18.327 Computational Inverse Problems – Spring 2018
Problem set 2 – Due 03/15/2018

Problems are labeled (∗) for easy, (★★) for medium, and (★★★) for hard. For homework 2, solve (at least) five stars worth of questions. Recommended exercises: 2, 4, 5, 6, 9.

1. (∗) Let $y \in \mathbb{R}^n$, and $B \in \mathbb{R}^{(n-2) \times n}$ be the second-difference (rectangular) matrix

$$
B = \begin{pmatrix}
1 & -2 & 1 & 1 \\
1 & -2 & 1 & \\
\vdots & \ddots & \ddots & \ddots \\
1 & -2 & 1
\end{pmatrix}.
$$

Show that the Tikhonov-regularized problem

$$
\min_x \|x - y\|^2 + \lambda \|Bx\|^2,
$$

in the limit $\lambda \to \infty$, becomes least-squares linear regression.

2. (∗) Consider the primal problem

$$
\min_x f_0(x) \quad \text{subject to} \quad Ax = y,
$$

for some convex and differentiable $f_0(x)$. Relax the constraint by considering the unconstrained Lagrangian formulation

$$
\min_x \phi(x), \quad \phi(x) = f_0(x) + \alpha \|Ax - y\|^2
$$

for some $\alpha > 0$. Show how a solution of (L) can provide a dual feasible vector for (P), and show how this dual feasible vector can be used to formulate a lower bound on the optimal value of (P).

3. (∗) Show that, for $x \in \mathbb{R}^n$, $\|x\|_{\ell_0} = \lim_{p \to 0} \|x\|^p_p$.

4. (∗) Show that $\min_{x \in \mathbb{R}^n} \|x\|_1$ subject to a single linear constraint $a^T x = b$ always has a one-sparse vector as minimizer. (This is perhaps the simplest illustration of the general phenomenon that solutions of underdetermined $\ell_1$ minimization problems tend to be sparse.)

5. (∗) Show that $\min_{a \in \mathbb{R}} \|x - a\|_1$ has the median of the vector $x$ as a minimizer. (This is perhaps the simplest illustration of the general phenomenon that measuring the data misfit in the $\ell_1$ norm tends to be robust to outliers.)

6. (∗) Show that the dual norm for the spectral norm is the nuclear norm.

7. (★★) Show that the nuclear norm is the so-called atomic norm for the dictionary of rank-1 normalized matrices, i.e.,

$$
\|X\|_* = \inf \{ \sum_i c_i : X = \sum_i c_i A_i, \ c_i > 0, \ \text{rank}(A_i) = 1, \ \|A_i\| = 1 \}.
$$

[Hint: Maryam Fazel’s PhD thesis, section 5.1.4]

8. (★★) Recall that $X > 0$ means positive definite, and $X \succeq 0$ means positive semi-definite.

a) Consider a block matrix

$$
M = \begin{pmatrix} A & B \\
B^T & C \end{pmatrix}
$$

with $A \succ 0$. Show that $M \succeq 0$ if and only if $S = C - B^T A^{-1} B \succeq 0$. [Hint: Boyd and Vandenberghe, p. 650, or else perform block-gaussian elimination. The matrix $S$ is called a Schur complement.]
b) Show that the constraint \( \|X\| \leq 1 \) in the spectral norm can be equivalently encoded by the linear matrix inequality
\[
\begin{pmatrix}
I & X \\
X^T & I
\end{pmatrix} \succeq 0.
\]
[Hint: use the result in part a).]

9. (⋆) Show that the “lifting trick” for the phase retrieval problem \( \min_x 0 : |a_i^Tx|^2 = y_i \) can be automatically derived from taking the dual of the dual. Show that strong duality holds. [Hint: the LMI \( X \succeq 0 \) is handled in the Lagrangian by means of a term \(-\langle M, X \rangle\) with \( M \succeq 0 \).]

10. (⋆⋆⋆) Characterize, in any way that you see fit, the convex envelope of the \( \ell_0 \) quasi-norm restricted to the set \( \|x\|_2 \leq 1 \).