# 18.325 - Waves and Imaging
# Fall 2012 - Class notes

Laurent Demanet

*Draft September 24, 2012*

# Preface

In this text we use the symbol (\$) to draw attention every time a physical assumption or simplification is made.

# Chapter 1

# Wave equations

## 1.1 Physical models

### 1.1.1 Acoustic waves

Acoustic waves are propagating pressure disturbances in a gas or liquid. With $p(x,t)$ the pressure fluctuation (a time-dependent scalar field) and $v(x,t)$ the particle velocity (a time-dependent vector field), the acoustic wave equations read

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\nabla p, \tag{1.1}$$

$$\frac{\partial p}{\partial t} = -\kappa_0 \nabla \cdot v. \tag{1.2}$$

The two quantities $\rho_0$ and $\kappa_0$ are the mass density and the bulk modulus, respectively. They are linked to the wave speed $c$ through $\kappa_0 = \rho_0 c^2$. Initial conditions on $p$ and $v$ must be supplied. A forcing term may be added to the dynamic balance equation (1.1) when external forces (rather than initial conditions) create the waves.

Let us now explain how these equations are obtained from a linearization of Euler's gas dynamics equations in a uniform background medium ($). Consider the mass density $\rho$ as a scalar field. In the inviscid case ($), conservation of momentum and mass respectively read

$$\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

An additional equation, called constitutive relation, must be added to close the system. It typically relates the pressure and the density in an algebraic way, and encodes a thermodynamic assumption about compression and dilation. For instance if the gas is assumed to be ideal, and if the compression-dilation process occurring in the wave is adiabatic reversible (no heat transfer), then $p \sim \rho^\gamma$, $\gamma = 1.4$, where $\sim$ indicates equality up to a dimensional constant. More generally, assume for the moment that the constitutive relation takes the form

$$p = f(\rho)$$

for some scalar function $f$, which we assume differentiable and strictly increasing ($f'(\rho) > 0$ for all $\rho > 0$).

Consider small disturbances off of an equilibrium state:

$$p = p_0 + p_1, \qquad \rho = \rho_0 + \rho_1, \qquad v = v_0 + v_1.$$

In what follows, neglect quadratic quantities of $p_1, \rho_1, v_1$. Consider a medium at rest (\$): $p_0, \rho_0$ independent of $t$, and $v_0 = 0$. After some algebraic simplification the conservation of momentum becomes

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_0 - \nabla p_1.$$

To zero-th order (i.e., at equilibrium, $p_1 = \rho_1 = v_1 = 0$,) we have

$$\nabla p_0 = 0 \qquad \Rightarrow \qquad p_0 \ \text{ constant in } x.$$

To first order, we get

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1,$$

which is exactly (1.1) after renaming $v_1 \to v$, $p_1 \to p$. The constitutive relation must hold at equilibrium, hence $p_0$ constant in $x$ implies that $\rho_0$ is also constant in $x$ (uniform). Conservation of mass becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot v_1 = 0.$$

Differentiate the constitutive relation to obtain $p_1 = f'(\rho_0)\rho_1$. Call $f'(\rho_0) = c^2$, a number that we assume positive. Then we can eliminate $\rho_1$ to get

$$\frac{\partial p_1}{\partial t} + \rho_0 c^2 \nabla \cdot v_1 = 0.$$

This is exactly (1.2) with $\kappa_0 = \rho_0 c^2$.

Conveniently, the equations for acoustic waves in a variable medium $\rho_0(x)$, $\kappa_0(x)$ are obvious modifications of (1.1), (1.2):

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0(x)}\nabla p, \tag{1.3}$$

$$\frac{\partial p}{\partial t} = -\kappa_0(x)\nabla \cdot v. \tag{1.4}$$

A different argument is needed to justify these equations, however. The previous reasoning does not leave room for variable $\rho_0(x)$ or $\kappa_0(x)$. Instead, it is necessary to introduce a more realistic constitutive relation

$$p = f(\rho, s),$$

where $s$ is the entropy. An additional equation for conservation of entropy needs to be considered. The new constitutive relation allows $\rho_0$ and $s_0$ to be functions of $x$ in tandem, although $p_0$ is still (necessarily) uniform in $x$. The reasoning leading to (1.3), (1.4) is the subject of an exercise in section 1.3.

Acoustic waves can take the form of a first-order system of equations, or else a second-order scalar equation. Combining (1.3), (1.4), we get

$$\frac{\partial^2 p}{\partial t^2} = \kappa_0(x)\nabla \cdot (\frac{1}{\rho_0(x)}\nabla p).$$

Initial conditions on both $p$ and $\partial p/\partial t$ must be supplied. This equation may come with a right-hand side $f(x,t)$ that indicates forcing. When $\rho_0$ and $\kappa_0$ are constant, the scalar wave equation reduces to

$$\frac{\partial^2 p}{\partial t^2} = c_0^2\,\Delta p.$$

Waves governed by (1.3), (1.4) belong in the category of hyperbolic waves because they obey conservation of energy. Define

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa_0\nabla\cdot & 0 \end{pmatrix}.$$

Then the acoustic system simply reads

$$\frac{\partial w}{\partial t} = Lw.$$

$L$ is called the generator of the evolution.

**Definition 1.** *The system $\frac{\partial w}{\partial t} = Lw$ is said to be hyperbolic if $L$ is a matrix of first-order differential operators, and there exists an inner product $\langle w, w' \rangle$ with respect to which $L^* = -L$, i.e., $L$ is anti-self-adjoint.*

An adjoint operator such as $L^*$ is defined through the equation[1]

$$\langle Lw, w' \rangle = \langle w, L^* w' \rangle, \qquad \text{for all } w, w'.$$

For instance, in the case of the acoustic system, the proper notion of inner product is (the factor 1/2 is optional)

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho_0 v \cdot v' + \frac{1}{\kappa_0} p p') \, dx.$$

It is an exercise in section 1.3 to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for that inner product, for all $w, w'$.

