

IC $\quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x)$

BC $\quad u(x,t) = g(x,t), \qquad x \in \partial\Omega, t > 0$

<u>Prop.</u> Information propagates along (charact.) lines $\underline{x} = \underline{x}_0 + ct\underline{e}$

Testbed for dispersion of schemes

Other ex: One-way (transport), Maxwell
Elastic wave eq., Helmholtz

Useful: Acoustic/elastic waves
(linearized gas dynamics)
Electromagnetic waves. Water waves
Tomographic imaging
(medical, seismic, radar, sonar)

4) Dispersive wave eq. (linear):

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi \quad ; \text{ Schrödinger (QM)}$$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + m^2 u \text{ : Klein-Gordon (RQM)}$$

$u_{tt} + \alpha u_t = u_{xx}$: Telegraph eq.

Prop. different wave vectors travel at different speeds.

• Semilinear:

$-\Delta u = f(u)$ nonlinear Poisson

$u_t - \Delta u = f(u)$ reaction-diffusion
(combustion, wild fires)

$u_{tt} - \Delta u = f(u)$ NLW
(conical refraction)

$iu_t + \Delta u = f(u)$ NLS eg $f(u) = u^3$
(nonlinear optics, solitons)

$u_t + u u_x = \alpha u_{xx}$ (Visc-Burgers)

$u_t + u u_x + u_{xxx} = 0$ (KdV, solitons)

- Quasilinear:
  - Burgers $u_t + u u_x = 0$
  - (Viscous B $u_t + u u_x = \alpha u_{xx}$ semilin.)
  - Porous medium $u_t - \Delta(u^\gamma) = 0$
  - Gas dynamics = 2D Burgers
  - Navier-Stokes $\begin{cases} u_t + u\nabla u = \alpha \Delta u - \nabla p \\ \nabla \cdot u = 0 \end{cases}$

    ($\alpha = 0$, Euler) (fluids)
  - many others
- Fully nonlinear:
  - Eikonal $|\nabla u| = n(x)$

    (distance, traveltimes)
  - p-Laplacian $\nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = 0$
  - minimal surfaces $\nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}} \right) = 0$
  - Monge-Ampère eq:

    $\det(\nabla\nabla u) = f$
  - HJ $u_t + H(\nabla u, x) = 0$

18.336 02/04/10

- Review of nonlinear PDE.
- Finite differences, order of accuracy
- Calculus of FD operators
  (Dahlquist and Björck, Numerical methods)

- Semilinear | highest-order der. appears linearly

$-\Delta u = f(u)$ nonlinear Poisson

$u_t - \Delta u = f(u)$ reaction-diffusion

$u_{tt} - \Delta u = f(u)$ NL Wave

$i u_t + \Delta u = f(u)$ NL Schrödinger

$u_t + u u_x = \alpha u_{xx}$ viscous Burgers

$u_t + u u_x + u_{xxx} = 0$ KdV.

- Quasilinear | highest-order der. appears near-linearly (coeff depends on l.o.d.)

$u_t + u u_x = 0$ Burgers

$u_t - \Delta(u^\gamma) = 0$ Porous medium

Gas dynamics, 2D Burgers

$$\begin{cases} u_t + u \cdot \nabla u = \alpha \Delta u - \nabla p \\ \nabla \cdot u = 0 \end{cases}$$

N.S., ($\alpha = 0$ E)

etc.

- Fully nonlinear

$$|\nabla u| = n(x) \qquad \text{eikonal}$$

$$\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) = 0 \qquad p\text{-laplacian}$$

$$\nabla\cdot\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) = 0 \qquad \text{minimal surfaces,}$$

$$\det(\nabla\nabla u) = f \qquad \text{MA.}$$

$$u_t + H(\nabla u, x) = 0 \qquad \text{HJ}$$

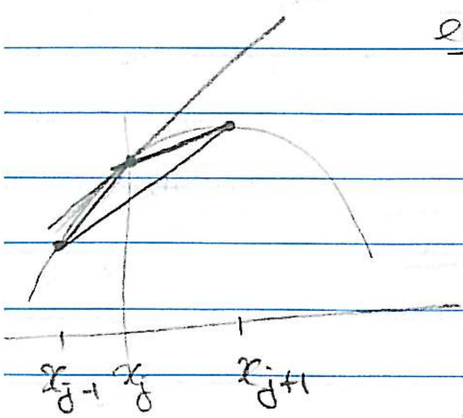
etc.

