18.327 Computational Inverse Problems – Spring 2018
Problem set 4 – Due 05/10/2018

Problems are labeled ($\star$) for easy, ($\star\star$) for medium, and ($\star\star\star$) for hard. For homework 4, solve (at least) five stars worth of questions. Recommended exercise: 6

1. ($\star$) Let
$$\sigma_k(x)_p = \inf_y \{\|x-y\|_p : \|y\|_{\ell_0} = k\}.$$
If $q > p > 0$, and $x \in \ell_p$, show that
$$\sigma_k(x)_q \leq \frac{1}{k^{\frac{1}{p}-\frac{1}{q}}} \|x\|_p.$$

2. ($\star$) Compute the proximal operator for the ℓ_p norm, when $1 < p < \infty$.

3. ($\star$) Show that the Chambolle-Pock primal-dual algorithm
$$\begin{aligned} y_{k+1} &= \mathrm{prox}_{\lambda f^*}(y_k + \lambda \overline{x}_k) \\ x_{k+1} &= \mathrm{prox}_{\lambda g}(x_k - \lambda y_{k+1}) \\ \overline{x}_{k+1} &= 2x_{k+1} - x_k \end{aligned}$$
is the same as Douglas-Rachford splitting
$$\begin{aligned} \tilde{x}_{k+1} &= \frac{1}{2}\left(I + \mathrm{rprox}_{\lambda f} \circ \mathrm{rprox}_{\lambda g}\right)(\tilde{x}_k) \\ x_{k+1} &= \mathrm{prox}_{\lambda g}(\tilde{x}_{k+1}) \end{aligned}$$
when $\tilde{x}_{k+1} = x_k - \lambda y_{k+1}$.

4. ($\star\star$) *Recovery of positive vectors*

   (a) Find the subdifferential $\partial f(x)$ for $f(x) = 1_{x\geq 0}(x)$, the indicator of the cone of vectors with non-negative components:
$$1_{x\geq 0}(x) = \begin{cases} 0 & \text{if } x_i \geq 0 \text{ for all } i; \\ \infty & \text{otherwise.} \end{cases}$$
   Justify your answer. [Hint: Your answer should involve I, the set of nonzero components of x.]

   (b) Consider the feasibility problem
$$\min 0 : \qquad Ax = y, \qquad x \geq 0.$$
   When $y = Ax_0$, the vector x_0 is a feasible point ("minimizer") provided there exists a dual certificate $\eta = A^T\lambda \in \partial f(x)$. Find a stronger condition on η, and a condition on A_I (the column restriction of A to the support of x_0), such that x_0 is *the unique* such feasible point. Justify your answer.

5. ($\star\star\star$) *Recovery of positive semidefinite matrices*

   (a) Find the subdifferential $\partial f(X)$ for $f(X) = 1_{X\succeq 0}(X)$, the indicator of the cone of positive semidefinite, symmetric matrices:
$$1_{X\succeq 0}(X) = \begin{cases} 0 & \text{if } X \succeq 0; \\ \infty & \text{otherwise.} \end{cases}$$
   Justify your answer. [Hint: Your answer should involve the set T defined in class.]

(b) Consider the feasibility problem

$$\min 0 : \qquad A(X) = y, \qquad X \succeq 0.$$

When $y = A(X_0)$, the matrix X_0 is a feasible point ("minimizer") provided there exists a dual certificate $Y \in \text{Ran } A^* \cap \partial f(X)$. Find a stronger condition on Y, and a condition on A_T (the restriction of A to matrices in T), such that X_0 is *the unique* such feasible point. Justify your answer.

(c) Generalize your answers to $f(X) = \text{tr}(X) + 1_{X \succeq 0}(X)$, and to

$$\min \text{tr}(X) : \qquad A(X) = y, \qquad X \succeq 0.$$

6. ($\star \star \star \star \star$) A much celebrated numerical result concerns the recovery of images from seemingly incomplete information in the Fourier domain, via total variation (TV) minimization. Magnetic resonance imaging is an area where this idea has been particularly powerful. With a (greyscale, large) image of your choice, a subsampling scheme of your choice (random or not) in the Fourier domain, and a proximal algorithm of your choice for TV minimization, illustrate the favorable behavior of TV regularization for the task of recovering the image from the subsampled Fourier measurements. Append your code to your writeup.