**Theorem 1.** *If $\frac{\partial w}{\partial t} = Lw$ is a hyperbolic system, then $E = \langle w, w \rangle$ is conserved in time.*

*Proof.*

$$\begin{aligned}
\frac{d}{dt} \langle w, w \rangle &= \langle \frac{\partial w}{\partial t}, w \rangle + \langle w, \frac{\partial w}{\partial t} \rangle \\
&= 2 \, \langle \frac{\partial w}{\partial t}, w \rangle \\
&= 2 \, \langle Lw, w \rangle \\
&= 2 \, \langle w, L^* w \rangle \\
&= 2 \, \langle w, (-L) w \rangle \\
&= -2 \, \langle Lw, w \rangle.
\end{aligned}$$

A quantity is equal to minus itself if and only if it is zero. □

In the case of acoustic waves,

$$E = \frac{1}{2} \int (\rho_0 v^2 + \frac{p^2}{\kappa}) \, dx,$$

[1] The existence of $L^*$ can be traced back to the Riesz representation theorem once $\langle Lw, w' \rangle$ is shown to be a continuous functional of $w$ in some adequate Hilbert space norm.

which can be understood as kinetic plus potential energy. We now see that the factor 1/2 was chosen to be consistent with the physicists' convention for energy.

In the presence of external forcings the hyperbolic system reads $\partial w/\partial t = Lw + f$: in that case the rate of change of energy is determined by $f$.

For reference, common boundary conditions for acoustic waves include

- Sound soft boundary condition: Dirichlet for the pressure, $p = 0$.
- Sound-hard boundary condition: Neumann for the pressure, $\frac{\partial p}{\partial n} = 0$, or equivalently $v \cdot n = 0$.

Another important physical quantity is related to acoustic waves: the acoustic impedance $Z = \sqrt{\rho_0 \kappa_0}$. We will see later that impedance jumps determine reflection and transmission coefficients at medium discontinuities.

### 1.1.2 Elastic waves

Elastic waves are propagating pressure disturbances in solids. The interesting physical variables are

- The displacement $u(x,t)$, a time-dependent vector field. In terms of $u$, the particle velocity is $v = \frac{\partial u}{\partial t}$.
- The strain tensor
$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T),$$
a symmetric time-dependent tensor field.
- The stress tensor $\sigma$, also a symmetric time-dependent tensor field.

For elastic waves, the density $\rho$ is very often assumed independent of $t$ along particle trajectories, namely $\rho_0(x,0) = \rho_0(x + u(x,t), t)$.

The equation of elastic waves in an isotropic medium (where all the waves travel at the same speed regardless of the direction in which they propagate) ($) reads
$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda \nabla \cdot u) + \nabla \cdot (\mu(\nabla u + (\nabla u)^T)). \tag{1.5}$$
where $\rho$, $\lambda$, and $\mu$ may possibly depend on $x$. As for acoustic waves, a forcing term is added to this equation when waves are generated from external forces.

To justify this equation, start by considering the equation of conservation of momentum ("$F = ma$"),

$$\rho \frac{\partial v}{\partial t} = \nabla \cdot \sigma,$$

possibly with an additional term $f(x,t)$ modeling external forces. The notation $\nabla \cdot$ indicates tensor divergence, namely $(\nabla \cdot \sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$. Stress and strain are linked by a constitutive relation called Hooke's law,

$$\sigma = C : \epsilon,$$

where $C$ is the 4-index elastic tensor. In three spatial dimensions, $C$ has 81 components. The colon indicates tensor contraction, so that $(C : \epsilon)_{ij} = \sum_{k\ell} C_{ijk\ell} \epsilon_{k\ell}$.

These equations form a closed system when they are complemented by

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

which holds by definition of $\epsilon$.

At this point we can check that the first-order system for $v$ and $\epsilon$ defined by the equations above is hyperbolic. Define

$$w = \begin{pmatrix} v \\ \epsilon \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix},$$

with

$$L_1 v = \frac{1}{2}(\nabla v + (\nabla v)^T), \qquad L_2 \epsilon = \frac{1}{\rho_0} \nabla \cdot (C : \epsilon).$$

Then, as previously, $\frac{\partial w}{\partial t} = Lw$. An exercise in section 1.3 asks to show that the matrix operator $L$ is anti-selfadjoint with respect to the inner product

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + \epsilon : C : \epsilon) \, dx.$$

The corresponding conserved elastic energy is $E = \langle w, w \rangle$.

Isotropic elasticity is obtained where $C$ takes a special form with 2 degrees of freedom rather than 81, namely

$$C_{ijk\ell} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{i\ell} \delta_{jk} + \delta_{ik} \delta_{j\ell}).$$

We are not delving into the justification of this equation. The two elastic parameters $\lambda$ and $\mu$ are also called Lamé parameters:

- $\lambda$ corresponds to longitudinal waves, also known as compressional, pressure waves (P).
- $\mu$ corresponds to transverse waves, also known as shear waves (S).

Originally, the denominations P and S come from "primary" and "secondary", as P waves tend to propagate faster, hence arrive earlier, than S waves.

