18.325 - Waves and Imaging
Fall 2012 - Class notes

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Preface

In this text we use the symbol ($) to draw attention every time a physical assumption or simplification is made.

Chapter 1

Wave equations

1.1 Acoustic waves

Acoustic waves are propagating pressure disturbances in a gas or liquid. With $p(x,t)$ the pressure fluctuation (a time-dependent scalar field) and $v(x,t)$ the particle velocity (a time-dependent vector field), the acoustic wave equations read

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\nabla p, \tag{1.1}$$

$$\frac{\partial p}{\partial t} = -\kappa_0 \nabla \cdot v. \tag{1.2}$$

The two quantities ρ_0 and κ_0 are the mass density and the bulk modulus, respectively. They are linked to the wave speed c through $\kappa_0 = \rho_0 c^2$. Initial conditions on p and v must be supplied. A forcing term may be added to the dynamic balance equation (1.1) when external forces (rather than initial conditions) create the waves.

Let us now explain how these equations are obtained from a linearization of Euler's gas dynamics equations in a uniform background medium (\$). Consider the mass density ρ as a scalar field. In the inviscid case (\$), conservation of momentum and mass respectively read

$$\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

An additional equation, called constitutive relation, must be added to close the system. It typically relates the pressure and the density in an algebraic way, and encodes a thermodynamic assumption about compression and dilation. For instance if the gas is assumed to be ideal, and if the compression-dilation process occurring in the wave is adiabatic reversible (no heat transfer), then $p \sim \rho^\gamma$, $\gamma = 1.4$, where $\sim$ indicates equality up to a dimensional constant. More generally, assume for the moment that the constitutive relation takes the form

$$p = f(\rho)$$

for some scalar function f, which we assume differentiable and strictly increasing ($f'(\rho) > 0$ for all $\rho > 0$).

Consider small disturbances off of an equilibrium state:

$$p = p_0 + p_1, \qquad \rho = \rho_0 + \rho_1, \qquad v = v_0 + v_1.$$

In what follows, neglect quadratic quantities of p_1, ρ_1, v_1. Consider a medium at rest (\$): p_0, ρ_0 independent of t, and $v_0 = 0$. After some algebraic simplification the conservation of momentum becomes

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_0 - \nabla p_1.$$

To zero-th order (i.e., at equilibrium, $p_1 = \rho_1 = v_1 = 0$,) we have

$$\nabla p_0 = 0 \qquad \Rightarrow \qquad p_0 \text{ constant in } x.$$

To first order, we get

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1,$$

which is exactly (1.1) after renaming $v_1 \to v$, $p_1 \to p$. The constitutive relation must hold at equilibrium, hence p_0 constant in x implies that ρ_0 is also constant in x (uniform). Conservation of mass becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot v_1 = 0.$$

Differentiate the constitutive relation to obtain $p_1 = f'(\rho_0)\rho_1$. Call $f'(\rho_0) = c^2$, a number that we assume positive. Then we can eliminate ρ_1 to get

$$\frac{\partial p_1}{\partial t} + \rho_0 c^2 \nabla \cdot v_1 = 0.$$

This is exactly (1.2) with $\kappa_0 = \rho_0 c^2$.

Conveniently, the equations for acoustic waves in a variable medium $\rho_0(x)$, $\kappa_0(x)$ are obvious modifications of (1.1), (1.2):

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0(x)}\nabla p, \tag{1.3}$$

$$\frac{\partial p}{\partial t} = -\kappa_0(x)\nabla \cdot v. \tag{1.4}$$

A different argument is needed to justify these equations, however. The previous reasoning does not leave room for variable $\rho_0(x)$ or $\kappa_0(x)$. Instead, it is necessary to introduce a more realistic constitutive relation

$$p = f(\rho, s),$$

where s is the entropy. An additional equation for conservation of entropy needs to be considered. The new constitutive relation allows ρ_0 and s_0 to be functions of x in tandem, although p_0 is still (necessarily) uniform in x. The reasoning leading to (1.3), (1.4) is the subject of an exercise in section 1.5.

Acoustic waves can take the form of a first-order system of equations, or else a second-order scalar equation. Combining (1.3), (1.4), we get

$$\frac{\partial^2 p}{\partial t^2} = \kappa_0(x)\nabla \cdot (\frac{1}{\rho_0(x)}\nabla p).$$

Initial conditions on both p and $\partial p/\partial t$ must be supplied. This equation may come with a right-hand side $f(x,t)$ that indicates forcing. When ρ_0 and κ_0 are constant, the scalar wave equation reduces to

$$\frac{\partial^2 p}{\partial t^2} = c_0^2\,\Delta p.$$

Waves governed by (1.3), (1.4) belong in the category of hyperbolic waves because they obey conservation of energy. Define

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa_0\nabla\cdot & 0 \end{pmatrix}.$$

Then the acoustic system simply reads

$$\frac{\partial w}{\partial t} = Lw.$$

L is called the generator of the evolution.