Finite differences.

Goal: pass from $x \in \mathbb{R}^n$ to x_j, $j = 1, .., N$ and $u_j \simeq u(x_j)$.

Obstacle: Get derivatives, e.g. $(\nabla u)(x_j)$ or $(\Delta u)(x_j)$

Solution: FD



ex $x_j = jh$, $j = 0, .., N$, $h = 1/N$ ($N+1$ pts)

Assume $u(x_j)$ is available

$$u'(x_j) \simeq D_+ u(x_j) = \frac{u(x_{j+1}) - u(x_j)}{h}$$

$$\text{or } D_- u(x_j) = \frac{u(x_j) - u(x_{j-1})}{h}$$

$\rightarrow$ one-sided differences

$$\text{or } D_0 u(x_j) = \frac{u(x_{j+1}) - u(x_{j-1})}{2h}$$

$\rightarrow$ centered difference.

D_+, D_- are slopes of secants on the right, left respectively.

D_0 is the slope of the secant from $(x_{j-1}, u(x_{j-1}))$ to $(x_{j+1}, u(x_{j+1}))$
(interpolating u at x_{j-1}, x_{j+1})

D_+, D_- are first-order accurate:

$$|D_\pm u(x_j) - u'(x_j)| \le C_\pm h$$
when $u \in C^2$

D_0 is second-order accurate:

$$|D_0 u(x_j) - u'(x_j)| \le C_0 h^2$$
when $u \in C^3$.

↳ also written

$$D_0 u(x_j) - u'(x_j) = O(h^2) \quad (h \to 0)$$

↳ big-Oh.

(on the order of h^2, of order h^2)

If error $= O(h^p)$, then method of order p.

<u>Truncation error</u>: Taylor expansion of u about the point x where the derivative is evaluated: (1D)

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{1}{6} h^3 u'''(x) + O(h^4)$$

$$u(x-h) = \quad - \quad + \quad - \quad + O(h^4)$$

$$D_+ u(x) = \frac{u(x+h) - u(x)}{h}$$

$\to \frac{h}{2} u''(\xi)$ $\quad \xi \in [x, x+h]$

$$= u'(x) + \boxed{\frac{h}{2} u''(x) + \ldots}$$

$\to$ OKAY

$\to$ indep. of h

$$= u'(x) + O(h) \qquad \text{order 1}$$

$$D_c u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

$$= u'(x) + \frac{h^2}{6} u'''(x) + \ldots$$

$$= u'(x) + O(h^2). \qquad \text{order 2}$$

Ex. $D_3 u(x) = \dfrac{2u(x+h) + 3u(x) - 6u(x-h) + u(x-2h)}{6h}$

$$= u'(x) + \frac{h^3}{12} u^{(4)}(x) + \ldots$$

$$= u'(x) + O(h^3) \qquad \text{order 3.}$$

Ex. Approximation of the 2^{nd} derivative.

$$D_{c,2} u(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

$$= u''(x) + \frac{h^2}{12} u^{(4)}(x) + \ldots$$

$$= u''(x) + O(h^2) \qquad \text{order 2.}$$

turns out $D_{c,2} u(x) = D_+ D_- u(x)$

Pf

$$D_+D_-u(x) = \frac{D_-u(x+h) - D_-u(x)}{h}$$

$$= \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x-h)-u(x)}{h}}{h}$$

$$= D_{c,2}u(x).$$

Also, let $\hat{D}_c u(x) = \dfrac{u(x+\frac{h}{2}) - u(x-\frac{h}{2})}{h}$

$$\hat{D}_c\hat{D}_c u(x) = \frac{\hat{D}_c u(x+\frac{h}{2}) - \hat{D}_c u(x-\frac{h}{2})}{h}$$

$$= \frac{\frac{u(x+h)-u(x)}{h} - \left(\frac{u(x) - u(x-\frac{h}{2})}{h}\right)}{h}$$

$$= D_{c,2}u(x).$$

How to derive FD approximations?