With this parametrization of $C$, it is easy to check that the elastic system reduces to the single equation (1.5). In index notation, it reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_i(\lambda \partial_j u_j) + \partial_j(\mu(\partial_i u_j + \partial_j u_i)).$$

For reference, the hyperbolic propagator $L_2$ reduces to

$$L_2 \epsilon = \frac{1}{\rho}(\nabla(\lambda \operatorname{tr} \epsilon) + 2\,\nabla \cdot (\mu \epsilon)), \qquad \operatorname{tr} \epsilon = \sum_i \epsilon_{ii},$$

and the energy inner product is

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + 2\,\mu \operatorname{tr}(\epsilon^T \epsilon') + \lambda (\operatorname{tr} \epsilon)(\operatorname{tr} \epsilon'))\, dx.$$

The elastic wave equation looks like an acoustic wave equation with "2 terms, hence 2 waves". To make this observation more precise, assume that $\lambda$ and $\mu$ are constant (\$). Use some vector identities[2] to reduce (1.5) to

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \nabla (\nabla \cdot u) + \mu \Delta u, \\ &= (\lambda + 2\mu) \nabla (\nabla \cdot u) - \mu \nabla \times \nabla \times u. \end{aligned}$$

Perform the Helmholtz (a.k.a. Hodge) decomposition of $u$ in terms of potentials $\phi$ and $\psi$:

$$u = \nabla \phi + \nabla \times \psi,$$

where $\phi$ is a scalar field and $\psi$ is a vector field[3]. These two potentials are determined up to a gauge choice, namely

$$\phi' = \phi + C, \qquad \psi' = \psi + \nabla f.$$

[2] In this section, we make use of $\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u$, $\nabla \cdot \nabla \times \psi = 0$, and $\nabla \times \nabla \psi = 0$.

[3] Normally the Helmholtz decomposition comes with a third term $h$ which obeys $\Delta h = 0$, i.e., $h$ is harmonic, but under suitable assumptions of decay at infinity the only solution to $\Delta h = 0$ is $h = 0$.

Choose $f$ such that $\psi'$ has zero divergence:

$$\nabla \cdot \psi' = 0 \qquad \Rightarrow \qquad \Delta f = -\nabla \cdot \psi.$$

This is a well-posed Poisson equation for $f$. With this choice of $\psi'$, it holds that

$$\nabla \cdot u = \Delta \phi, \qquad \nabla \times u = \nabla \times \nabla \times u = -\Delta \psi.$$

The elastic wave equation can then be rewritten in terms of $\phi$, $\psi$ as

$$\nabla \left[ \rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi \right] + \nabla \times \left[ \rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi \right] = 0.$$

Take the gradient of this equation to conclude that (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi = \text{ harmonic } = 0.$$

Now that the first term is zero, we get (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi = \nabla (\text{something}) = 0.$$

Hence each potential $\phi$ and $\psi$ solve their own scalar wave equation: one for the longitudinal waves ($\phi$) and one for the transverse waves ($\phi$). They obey a superposition principle. The two corresponding wave speeds are

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \qquad c_S = \sqrt{\frac{\mu}{\rho_0}}.$$

In the limit $\mu \to 0$, we see that only the longitudinal wave remains, and $\lambda$ reduces to the bulk modulus. In all cases, since $\lambda \geq 0$ we always have $c_P \geq \sqrt{2} c_S$: the P waves are indeed always faster (by a factor at least $\sqrt{2}$) than the S waves.

The assumption that $\lambda$ and $\mu$ are constant is a very strong one: there is a lot of physics in the coupling of $\phi$ and $\psi$ that the reasoning above does not capture. Most important is mode conversion as a result of wave reflection at discontinuity interfaces of $\lambda(x)$ and/or $\mu(x)$.

### 1.1.3 Electromagnetic waves

The quantities of interest for electromagnetic waves are:

- Physical fields: the electric field $E$, and the magnetic field $H$,
- Medium parameters: the electric permittivity $\epsilon$ and the magnetic permeability $\mu$,
- Forcings: electric currents $j$ and electric charges $\rho$.

The electric displacement field $D$ and the magnetic induction field $B$ are also considered. In the linearized regime ($), they are assumed to be linked to the usual fields $E$ and $H$ by the constitutive relations

$$D = \epsilon E, \qquad B = \mu H.$$

Maxwell's equations in a medium with possible space-varying parameters $\epsilon$ and $\mu$ read

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's law)} \tag{1.6}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + j \qquad \text{(Ampère's law with Maxwell's correction)} \tag{1.7}$$

$$\nabla \cdot D = \rho \qquad \text{(Gauss's law for the electric field)} \tag{1.8}$$

$$\nabla \cdot B = 0 \qquad \text{(Gauss's law for the magnetic field)} \tag{1.9}$$

The integral forms of these equations are obtained by a volume integral, followed by a reduction to surface equations by Stokes's theorem for (1.6), (1.7) and the divergence (Gauss's) theorem for (1.8), (1.9). The integral equations are valid when $\epsilon$ and $\mu$ are discontinuous, whereas the differential equations strictly speaking are not.

The total charge in a volume $V$ is $\int_V \rho dV$, while the total current through a surface $S$ is $\int_S j \cdot dS$. Conservation of charge follows by taking the divergence of (1.7) and using (1.8):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

In vacuum, or dry air, the parameters are constant and denoted $\epsilon = \epsilon_0$, $\mu = \mu_0$. They have specific numerical values in adequate units.

We now take the viewpoint that (1.6) and (1.7) are evolution equations for $E$ and $H$ (or $D$ and $B$) that fully determine the fields when they are solved forward (or backward) in time. In that setting, the other two equations (1.8) and (1.9) are simply constraints on the initial (or final) condition at $t = 0$. As previously, we may write Maxwell's equations in the more concise hyperbolic form

$$\frac{\partial w}{\partial t} = Lw + \begin{pmatrix} -j/\epsilon \\ 0 \end{pmatrix}, \qquad \text{with } w = \begin{pmatrix} E \\ H \end{pmatrix},$$

provided

$$L = \begin{pmatrix} 0 & \frac{1}{\epsilon}\nabla\times \\ -\frac{1}{\mu}\nabla\times & 0 \end{pmatrix}.$$

The "physical" inner product that makes $L^* = -L$ is

$$\langle w, w' \rangle = \frac{1}{2}\int (\epsilon E E' + \mu H H')\, dx.$$

The electromagnetic energy $E = \langle w, w \rangle$ is conserved when $j = 0$.