Definition 1. *The system* $\frac{\partial w}{\partial t} = Lw$ *is said to be hyperbolic if* L *is a matrix of first-order differential operators, and there exists an inner product* $\langle w, w' \rangle$ *with respect to which* $L^* = -L$*, i.e.,* L *is anti-self-adjoint.*

An adjoint operator such as L^* is defined through the equation[1]

$$\langle Lw, w' \rangle = \langle w, L^* w' \rangle, \qquad \text{for all } w, w'.$$

For instance, in the case of the acoustic system, the proper notion of inner product is (the factor 1/2 is optional)

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho_0 v \cdot v' + \frac{1}{\kappa_0} p p') \, dx.$$

It is an exercise in section 1.5 to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for that inner product, for all w, w'.

Theorem 1. *If* $\frac{\partial w}{\partial t} = Lw$ *is a hyperbolic system, then* $E = \langle w, w \rangle$ *is conserved in time.*

Proof.

$$\begin{aligned} \frac{d}{dt} \langle w, w \rangle &= \langle \frac{\partial w}{\partial t}, w \rangle + \langle w, \frac{\partial w}{\partial t} \rangle \\ &= 2 \, \langle \frac{\partial w}{\partial t}, w \rangle \\ &= 2 \, \langle Lw, w \rangle \\ &= 2 \, \langle w, L^* w \rangle \\ &= 2 \, \langle w, (-L) w \rangle \\ &= -2 \, \langle Lw, w \rangle. \end{aligned}$$

A quantity is equal to minus itself if and only if it is zero. □

In the case of acoustic waves,

$$E = \frac{1}{2} \int (\rho_0 v^2 + \frac{p^2}{\kappa}) \, dx,$$

[1] The existence of L^* can be traced back to the Riesz representation theorem once $\langle Lw, w' \rangle$ is shown to be a continuous functional of w in some adequate Hilbert space norm.

which can be understood as kinetic plus potential energy. We now see that the factor 1/2 was chosen to be consistent with the physicists' convention for energy.

In the presence of external forcings the hyperbolic system reads $\partial w/\partial t = Lw + f$: in that case the rate of change of energy is determined by f.

For reference, common boundary conditions for acoustic waves include

- Sound soft boundary condition: Dirichlet for the pressure, $p = 0$.
- Sound-hard boundary condition: Neumann for the pressure, $\frac{\partial p}{\partial n} = 0$, or equivalently $v \cdot n = 0$.

Another important physical quantity is related to acoustic waves: the acoustic impedance $Z = \sqrt{\rho_0 \kappa_0}$. We will see later that impedance jumps determine reflection and transmission coefficients at medium discontinuities.

1.2 Elastic waves

Elastic waves are propagating pressure disturbances in solids. The interesting physical variables are

- The displacement $u(x, t)$, a time-dependent vector field. In terms of u, the particle velocity is $v = \frac{\partial u}{\partial t}$.
- The strain tensor
$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T),$$
a symmetric time-dependent tensor field.
- The stress tensor σ, also a symmetric time-dependent tensor field.

For elastic waves, the density ρ is very often assumed independent of t along particle trajectories, namely $\rho_0(x, 0) = \rho_0(x + u(x, t), t)$.

The equation of elastic waves in an isotropic medium (where all the waves travel at the same speed regardless of the direction in which they propagate) ($) reads
$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda \nabla \cdot u) + \nabla \cdot (\mu(\nabla u + (\nabla u)^T)). \tag{1.5}$$
where ρ, λ, and μ may possibly depend on x. As for acoustic waves, a forcing term is added to this equation when waves are generated from external forces.

To justify this equation, start by considering the equation of conservation of momentum ("$F = ma$"),

$$\rho \frac{\partial v}{\partial t} = \nabla \cdot \sigma,$$

possibly with an additional term $f(x,t)$ modeling external forces. The notation $\nabla\cdot$ indicates tensor divergence, namely $(\nabla \cdot \sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$. Stress and strain are linked by a constitutive relation called Hooke's law,

$$\sigma = C : \epsilon,$$

where C is the 4-index elastic tensor. In three spatial dimensions, C has 81 components. The colon indicates tensor contraction, so that $(C : \epsilon)_{ij} = \sum_{k\ell} C_{ijk\ell}\epsilon_{k\ell}$.

These equations form a closed system when they are complemented by

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

which holds by definition of ϵ.

At this point we can check that the first-order system for v and ϵ defined by the equations above is hyperbolic. Define

$$w = \begin{pmatrix} v \\ \epsilon \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix},$$

with

$$L_1 v = \frac{1}{2}(\nabla v + (\nabla v)^T), \qquad L_2 \epsilon = \frac{1}{\rho_0} \nabla \cdot (C : \epsilon).$$

Then, as previously, $\frac{\partial w}{\partial t} = Lw$. An exercise in section 1.5 asks to show that the matrix operator L is anti-selfadjoint with respect to the inner product

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + \epsilon : C : \epsilon) \, dx.$$

The corresponding conserved elastic energy is $E = \langle w, w \rangle$.