(a) method of undetermined coefficients

$$D_3 u(x) = au(x+h) + bu(x) + cu(x-h) + du(x-2h)$$

$\rightarrow$ det a, b, c, d s.t. order 3.

(b) Calculus of FD operators

Let $Du(x) = u'(x)$

$$D_+u(x) = \frac{u(x+h)-u(x)}{h}$$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x)$$

$$= \left(1 + hD + \frac{h^2D^2}{2} + \frac{h^3D^3}{3!} + \dots\right)u(x)$$

$$= \exp(hD)\, u(x)$$

↳ exponential of an operator (definition)

Compare with $u_t(x,t) = A\, u(x,t)$, A = generator

$\Rightarrow u(t) = e^{tA} u$

(operator exponential

Here: $t = h$, $A = D$

$u_t(x,t) = u_x(x,t)$ one-way wave eq.

$\Rightarrow u(x,t) = f(x-t)$

D = generator of translations.

$$u(x+h) = e^{hD} u(x)$$

$$u(x+h) = u(x) + h D_+ u(x)$$

$$= (1 + hD_+)\, u(x)$$

$$\Rightarrow e^{hD} = 1 + hD_+$$

$$hD = \log(1 + hD_+) \qquad (\$)$$

operator log

Rmk: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$\Rightarrow hD = hD_+ - \frac{(hD_+)^2}{2} + \frac{(hD_+)^3}{3} + O(h^4)$$

$$u'(x) = \underbrace{\underbrace{(D_+ u)(x)}_{1^{st}\text{ order}} - \frac{h}{2} D_+^2 u(x)}_{2^{nd}\text{ order}} + \frac{h^2}{3} D_+^3 u + O(h^3) \qquad (\$)$$

eg. (2nd order)

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{1}{2}\left(D_+ u(x+h) - D_+ u(x)\right) + O(h^2)$$

$$= \frac{u(x+h) - u(x)}{2} - \frac{1}{2}\left(u(x+2h) - u(x+h) - (u(x+h) - u(x))\right)/h$$

$$= \frac{-\frac{1}{2}u(x+2h) + 2u(x+h) - \frac{3}{2}u(x) + O(h^2)}{h}$$

$\to$ order 2.

why? $\to$ check works when $u = x^p$, only need p terms to get exact result.

$$\log(1 + hD)\, x^p = \sum_{n=1}^{p} \frac{(hD_+)^n}{n} x^p .$$

$\to$ then $u(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} u^{(k)}(x)$,

only $\sum_{p+1}^{\infty} (\ldots)$ is ignored by truncating the log series

" $O(h^{p+1})$

$\hookrightarrow$ this is interesting but not part of the material

02/09.

FD Operator calculus.

Q. Derive higher-order formula that generalize centered differences → HW1.

Q. $$D_{c,2}\, u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$ for $u''(x)$.

$x-h \quad x \quad x+h$

Introduce $\delta u(x) = u(x+\frac{h}{2}) - u(x-\frac{h}{2})$ not on grid!

Then
$$\delta^2 u(x) = \delta u(x+\tfrac{h}{2}) - \delta u(x-\tfrac{h}{2})$$
$$= u(x+h) - u(x) - u(x) + u(x-h)$$
$$= h^2 D_{c,2}\, u(x).$$

Recall $u(x+h) = (e^{hD} u)(x)$ (Taylor exp.)
$$\delta u(x) = (e^{hD/2} u)(x) - (e^{-hD/2} u)(x)$$
$$= (e^{hD/2} - e^{-hD/2})\, u(x)$$
$$= 2 \sinh\left(\frac{hD}{2}\right) u(x)$$

$$\frac{hD}{2} = \sinh^{-1}\left(\frac{\delta}{2}\right)$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

$$\Rightarrow (\ldots) \quad hD = \delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + \cdots$$

(not very useful! not on grid)

$$\boxed{(hD)^2 = \delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} - \frac{\delta^8}{560} + \cdots}$$

$h^2 D_{S2}$

2nd order

4th order

6th order

Concl : (1) Can do one-sided 1st derivative

(2) —— centered 2nd der.

