18.327 Computational Inverse Problems – Spring 2018
Problem set 1 – Due 03/01/2018

Problems are labeled ($\star$) for easy, or ($\star\star$) for medium. For homework 1, solve (at least) five stars worth of questions. Recommended exercises: 3, 5, 6, 7.

1. ($\star$) Show that the set of vectors normal to (i.e., orthogonal to) $\{x : Ax = b\}$ is the range of A^T. [Hint: you may take for granted the properties of the four fundamental subspaces of A.]

2. ($\star\star$) Consider the set $S = \{x : \|Ax - b\| \leq \epsilon\}$ (an ellipsoid or a tube), and let v be a vector outside of S. Find a formula, or formulate an algorithm, to find the orthogonal projection of v onto S.

3. ($\star$) Consider $\min_x \frac{1}{2}\|Bx - c\|^2$ subject to $Ax = b$, where $\|\cdot\|$ denotes the Euclidean norm. Assuming that the solution is unique, find a formula for it. [Hint: the argument can either be geometric, or can use Lagrange multipliers.]

4. ($\star\star$) Consider $\min_x x^T Ax$ subject to $x^T x \leq 1$, with A symmetric, but not necessarily positive semidefinite. Write the dual problem, and show that strong duality holds. (This is a rare example of nonconvex program where one can directly show strong duality.)

5. ($\star$) If f and g are convex, and ϕ is convex increasing, show that $f + g$, $\max\{f, g\}$, and $\phi \circ f$ (i.e., ϕ composed with f) are convex.

6. ($\star$) Consider $\min_x f_0(x)$ subject to $f_1(x) \leq u$, with f_0 and f_1 convex. Denote by $p(u)$ the optimal objective value, as a function of the parameter u. Show that $p(u)$ is convex.

7. ($\star\star$) Let x and y be points on a grid, let f and g be given nonnegative functions obeying $\sum_x f(x) = \sum_y g(y)$, and consider the optimal transport program

$$\begin{aligned}
\min_{p(x,y)} \sum_{x,y} |x-y|\, p(x,y),& \\
\sum_y p(x,y) = f(x),& \\
\sum_x p(x,y) = g(y),& \\
p(x,y) \geq 0.&
\end{aligned}$$

   Write the dual problem, and show strong duality. [Hint: Slater.] (Intuitively, this program finds the optimal way to "move" a mass density distributed as $f(x)$, to a mass density distributed as $g(y)$. The cost of moving one unit from x to y (or vice-versa) is $|x - y|$, the transport plan is $p(x, y)$, and the overall transport cost is the objective function. The optimal value of the objective is called Earth-mover's distance (EMD), or Wasserstein W_1 distance. Optimal transport is a linear program (LP) with n^2 primal variables, if x and y both take on n values. Passing to the dual is a much more economical way of computing the EMD, because there are (effectively) only $2n$ dual variables. This construction goes by the name Kantorovich duality.)

8. ($\star\star$) Show that the only notion of distance function $d(x, y)$ (nonnegative, symmetric, defined over the positive reals) such that

   - $d(x, y) + d(y, z) = d(x, z)$ for all $x < y < z$,
   - $d(ax, ay) = d(x, y)$ for all $a > 0$,
   - d is twice differentiable when $x \neq y$, and otherwise continuous,

   is $d(x, y) = |\log(x/y)|$, or any positive multiple of it. (In inverse problems, it is sometimes important to choose model quantities so that their pairwise comparisons do not depend on the choice of units, or stays unchanged under the $1/x$ operation. Taking a logarithm before subtracting the quantities has this beneficial effect. For more information, check what Tarantola calls "Jeffreys parameters" in his book "Inverse problems".)