18.336 Homework 1 :: Spring 2010 :: Due February 25

1. (30 pts) In this question you will generalize the centered difference scheme for the first derivative to arbitrarily high orders. Assume that the samples of a function are given on a Cartesian grid with spacing h, and assume that the point x is on the grid. As in class, define the operator

$$\delta u(x) = u(x + h/2) - u(x - h/2).$$

Also define

$$\mu u(x) = \frac{u(x + h/2) + u(x - h/2)}{2}.$$

Notice that μ is a handy way to fall back on the grid right after δ is taken. In fact, $\mu\delta u(x) = hD_c u(x)$, where D_c is the centered difference scheme.

   (a) Show that

$$\mu \left(1 + \frac{1}{4}\delta^2\right)^{-1/2} = 1.$$

   (b) Recall that we saw in class that hD and δ are related by

$$hD = \delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + \ldots$$

   Why does this equation, in itself, not give proper FD schemes after truncation?

   (c) Using the results in parts (a) and (b), derive a series for hD as a function of μ and δ, which gives the desired FD schemes upon truncation. It is fine to only write down the first two terms of this series. [Hint: Pre-multiply the whole equation in part (b) by $\mu\left(1 + \frac{1}{4}\delta^2\right)^{-1/2}$. To treat the left-hand side, use the result in part (a). To treat the right-hand-side, expand the parenthesis in a (Taylor) series, and conclude by multiplying the two series.]

2. (70 pts) In this question you will solve the 2D Poisson equation of electrostatics using finite differences. Consider that the square $[0, 1]^2$ is made of a dielectric material with permittivity 1. It is subjected to (1) a certain pattern of electric potential $f(y)$ on its left side, (2) it is grounded on its right side, and (3) it is insulated on its top and bottom sides. The resulting equation for the potential $u(x, y)$ inside the square, as a function of the excitation $f(y)$, is

$$\Delta u(x, y) = 0, \qquad x \in [0, 1]^2,$$

$$u(0, y) = f(y), \qquad u(1, y) = 0, \qquad 0 \le y \le 1,$$

$$\frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, 1) = 0, \qquad 0 \le x \le 1.$$

As usual, Δ is the Laplacian. The boundary conditions are of mixed Dirichlet-Neumann type. Unless otherwise stated, assume that

$$f(y) = \cos(2\pi y).$$

   (a) Write down and implement a second-order accurate numerical scheme for $u(x, y)$. Solve the linear system by an interative method of your choice – not a direct method. Illustrate the convergence of your numerical scheme in a log-log plot of some norm of the error vs. the grid spacing h. Check that the slope is approximately -2 in this graph. [Hint: the exact solution may be unavailable for comparison, but you can use a numerical solution on a fine grid instead. You may find it useful to use nested grids for the different values of N, so that the points on a coarse grid are a subset of the points on a finer grid.]

(b) Argue the consistency of your scheme. It is fine to assume that the solution u is infinitely differentiable.

(c) Argue the stability and the convergence of your scheme.

(d) How many steps does the iterative method of your choice require for convergence to within a given error tolerance? [Hint: see pages 74 and 75 of LeVeque 2007.]

(e) Implement a multigrid method of your choice to improve the convergence. Illustrate the gain in convergence speed by some plot of the logarithm of the error as a function of the number of iterations.

(f) Replace $f(y)$ given above by

$$\tilde{f}(y) = \text{sgn}(cos(2\pi y)),$$

for the left boundary condition, where sgn$(x) = 1$ if $x \geq 0$, and -1 if $x < 0$. Run your code again. What does the order of convergence become? How do you explain this behavior?

(g) (Bonus, 10 pts). Make your explanation in point (f) quantitative, i.e., repeat the error analysis with $\tilde{f}(y)$ in place of $f(y)$.