(3) —— centered 1st der. (Hw)

Chap 1

Chap. 2 Example: Boundary-value problem (BVP)

Steady-state eq. for heat equation

u = temperature profile [K]

ψ = heat source/sink [W]

$$\underbrace{u_t(x,t)}_{\substack{\text{say} = 0 \\ \text{(steady-state)}}} = \underbrace{(\kappa(x)\, u_x(x,t))_x}_{\text{say } \kappa(x) = \kappa} + \underbrace{\psi(x,t)}_{\text{say } \psi(x)}$$

Put $f(x) = -\frac{\psi(x)}{\kappa}$

$\Rightarrow \boxed{u''(x) = f(x)}$ f given, want u.

Also specify $x \in [0,1]$,

$\boxed{u(0) = \alpha,\ u(1) = \beta}$

$\Rightarrow$ 2-point BVP

[If specify $u(0)=\alpha$, $u'(0)=\sigma$,
then initial-value problem (IVP)
$\rightarrow$ march forward in x]

- Use heat equation? not very efficient.
- Solve explicitly? Integrate twice, find constants of integration. Not very representative
- Finite differences: good problem for studying relationship of local vs. global errors.

$x_j = jh$, $\quad h = \dfrac{1}{N+1}$ $\left\{\begin{array}{l|l} \text{mesh} & \text{width} \\ \text{grid} & \text{spacing} \end{array}\right.$

$j = 0, 1, \dots, N, N+1$

$x_0 = 0$, $U_0 = \alpha$

$\hookrightarrow x_{N+1} = 1$, $U_{N+1} = \beta$

Use $D_{c,2} U_j = \dfrac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f(x_j)$, $\quad j = 1, \dots, N$

$\Rightarrow AU = F$,

$$\underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) - \frac{\beta}{h^2} \end{bmatrix}}_{F}$$

Get U by solving this tri-diagonal system.

① How? Gaussian elimination : $A = LM$

$$\begin{bmatrix} -2 & +1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \times & \ddots & & \\ & \ddots & \ddots & \\ & & \times & 1 \end{bmatrix} \begin{bmatrix} -\times & \times & & \\ & \ddots & \ddots & \\ & & \ddots & \times \\ & & & \times \end{bmatrix}$$

somethings

$\Rightarrow$ not representative
(fillup of L and M in more general cases)

$\rightarrow$ iterative methods. (how fast)

② How good of an approximation?

Form $\hat{U} = \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix}$, $\quad E = U - \hat{U}$

Some norm:

$$\|E\|_\infty = \max_{1 \le j \le N} |E_j|$$

$$\text{or } \|E\|_1 = h \sum_{j=1}^{N} |E_j|$$

$$\text{or } \|E\|_2 = \left[h \sum_{j=1}^{N} |E_j|^2 \right]^{1/2}$$

Study E :

a) – local truncation error (consistency)
b) – stability
c) $\rightarrow$ convergence.

a) Local truncation error - consistency.

U_j are not the right values for the exact eq.
$u(x_j)$ ——————— num eq.

LTE:

$$\tau_j = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} - f(x_j)$$

$$= u''(x_j) + \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4) - f(x_j)$$

$$= \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4)$$

$$= O(h^2) \quad \text{as } h \to 0.$$

$$\tau = A\hat{U} - F \Rightarrow A\hat{U} = F + \tau \quad (-1)$$

$$AU = F \quad (+1)$$

$$\overline{\quad AE = -\tau \quad}$$

$$\Rightarrow \left| \begin{array}{l} \frac{1}{h^2}\left(E_{j-1} - 2E_j + E_{j+1}\right) = -\tau(x_j) \qquad j=1,..,N \\ \text{with } E_0 = E_{N+1} = 0 \end{array} \right.$$

This system has exactly the same form as before.