It is the balanced coupling of $E$ and $H$ through (1.6) and (1.7) that creates wave-like solutions to Maxwell's equations (and prompts calling the physical phenomenon electromagnetism rather than just electricity and magnetism.) Combining both equations, we obtain

$$\frac{\partial^2 E}{\partial t^2} = -\frac{1}{\epsilon}\nabla \times (\frac{1}{\mu}\nabla \times E),$$

$$\frac{\partial^2 H}{\partial t^2} = -\frac{1}{\mu}\nabla \times (\frac{1}{\epsilon}\nabla \times H).$$

These wave equations may be stand-alone but $E$ and $H$ are still subject to essential couplings.

A bit of algebra[4] reveals the more familiar form

$$\Delta E - \epsilon\mu\frac{\partial^2 E}{\partial t^2} + \frac{\nabla\mu}{\mu} \times (\nabla \times E) + \nabla(E \cdot \frac{\nabla\epsilon}{\epsilon}) = 0.$$

We now see that in a uniform medium, $\epsilon$ and $\mu$ are constant and the last two terms drop, revealing a wave equation with speed

$$c = \frac{1}{\sqrt{\epsilon\mu}}.$$

[4]Using the relations $\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - \Delta F$ again, as well as $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$.

The speed of light is $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. Even when $\epsilon$ and $\mu$ vary in $x$, the last two terms are kinematically much less important than the first two because they involve lower-order derivatives of $E$. They would not, for instance, change the path of the "light rays", a concept that we'll make clear later.

For reference, we now list the jump conditions that the electric and magnetic fields obey at a dielectric interface. These relations can be obtained from the integral form of Maxwell's equations, posed over a thin volume straddling the interface. Let $n$ be the vector normal to a dielectric interface.

$$n \times E_1 = n \times E_2 \qquad \text{(continuous tangential components)}$$

$$n \times H_1 = n \times H_2 + j_S$$

$$n \cdot D_1 = n \cdot D_2 + \rho_S$$

$$n \cdot H_1 = n \cdot H_2 \qquad \text{(continuous normal component)}$$

We have used $j_S$ and $\rho_S$ for surface currents and surface charges respectively. If the two dielectrics correspond to finite parameters $\epsilon_1, \epsilon_2$ and $\mu_1, \mu_2$, then these currents are zero. If material 2 is a perfect electric conductor however, then these currents are not zero, but the fields $E_2$, $H_2$, $D_2$ and $H_2$ are zero. This results in the conditions $n \times E = 0$ ($E$ perpendicular to the interface) and $n \times H = 0$ ($H$ parallel to the interface) in the vicinity of a perfect conductor.

Materials conducting current are best described by a complex electric permittivity $\epsilon = \epsilon' + i\sigma/\omega$, where $\sigma$ is called the conductivity. All these quantities could be frequency-dependent. It is the ratio $\sigma/\epsilon'$ that tends to infinity when the conductor is "perfect". Materials for which $\epsilon$ is real are called "perfect dielectrics": no conduction occurs and the material behaves like a capacitor. We will only consider perfect dielectrics in this class. When conduction is present, loss is also present, and electromagnetic waves tend to be inhibited. Notice that the imaginary part of the permittivity is $\sigma/\omega$, and not just $\sigma$, because we want Ampère's law to reduce to $j = \sigma E$ (the differential version of Ohm's law) in the time-harmonic case and when $B = 0$.

## 1.2 Special solutions

### 1.2.1 Plane waves, dispersion relations

In this section we study special solutions of wave equations that depend on $x$ like $e^{ikx}$. These solutions are obtained if we assume that the time dependence is harmonic, namely if the unknown is $w(x,t)$, then we assume (\$)

$$w(x,t) = e^{-i\omega t} f_\omega(x), \qquad \omega \in \mathbb{R}.$$

The number $\omega$ is called angular frequency, or simply frequency. Choosing $e^{+i\omega t}$ instead makes no difference down the road. Under the time-harmonic assumption, the evolution problem $\frac{\partial w}{\partial t} = Lw$ becomes an eigenvalue problem:

$$-i\omega f_\omega = L f_\omega.$$

Not all solutions are time-harmonic, but all solutions are *superpositions* of harmonic waves at different frequencies $\omega$. Indeed, if $w(x,t)$ is a solution, consider it as the inverse Fourier transform of some $\hat{w}(x,\omega)$:

$$w(x,t) = \frac{1}{2\pi} \int e^{-i\omega t} \hat{w}(x,\omega) d\omega.$$

Then each $\hat{w}(x,\omega)$ is what we called $f_\omega(x)$ above. Hence there is no loss of generality in considering time-harmonic solutions.

Consider the following examples.

- The one-way, one-dimensional wave equation

  $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R}.$$

  Time harmonic solutions $u(x,t) = e^{-i\omega t} f_\omega(x)$ obey

  $$i\frac{\omega}{c} f_\omega = f'_\omega, \qquad x \in \mathbb{R}.$$

  The solution to this equation is

  $$f_\omega(x) = e^{ikx}, \qquad k = \frac{\omega}{c} \in \mathbb{R}.$$

  Evanescent waves corresponding to decaying exponentials in $x$ and $t$ are also solutions over a half-line, say, but they are ruled out by our assumption (\$) that $\omega \in \mathbb{R}$.

