18.336 Homework 2 :: Spring 2010 :: Due March 18

1. (10 pts) Consider the function
$$f(x) = e^{-400(x-0.5)^2}$$
for $x \in [0,1]$. Sample it on a grid $x_j = jh$ with $h = 1/N$ and $0 \leq j \leq N$, for some N to be determined. Consider the linear interpolant of f computed from the $N+1$ samples $f(x_j)$. Numerically, find the smallest value of N such that f differs from its linear interpolant by at most 10^{-2} in the uniform sense (infinity norm).

2. (30 pts) The hyperbolic system for 2D acoustics (without the physical constants) is
$$u_t = -p_x,$$
$$v_t = -p_y,$$
$$p_t = -u_x - v_y.$$

Fix the initial conditions to be
$$u(x,y,0) = 0, \qquad v(x,y,0) = 0, \qquad p(x,y,0) = f(x)f(y),$$
where f was defined in problem 1. For the boundary conditions,

- Set $p = 0$ everywhere on the boundary (homogeneous Dirichlet B.C.)
- Set $u = 0$ at $y = 0, 1$, and set $u_x = 0$ at $x = 0, 1$.
- Set $v = 0$ at $x = 0, 1$, and set $v_y = 0$ at $y = 0, 1$.

Consider a spatial grid $\mathbf{x_j} = (x_{j_1}, y_{j_2}) = (j_1 \Delta x, j_2 \Delta x)$ with Δx small enough to resolve the initial condition, in the sense of problem 1. (I.e. take Δx less than $h = 1/N$, where the critical N was found in problem 1.)

   (a) The "simplest" numerical method for this system uses centered differences in space, and uses the Euler explicit method for the time discretization. For instance, the first equation is discretized as
$$U_{j_1,j_2}^{n+1} = U_{j_1,j_2}^{n} - \frac{\Delta t}{2\Delta x}(P_{j_1+1,j_2}^{n} - P_{j_1-1,j_2}^{n}). \tag{1}$$
   Implement this scheme and test it. Would you recommend it to a friend?

   (b) By looking at the local truncation error alone, what are the orders of accuracy in time and in space for this method?

   (c) Perform the von Neumann stability analysis for this method. Can the method be made stable by taking Δt adequately small? Does your answer explain the behavior observed in part (a)?

3. (25 pts) Consider the Lax-Friedrichs modification of the scheme in problem 2, i.e. for the first equation write instead
$$U_{j_1,j_2}^{n+1} = \frac{U_{j_1-1,j_2-1}^{n} + U_{j_1+1,j_2-1}^{n} + U_{j_1-1,j_2+1}^{n} + U_{j_1+1,j_2+1}^{n}}{4} - \frac{\Delta t}{2\Delta x}(P_{j_1+1,j_2}^{n} - P_{j_1-1,j_2}^{n}).$$

The first term in the right-hand side of (1) is replaced by its average over the four neighboring points. Consider the analogous modifications for the other two equations.

Implement this scheme. What are the orders of accuracy in space and in time? Repeat the stability analysis. Produce a log-log plot of the error for the solution at time $T = 0.75$, as a function of decreasing grid spacings. (This involves making a decision about Δt.) What do you like and what do you dislike about the Lax-Friedrichs method?

4. (25 pts) Consider the leap-frog version of the scheme in problem 2, i.e. for the first equation write instead

$$U_{j_1,j_2}^{n+1} = U_{j_1,j_2}^{n-1} - \frac{\Delta t}{\Delta x}(P_{j_1+1,j_2}^n - P_{j_1-1,j_2}^n).$$

The first term in the right-hand side is now at time t_{n-1}. Notice how the factor 2 at the denominator is gone in the second term of the right-hand side. Consider the analogous modifications for the other two equations.

Implement this scheme. Repeat the stability analysis. Produce a log-log plot of the error for the solution at time $T = 0.75$ as a function of decreasing grid spacings. (This involves making a decision about Δt.) What are the orders of accuracy in space and in time? What do you like and what do you dislike about the leap-frog method?

5. (10 pts) Produce one picture of what you believe is a good approximation of the solution to the equations of acoustics at time $T = 0.75$, using either the Lax-Friedrichs or the leap-frog method. Use the initial conditions provided.

6. (Bonus, 5 pts) A von Neumann analysis that uses plane waves $e^{i\mathbf{k}\cdot\mathbf{x}_j}$ is not rigorously valid, because plane waves do not obey the Dirichlet boundary conditions. Comment briefly on how you would modify the von Neumann analysis so that it would work for zero Dirichlet boundary conditions.

7. (Bonus, 5 pts) Find the explicit solution of the non-discretized acoustics problem.

8. (Bonus, 5 pts) Implement a scheme which is 2nd order in time and 4th order in space for the acoustics problem. Show a log-log plot of the error to back up your claim.

Homework 3 will have a question that requires comparing the finite difference schemes developed here with a spectral method, so please consider saving your scripts.

The methods in problems 3 and 4 are built on staggered grids: the stencils for two neighboring grid points do not overlap in space-time. As a result, there are two disjoint subsets of points in space-time that are not connected by the numerical evolution. This feature of staggered grids is common to most finite difference schemes used in practice for solving linear hyperbolic problems such as those governing acoustic waves, elastic waves, and electromagnetic waves (Maxwell).