Intuitively, discretization of $\begin{cases} e''(x) = -\tau(x) \\ x \in (0,1) \\ e(0) = e(1) = 0 \end{cases}$

Integrate twice: $\tau(x) \simeq \frac{1}{12}h^2 u^{(4)}(x)$

$\Rightarrow$

$$e(x) \simeq -\frac{1}{12}h^2 u''(x) + \frac{1}{12}h^2\left(u''(0) + x(u''(1) - u''(0))\right)$$

$$= O(h^2) \qquad (8)$$

→ circular argument! but $E = O(h^2)$ is correct.

How to make rigorous?

b) Stability: make explicit that we are on a grid w/ spacing h:

$$A^h E^h = -\tau^h$$
$$E^h = -(A^h)^{-1}\tau^h$$
$$\|E^h\| \le \|(A^h)^{-1}\| \, \|\tau^h\|$$

Have control if $\|(A^h)^{-1}\| \le C$ indep. of h.

$$\Rightarrow \|E^h\| \le C\|\tau^h\|$$
$$= O(h^2)$$

Def. Suppose a FD scheme for a BVP gives a sequence $A^h U^h = F^h$. The scheme is stable if $(A^h)^{-1}$ exists for h sufficiently small ($h < h_0$), and $\exists\ C > 0$ s.t. $\|(A_h)^{-1}\| \le C \quad \forall h < h_0$

Def. (Consistency) Same setup, the scheme is consistent if $\|\tau^h\| \to 0$ as $h \to 0$.

c) **Def.** (Convergence) Same setup, the scheme is <u>convergent</u> if
$$\|E^h\| \to 0 \quad \text{as } h \to 0.$$

Prop. If a scheme is consistent & stable, then it is convergent.

Proof. $\|E^h\| = \|(A^h)^{-1} \tau^h\|$
$$\leq C\|\tau^h\| \to 0. \quad \square$$

Called the "fundamental theorem of FD".

$O(h^p)$ local error (LTE) + stability (hard (understand PDE)) $= O(h^p)$ global error

⚠ Haven't shown $\|(A^h)^{-1}\| \leq C$ yet for the 2-point BVP.
→ choose $\|\cdot\|_2$, the 2-norm

d) $$\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$
$$= \max_{1 \leq p \leq N} |\lambda_p(A)| = \rho(A) \quad \text{(spectral radius)}$$
→ only when A is symmetric!

$$\|A^{-1}\|_2 = \max_p |\lambda_p(A^{-1})| = \left[\min_p |\lambda_p(A)|\right]^{-1}$$

→ show smallest eigval is bdd away from zero.

02/11 Example: order of accuracy.

```
N=1000;
h=1/N;
x=0:h:1;                      % N+1 points
u=x-x.^3;   plot(x,u)
dudx = 1-3*x.^2;
I = 2:N;
v= (u(I)-u(I-1))/h;
w=(u(I+1)-u(I-1))/(2*h);
errv= norm(dudx-v, inf)
errw= ________ w ___

errv=[]; errw=[]; hs=[];
for k=1:5
    N= 10^k;   h=1/N;
    hs(k)= h;
    x, u, dudx, I, v, w   as before
    errv(k) = norm(dudx-v, inf);
    errw(k) ________ w ____
end
clf;
loglog(hs, errv, 'b-')
hold on
loglog(hs, errw, 'r-')
```

Test slopes:

```
(log(errs(2:end)) - log(errs(1:end-1))) ./
    (log(hs(2:end)) - log(hs(1:end-1)))
```

(same with w)

```
N = 1e3;   h = 1/N ;  x = [h:h:(1-h)]';
f = ones(N-1,1);                    % right-hand side
e = ones(N-1,1);
A = spdiags([e -2*e e], -1:1, N-1, N-1)/h^2;
A(1:5, 1:5)
B = full(A(1:5,1:5))
u = A \ f;                          % only interior points
                                    %   solved for
clf; plot(x, u)
u_exact = x.^2/2 - x/2 ;
err = u - u_exact;
plot(x, err)
```