  While $\omega$ is the angular frequency (equal to $2\pi/T$ where $T$ is the period), $k$ is called the wave number (equal to $2\pi/\lambda$ where $\lambda$ is the wavelength.) It is like a "spatial frequency", though it is prudent to reserve the word frequency for the variable dual to time. The quantity measured in Hertz [1/s] and also called frequency is $\nu = \omega/(2\pi)$.

The full solution then takes the form

$$u(x,t) = e^{i(kx-\omega t)} = e^{ik(x-ct)},$$

manifestly a right-going wave at speed $c$. If the equation had been $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$ instead, the wave would have been left-going: $u(x,t) = e^{ik(x+ct)}$.

- The $n$-dimensional wave equation in a uniform medium,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \qquad x \in \mathbb{R}^n.$$

When $u(x,t) = e^{-i\omega t} f_\omega(x)$, the eigenvalue problem is called the Helmholtz equation. It is

$$-\omega^2 f_\omega(x) = \Delta f_\omega(x), \qquad x \in \mathbb{R}^n. \tag{1.10}$$

Again, plane waves are solutions to this equation:

$$f_\omega(x) = e^{ik\cdot x},$$

provided $\omega^2 = |k|^2 c^2$, i.e., $\omega = \pm |k| c$. Hence $f_\omega$ is a function that oscillates in the direction parallel to $k$. The full solution is

$$u(x,t) = e^{i(k\cdot x - \omega t)},$$

which are plane waves traveling with speed $c$, along the direction $k$. We call $k$ the *wave vector* and $|k|$ the *wave number*. The wavelength is still $2\pi/|k|$. The relation $\omega^2 = |k|^2 c^2$ linking $\omega$ and $k$, and encoding the fact that the waves travel with velocity $c$, is called the *dispersion relation* of the wave equation.

Note that $e^{ik\cdot x}$ are not the only (non-growing) solutions of the Helmholtz equation in free space; so is any linear combination of $e^{ik\cdot x}$ that share the same wave vector $|k|$. This superposition can be a discrete sum or a continuous integral. An exercise in section 1.3 deals with the continuous superposition with constant weight of all the plane waves with same wave vector $|k|$.

Consider now the general case of a hyperbolic system $\frac{\partial w}{\partial t} = Lw$, with $L^* = -L$. The eigenvalue problem is $-i\omega f_\omega = Lf_\omega$. It is fine to assume $\omega$ real: since $L$ is antiselfadjoint, $iL$ is selfadjoint (Hermitian), hence all the eigenvalues of $L$ are purely imaginary. This is sometimes how hyperbolic systems are defined — by assuming that the eigenvalues of the generator $L$ are purely imaginary.

We still look for eigenfunctions with a $e^{ik\cdot x}$ dependence, but since $w$ and $f_\omega$ may now be vectors with $m$ components, we should make sure to consider

$$f_\omega(x) = e^{ik\cdot x} r, \qquad r \in \mathbb{R}^m.$$

However, such $f_\omega$ cannot in general expected to be eigenvectors of $L$. It is only when the equation is *translation-invariant* that they will be. This means that the generator $L$ is a matrix of differential operators with constant coefficients – no variability as a function of $x$ is allowed. In this translation-invariant setting, and only in this setting, $L$ is written as a multiplication by some matrix $P(k)$ in the Fourier domain. Say that $f$ has $m$ components $(f_1, \ldots f_m)$; then

$$Lf(x) = \frac{1}{(2\pi)^n} \int e^{ik\cdot x} P(k) \hat{f}(k) dk,$$

where $P(k)$ is an $m$-by-$m$ matrix for each $k$. Here $P(k)$ is called the dispersion matrix. We operators such as $L$ *diagonal* in the Fourier domain, with respect to the $k$ variable, because they act like a "diagonal matrix" on vectors of the continuous index $k$ — although for each $k$ the small matrix $P(k)$ is not in general diagonal[5]. In pure math, $P(k)$ is called the multiplier, and $L$ is said to be a multiplication operator in the Fourier domain.

For illustration, let us specialize our equations to the 2D acoustic system with $\rho_0 = \kappa_0 = c = 1$, where

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & 0 \end{pmatrix}.$$

[5]Non-diagonal, translation-variant operators would require yet another integral over a $k'$ variable, and would read $Lf(x) = \frac{1}{(2\pi)^n} \int \int e^{ik\cdot x} Q(k,k') \hat{f}(k') dk'$, for some more complicated object $Q(k,k') \in \mathbb{R}^{m\times m}$. The name "diagonal" comes from the fact that $Q(k,k')$ simplifies as $P(k)\delta(k-k')$ in the translation-invariant case. You can think of $P(k)\delta(k-k')$ as the continuous analogue of $d_i\delta_{ij}$: it is a "diagonal continuous matrix" as a function of $k$ (continuous row index) and $k'$ (continuous column index).

It can be readily checked that

$$P(k) = \begin{pmatrix} 0 & 0 & -ik_1 \\ 0 & 0 & -ik_2 \\ & & \\ -ik_1 & -ik_2 & 0 \end{pmatrix},$$

from which it is apparent that $P(k)$ is a skew-Hermitian matrix: $P^*(k) = -P(k)$.