Convergence analysis for 2-point BVP

$$\begin{cases} u'' = f, & x \in [0,1] \\ u(0) = \alpha, & u(1) = \beta \end{cases}$$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = F_j$$

$\hookrightarrow$ contains f, α, β

$AU = F$, with

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \times \frac{1}{h^2}$$

Convergence $\Leftarrow$ Stability & Consistency

$$\|E^h\| \leq \|(A^h)^{-1}\| \cdot \|\tau^h\|$$

with $\tau^h = O(h^2)$ when $u(x)$ is smooth (C^2)

Stability?

Analysis in the 2-norm: want $\|(A^h)^{-1}\|_2 \leq C$ indep. h

$$\|M\|_2 = \max_{\|x\|_2=1} \|Mx\|_2 \qquad M \in \mathbb{R}^{m\times m}$$

$$= \max_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2}$$

$$= \max_{1\leq j\leq m} |\lambda_j(M)| = \rho(M) \text{ when } M = M^T$$

$$= \max_{1\leq j\leq m} \underbrace{\sqrt{|\lambda_j(M^TM)|}}_{\sigma_j(M),\text{ singular values}}$$

What about M^{-1}?

(i) Is M^{-1} symmetric?

$$MM^{-1} = I = (MM^{-1})^T$$
$$= (M^{-1})^T M^T$$
$$= (M^{-1})^T M$$
$$\Rightarrow (M^{-1})^T = M^{-1} \quad ✓$$

(ii) λ eigenvalue of M $\quad (\lambda \neq 0)$

$\iff$ $1/\lambda$ eigenvalue of M^{-1}

$$Mv = \lambda v \iff \lambda^{-1} v = M^{-1} v.$$

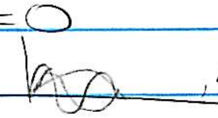
$$\|M^{-1}\|_2 = \max_j |\lambda_j(M^{-1})| = |\lambda|_{max}(M^{-1})$$

$$= 1/|\lambda|_{min}(M)$$

$\Rightarrow$ Show $\lambda_{min}(M)$ bounded away from zero

Eigen-decomposition of A^h.

Continuous problem: $\begin{cases} u'' = \lambda u \\ u(0) = u(1) = 0 \end{cases}$

$$\Rightarrow u_n(x) = \sin(n\pi x)$$  $n \geq 1$

$$\lambda_n = -n^2\pi^2.$$

$$|\lambda|_{min} = \pi^2$$

$$1/|\lambda|_{min} = 1/\pi^2 < \infty.$$

Discrete problem: $\begin{cases} \dfrac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = \lambda U_j \\ U_0 = U_{N+1} = 0 \end{cases}$

$\hookrightarrow j = 0, \underbrace{1, \ldots, N}_{\text{unknowns}}, N+1$

Try $U_j = \sin(n\pi jh)$ $\quad x_j = jh$, $h = \dfrac{1}{N+1}$

$$\Rightarrow \lambda_n = \frac{2}{h^2}\left(\cos(n\pi h) - 1\right)$$

$$= -\frac{4}{h^2} \sin^2\left(\frac{n\pi h}{2}\right)$$

with $\sin x = x - \frac{x^3}{3!} + O(x^5)$

$$\sin^2 x = x^2 + O(x^4)$$

$$\lambda_m = \frac{-4}{h^2}\left(\frac{m\pi h}{2}\right)^2 + O\left(\frac{m^4 h^4}{h^2}\right)$$