We can now study the conditions under which $-i\omega f_\omega = L f_\omega$: we compute (recall that $r$ is a fixed vector)

$$L(e^{ik\cdot x}r) = \frac{1}{(2\pi)^n}\int e^{ik'\cdot x}P(k')[\widehat{e^{ik\cdot x}r}](k')dk',$$
$$= \frac{1}{(2\pi)^n}\int e^{ik'\cdot x}P(k')(2\pi)^n\delta(k-k')r\,dk', \qquad = e^{ik\cdot x}P(k)r.$$

In order for this quantity to equal $-i\omega e^{ik\cdot x}r$ for all $x$, we require (at $x = 0$)

$$P(k)\,r = -i\omega\, r.$$

This is just the condition that $-i\omega$ is an eigenvalue of $P(k)$, with eigenvector $r$. We should expect both $\omega$ and $r$ to depend on $k$. For instance, in the 2D acoustic case, the eigen-decomposition of $P(k)$ is

$$\lambda_0(k) = -i\omega_0(k) = 0, \qquad r_0(k) = \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}$$

and

$$\lambda_\pm(k) = -i\omega_\pm(k) = -i|k|, \qquad r_\pm(k) = \begin{pmatrix} \pm k_1/|k| \\ \pm k_2/|k| \\ |k| \end{pmatrix}.$$

Only the last two eigenvalues correspond to physical waves: they lead to the usual dispersion relations $\omega(k) = \pm|k|$ in the case $c = 1$. Recall that the first two components of $r$ are particle velocity components: the form of the eigenvector indicates that those components are aligned with the direction $k$ of the wave, i.e., acoustic waves can only be longitudinal.

The general definition of dispersion relation follows this line of reasoning: there exists one dispersion relation for each eigenvalue $\lambda_j$ of $P(k)$, and $-i\omega_j(k) = \lambda_j(k)$; for short

$$\det\left[i\omega I + P(k)\right] = 0.$$

### 1.2.2 Traveling waves, characteristic equations

We now consider a few examples that build up to the notion of characteristic curve/surface.

- Let us give a complete solution to the one-way wave equation of one space variable in a uniform medium:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x). \tag{1.11}$$

The study of plane wave solutions in the previous section suggests that the variable $x - ct$ may play a role. Let us perform the change of variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

It inverts as

$$x = \frac{\xi + \eta}{2}, \qquad t = \frac{\eta - \xi}{2c}.$$

By the chain rule, e.g.,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t},$$

we get

$$-2c\frac{\partial}{\partial \xi} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}, \qquad 2c\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

With $U(\xi,\eta) = u(x,t)$, the wave equation simply becomes

$$\frac{\partial U}{\partial \eta} = 0,$$

whose general solution is $U(\xi,\eta) = F(\xi)$ for some differentiable function $F$. Hence $u(x,t) = F(x - ct)$. In view of the initial condition, this is

$$u(x,t) = u_0(x - ct).$$

The solutions to (1.11) are all the right-going waves with speed $c$, and nothing else.

The wave propagate along the lines $\xi(x,t) = x - ct =$const. in the $(x,t)$ plane. For this reason, we call $\xi$ the *characteristic coordinate*, and we call the lines $\xi(x,t) =$ const. *characteristic curves.*

Notice that imposing a boundary condition $u(0,t) = v_0(t)$ rather than an initial condition is also fine, and would result in a solution $u(x,t) = v_0(t - x/c)$. Other choices are possible; they are called Cauchy data. However, a problem occurs if we try to specify Cauchy data along a characteristic curve $\xi =$ constant, as $v_0(\eta)$:

1. this choice is not in general compatible with the property that the solution should be constant along the characteristic curves; and furthermore
2. it fails to determine the solution away from the characteristic curve.

In other words, there is a problem with both existence and uniqueness when we try to prescribe Cauchy data on a characteristic curve. This fact will be used in the sequel to define these curves when their geometric intuition becomes less clear.

- Using similar ideas, let us describe the full solution of the (two-way) wave equation in one space dimension,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

The same change of variables leads to the equation

$$\frac{\partial U}{\partial \xi \partial \eta} = 0,$$

which is solved via

$$\frac{\partial U}{\partial \eta}(\xi,\eta) = f(\xi), \qquad U(\xi,\eta) = \int^{\xi} f(\xi')d\xi' + G(\eta) = F(\xi) + G(\eta).$$

The resulting general solution is a superposition of a left-going wave and a right-going wave:

$$u(x,t) = F(x - ct) + G(x + ct).$$

Matching the initial conditions yields d'Alembert's formula (1746):

$$u(x,t) = \frac{1}{2}(u_0(x - ct) + u_0(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy.$$

It is the complete solution to the 1D wave equation in a uniform wave speed $c$. Notice that we now have two families of criss-crossing characteristic curves, given by $\xi(x,t) = \text{const.}$ and $\eta(x,t) = \text{const.}$ Cauchy data cannot be prescribed on either type of characteristics.

- Consider now the wave equation in a variable medium $c(x)$ (technically, acoustic waves on an infinite string with variable bulk modulus):

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

We will no longer be able to give an explicit solution to this problem, but the notion of characteristic curve remains very relevant. Consider an as-yet-undetermined change of coordinates $(x,t) \mapsto (\xi,\eta)$, which generically changes the wave equation into

$$\alpha(x)\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \xi \partial \eta} + \beta(x)\frac{\partial^2 U}{\partial \eta^2} + \left[p(x)\frac{\partial U}{\partial \xi} + q(x)\frac{\partial U}{\partial \eta} + r(x)U\right] = 0,$$

with

$$\alpha(x) = \left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \xi}{\partial x}\right)^2,$$

$$\beta(x) = \left(\frac{\partial \eta}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \eta}{\partial x}\right)^2.$$

The lower-order terms in the square brackets are kinematically less important than the first three terms[6]. We wish to define characteristic coordinates as those along which

$$U(\xi,\eta) \simeq F(\xi) + G(\eta),$$

i.e., "directions in which the waves travel" in space-time. It is in general impossible to turn this approximate equality into an actual equality (because of the terms in the square brackets), but it is certainly possible to choose the characteristic coordinates so that the $\frac{\partial^2 U}{\partial \xi^2}$ and $\frac{\partial^2 U}{\partial \eta^2}$ vanish. Choosing $\alpha(x) = \beta(x) = 0$ yields the same equation for both $\xi$ and $\eta$, here expressed in terms of $\xi$:

$$\left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \xi}{\partial x}\right)^2 = 0. \tag{1.12}$$

[6]In a sense that we are not yet ready to make precise. Qualitatively, they affect the shape of the wave, but not the character that the waves travel with local speed $c(x)$.

This relation is called the *characteristic equation*. Notice that $\xi = x - ct$ and $\eta = x + ct$ are both solutions to this equation in the case when $c(x) = c$ is a constant. But it can be checked that $\xi = x \pm c(x)t$ is otherwise not a solution of (1.12). Instead, refer to the exercise section for a class of solutions to (1.12).

- Consider now the $n$ dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\Delta u = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

A change of variables would now read $(x_1, \dots, x_n, t) \mapsto (\xi, \eta_1, \dots, \eta_n)$. The variable $\xi$ is called characteristic when the coefficient of the leading term $\frac{\partial^2 U}{\partial \xi^2}$ vanishes in the expression of the equation in the new coordinates. This condition leads to the $n$-dimensional version of the characteristic equation

$$\left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)|\nabla_x \xi|^2 = 0. \tag{1.13}$$

The same relations should hold for the other coordinates $\eta_1, \dots, \eta_n$ if they are to be characteristic as well. Equation (1.13) is called a *Hamilton-Jacobi* equation. We now speak of characteristic surfaces $\xi(x,t) = \text{const.}$, rather than curves.

The set of solutions to (1.13) is very large. In the case of constant $c$, we can check that possible solutions are

$$\xi(x,t) = x \cdot k \pm \omega t, \qquad \omega = |k|c,$$

corresponding to more general plane waves $u(x,t) = F(x \cdot k \pm \omega t)$ (which the reader can check are indeed solutions of the $n$-dimensional wave equation for smooth $F$), and

$$\xi(x,t) = \|x - y\| \pm ct, \qquad \text{for some fixed } y, \text{ and } x \neq y,$$

corresponding to concentric spherical waves originating from $y$. We describe spherical waves in more details in the next section. Notice that both formulas for $\xi$ reduce in some sense to $x \pm ct$ in the one-dimensional case.

The choice of characteristic coordinates led to the reduced equation

$$\frac{\partial^2 U}{\partial \xi \partial \eta} + \text{ lower order terms } = 0,$$

sometimes called "first fundamental form" of the wave equation, on the intuitive basis that solutions (approximately) of the form $F(\xi) + G(\eta)$ should travel along the curves $\xi =$ const. and $\eta =$ const. Let us now motivate this choice of the reduced equation in more precise terms, by linking it to the idea that *Cauchy data cannot be prescribed on a characteristic curve.*

Consider $u_{tt} = c^2 u_{xx}$. Prescribing initial conditions $u(x, 0) = u_0$, $u_t(x, 0) = u_1$ is perfectly acceptable, as this completely and uniquely determines all the partial derivatives of $u$ at $t = 0$. Indeed, $u$ is specified through $u_0$, and all its $x$-partials $u_x, u_{xx}, u_{xxx}, \ldots$ are obtained from the $x$-partials of $u_0$. The first time derivative $u_t$ at $t = 0$ is obtained from $u_1$, and so are $u_{tx}, u_{txx}, \ldots$ by further $x$-differentiation. As for the second derivative $u_{tt}$ at $t = 0$, we obtain it from the wave equation as $c^2 u_{xx} = c^2 (u_0)_{xx}$. Again, this also determines $u_{ttx}, u_{ttxx}, \ldots$ The third derivative $u_{ttt}$ is simply $c^2 u_{txx} = c^2 (u_1)_{xx}$. For the fourth derivative $u_{tttt}$, apply the wave equation twice and get it as $c^4 (u_0)_{xxxx}$. And so on. Once the partial derivatives are known, so is $u$ itself in a neighborhood of $t = 0$ by a Taylor expansion — this is the original argument behind the Cauchy-Kowalevsky theorem.

The same argument fails in characteristic coordinates. Indeed, assume that the equation is $u_{\xi\eta} + p u_\xi + q u_\eta + ru = 0$, and that the Cauchy data is $u(\xi, 0) = v_0(\xi)$, $u_\eta(\xi, 0) = v_1(\eta)$. Are the partial derivatives of $u$ all determined in a unique manner at $\eta = 0$? We get $u$ from $v_0$, as well as $u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots$ by further $\xi$ differentiation. We get $u_\eta$ from $v_1$, as well as $u_{\eta\xi}, u_{\eta\xi\xi}, \ldots$ by further $\xi$ differentiation. To make progress, we now need to consider the equation $u_{\xi\eta} + (\text{l.o.t.}) = 0$, but two problems arise:

- First, all the derivatives appearing in the equation have already been determined in terms of $v_0$ and $v_1$, and there is no reason to believe that this choice is compatible with the equation. In general, it isn't. There is a problem of existence.

- Second, there is no way to determine $u_{\eta\eta}$ from the equation, as this term does not appear. Hence additional data would be needed to determine this partial derivative. There is a problem of uniqueness.

The only way to redeem this existence-uniqueness argument is by making sure that the equation contains a $u_{\eta\eta}$ term, i.e., by making sure that $\eta$ is *non*-characteristic.