$$= -m^2\pi^2 + O(m^4 h^2)$$

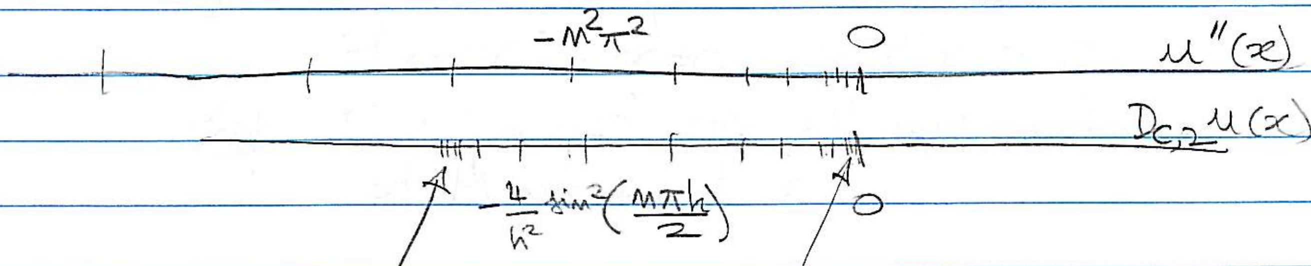
$$|\lambda|_{min} = \pi^2 + O(h^2)$$

$$1/|\lambda|_{min} = 1/\pi^2 + O(h^2) \quad < \infty$$

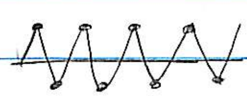
$$\Rightarrow \|(A^h)^{-1}\|_2 = 1/\pi^2 + O(h^2) < \infty$$

stability OK.

Spectrum of the 2^{nd} derivative.



$|\lambda|_{max} \approx 4/h^2$

$U_j^{(m)} =$ 

fast modes
barely resolved
($D_{c,2}$ inaccurate)

$\lambda_m \approx -m^2\pi^2$, same as continuous

$U_j^{(m)} =$ 

slow modes
well resolved.
($D_{c,2}$ accurate)

so $|\lambda|_{min} \simeq \pi^2$

the smoothest nonzero fn. that takes on zero values at the endpoints

(π^2 = Poincaré constant)

→ insensitive to whether the problem is discrete or not.

$|\lambda|_{max} \simeq 4/h^2$..

the most oscillatory function supported on the grid (and which satisfies the BC)

→ depends crucially on h.

also, all $\lambda_j < 0$ $\Rightarrow$ $\frac{d^2}{dx^2}$, A^h are negative definite

Def. $M \in \mathbb{R}^{m \times m}$ is positive definite if $\forall x \in \mathbb{R}^m$, $x \neq 0$, $x^T M x > 0$.

Prop. M is pos def. $\Leftrightarrow$ all eigenvalues of M are positive.

Consistency: $\tau_j^h = D_{C2}\, u(x_j) - f(x_j)$

$$\simeq \frac{h^2}{12} u^{(4)}(x_j)$$

$$= \frac{h^2}{12} f''(x_j) \quad \text{because } u'' = f.$$

→ need $f \in C^2$ for 2^{nd} order

Convergence: $\|E^h\|_2 \leq \|(A^h)^{-1}\|_2 \, \|\tau^h\|_2$

$$\leq \frac{h^2}{12\pi^2} \|f''\|_2 + O(h^4)$$

Chap. 2, section 2.12. Dirichlet B.C.: $u(0)=\alpha$, $u(1)=\beta$.

Neumann boundary cond.: $\begin{cases} u'(0)=\sigma & (N) \\ u(1)=\beta & (D) \end{cases}$

U_j, $j=0,\dots,N+1$

$U_{N+1}=\beta$

U_0 is our unknown.

Impose e.g. $\frac{U_1-U_0}{h}=\sigma$ => add a row

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & 0 & \cdots & 0 \\ -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & 1 & -2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \\ U_N \end{bmatrix}}_{U}=\underbrace{\begin{bmatrix} \sigma \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ f(x_N)-\frac{\beta}{h^2} \end{bmatrix}}_{F}$$

Only 1st order accurate: LTE for the 1st row

$$\tau_0=\frac{1}{h}\left(u(x_1)-u(x_0)\right)-\sigma$$

$$=u'(x_0)+\frac{1}{2}hu''(x_0)+O(h^2)-\sigma$$

$$=\frac{1}{2}hu''(x_0)+O(h^2)$$

=> $E_j \sim h$ as well.

-> More accurate one-sided FD.

Use U_{-1}, impose eq. at U_0, centered diff.