Please refer to the exercise section for a link between characteristic equations, and the notions of traveltime and (light, sound) ray. We will return to such topics in the scope of geometrical optics, in chapter 6.

### 1.2.3 Spherical waves, Green's functions

Consider $x \in \mathbb{R}^3$ and $c$ constant. We will only be dealing with solutions in 3 spatial dimensions for now. We seek radially symmetric solutions of the wave equation. In spherical coordinate $(r, \theta, \phi)$, the Laplacian reads

$$\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \text{ angular terms.}$$

For radially symmetric solutions of the wave equation, therefore,

$$\frac{\partial^2}{\partial t^2}(ru) = \frac{\partial^2}{\partial r^2}(ru).$$

This is a one-dimensional wave equation in the $r$ variable, whose solution we derived earlier:

$$ru(r,t) = F(r-ct) + G(r+ct) \qquad \Rightarrow \qquad u(r,t) = \frac{F(r-ct)}{r} + \frac{G(r+ct)}{r}.$$

Spherical waves corresponding to the $F$ term are called *outgoing*, while waves corresponding to the $G$ term are called *incoming*. More generally, spherical waves can be outgoing/incoming with respect to any point $y \in \mathbb{R}^3$, for instance

$$u(x,t) = \frac{F(\|x-y\| - ct)}{\|x-y\|}.$$

Notice that we had already seen that $\|x-y\| \pm ct$ is a characteristic variable for the wave equation, in the previous section. The surfaces $\|x-y\| = ct+$ const. are often called *light cones* in the setting of electromagnetic waves.

In what follows we will be interested in the special case $F(r) = \delta(r)$, the Dirac delta, for which the wave equation is only satisfied in a distributional sense. Superpositions of such spherical waves are still solutions of the wave equation. Let us now see that the general solution of the wave equation in

$\mathbb{R}^3$, with constant $c$, can be written in terms of a *spherical means* formula. For now, consider

$$u(x,t) = \int_{\mathbb{R}^3} \frac{\delta(\|x-y\| - ct)}{\|x-y\|} \psi(y) dy.$$

Since $\|x-y\| = ct$ on the support of the delta function, the denominator can be written $ct$. Denoting by $B_x(ct)$ the ball centered at $x$ and with radius $ct$, we can rewrite

$$u(x,t) = \frac{1}{ct} \int_{\partial B_x(ct)} \psi(y) dy.$$

hence the name spherical means (note that the argument of $\delta$ has derivative 1 in the radial variable — no Jacobian is needed.) The interesting question is that of matching $u(x,t)$ given by such a formula, with the initial conditions.

### 1.2.4 Reflected waves

Impedance jump

nD version, angle of incidence

### 1.2.5 The Helmholtz equation

Green's function for Helmholtz

Sommerfeld RBC

## 1.3 Exercises

1. Continue the reasoning in section 1.1.1 with the entropy to justify the equations of variable-density acoustics. [Hints: conservation of entropy reads $\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$. Continue assuming that the background velocity field is $v_0 = 0$. Assume a fixed, variable background density $\rho_0(x)$. The new constitutive relation is $p = f(\rho, s)$. Consider defining $c^2(x) = \frac{\partial f}{\partial \rho}(\rho_0(x), s_0(x))$.]

2. First, show the multivariable rule of integration by parts $\int \nabla f \cdot g = -\int f \nabla \cdot g$, when $f$ and $g$ are smooth and decay fast at infinity, by invoking the divergence theorem. Second, use this result to show that $L^* = -L$ for variable-density acoustics (section 1.1.1), i.e., show that

$\langle Lw, w' \rangle = -\langle w, Lw' \rangle$ for all reasonable functions $w$ and $w'$, and where $\langle \cdot, \cdot \rangle$ is the adequate notion of inner product seen in section 1.1.1.

3. Show that $\langle Lw, w' \rangle = -\langle w, Lw' \rangle$ for general elastic waves.

4. In $\mathbb{R}^2$, consider

$$f_\omega(x) = \int_0^{2\pi} e^{ik_\theta \cdot x} d\theta, \qquad k_\theta = |k| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

with $|k| = \omega/c$. Show that $f_\omega$ is a solution of the Helmholtz equation (1.10) with constant $c$, and simplify the expression of $f_\omega$ by means of a Bessel function. [Hint: show first that $f_\omega$ is radially symmetric.]

5. Find all the functions $\tau(x)$ for which

$$\xi(x,t) = \tau(x) - t$$

is a solution of (1.12) in the case $x \in \mathbb{R}$.

The function $\tau(x)$ has the interpretation of a *traveltime*.

6. Consider a characteristic curve as the level set $\xi(x,t)$ =const., where $\xi$ is a characteristic coordinate obeying (1.12). Express this curve parametrically as $(X(t), t)$, and find a differential equation for $X(t)$ of the form $\dot{X}(t) = \ldots$ How do you relate this $X(t)$ to the traveltime function $\tau(x)$ of the previous exercise? Justify your answer.

Such functions $X(t)$ are exactly the *rays* — light rays or sound rays. They encode the idea that waves propagate with local speed $c(x)$.

7. Give a complete solution to the wave equation in $\mathbb{R}^n$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

by Fourier-transforming $u(x,t)$ in the $x$-variable, solving the resulting ODE to obtain the $e^{\pm i|k|/ct}$ time dependencies, matching the initial conditions, and finishing with an inverse Fourier transform. The resulting formula is a generalization of d'Alembert's formula.

# Chapter 2

# Scattering series

# Chapter 3

# Adjoint-state methods

# Chapter 4

# Synthetic-aperture radar

# Chapter 5

# Computerized tomography

# Chapter 6

# Seismic imaging

# Chapter 7

# Optimization

# Appendix A

# Vector and tensor calculus