Isotropic elasticity is obtained where C takes a special form with 2 degrees of freedom rather than 81, namely

$$C_{ijk\ell} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}).$$

We are not delving into the justification of this equation. The two elastic parameters λ and μ are also called Lamé parameters:

- λ corresponds to longitudinal waves, also known as compressional, pressure waves (P).
- μ corresponds to transverse waves, also known as shear waves (S).

Originally, the denominations P and S come from "primary" and "secondary", as P waves tend to propagate faster, hence arrive earlier, than S waves.

With this parametrization of C, it is easy to check that the elastic system reduces to the single equation (1.5). In index notation, it reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_i(\lambda \partial_j u_j) + \partial_j(\mu(\partial_i u_j + \partial_j u_i)).$$

For reference, the hyperbolic propagator L_2 reduces to

$$L_2 \epsilon = \frac{1}{\rho}(\nabla(\lambda \operatorname{tr} \epsilon) + 2\, \nabla \cdot (\mu \epsilon)), \qquad \operatorname{tr} \epsilon = \sum_i \epsilon_{ii},$$

and the energy inner product is

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + 2\, \mu \operatorname{tr}(\epsilon^T \epsilon') + \lambda (\operatorname{tr} \epsilon)(\operatorname{tr} \epsilon'))\, dx.$$

The elastic wave equation looks like an acoustic wave equation with "2 terms, hence 2 waves". To make this observation more precise, assume that λ and μ are constant ($). Use some vector identities[2] to reduce (1.5) to

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \nabla (\nabla \cdot u) + \mu \Delta u, \\ &= (\lambda + 2\mu) \nabla (\nabla \cdot u) - \mu \nabla \times \nabla \times u. \end{aligned}$$

Perform the Helmholtz (a.k.a. Hodge) decomposition of u in terms of potentials ϕ and ψ:

$$u = \nabla \phi + \nabla \times \psi,$$

where ϕ is a scalar field and ψ is a vector field[3]. These two potentials are determined up to a gauge choice, namely

$$\phi' = \phi + C, \qquad \psi' = \psi + \nabla f.$$

[2]In this section, we make use of $\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u$, $\nabla \cdot \nabla \times \psi = 0$, and $\nabla \times \nabla \psi = 0$.

[3]Normally the Helmholtz decomposition comes with a third term h which obeys $\Delta h = 0$, i.e., h is harmonic, but under suitable assumptions of decay at infinity the only solution to $\Delta h = 0$ is $h = 0$.

Choose f such that ψ' has zero divergence:

$$\nabla \cdot \psi' = 0 \qquad \Rightarrow \qquad \Delta f = -\nabla \cdot \psi.$$

This is a well-posed Poisson equation for f. With this choice of ψ', it holds that

$$\nabla \cdot u = \Delta \phi, \qquad \nabla \times u = \nabla \times \nabla \times u = -\Delta \psi.$$

The elastic wave equation can then be rewritten in terms of ϕ, ψ as

$$\nabla \left[\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi \right] + \nabla \times \left[\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi \right] = 0.$$

Take the gradient of this equation to conclude that (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi = \text{ harmonic } = 0.$$

Now that the first term is zero, we get (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi = \nabla(\text{something}) = 0.$$

Hence each potential ϕ and ψ solve their own scalar wave equation: one for the longitudinal waves (ϕ) and one for the transverse waves (ϕ). They obey a superposition principle. The two corresponding wave speeds are

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \qquad c_S = \sqrt{\frac{\mu}{\rho_0}}.$$

In the limit $\mu \to 0$, we see that only the longitudinal wave remains, and λ reduces to the bulk modulus. In all cases, since $\lambda \geq 0$ we always have $c_P \geq \sqrt{2} c_S$: the P waves are indeed always faster (by a factor at least $\sqrt{2}$) than the S waves.

The assumption that λ and μ are constant is a very strong one: there is a lot of physics in the coupling of ϕ and ψ that the reasoning above does not capture. Most important is mode conversion as a result of wave reflection at discontinuity interfaces of $\lambda(x)$ and/or $\mu(x)$.

1.3 Electromagnetic waves

The quantities of interest for electromagnetic waves are:

- Physical fields: the electric field E, and the magnetic field H,
- Medium parameters: the electric permittivity ϵ and the magnetic permeability μ,
- Forcings: electric currents j and electric charges ρ.

The electric displacement field D and the magnetic induction field B are also considered. In the linearized regime ($), they are assumed to be linked to the usual fields E and H by the constitutive relations

$$D = \epsilon E, \qquad B = \mu H.$$

Maxwell's equations in a medium with possible space-varying parameters ϵ and μ read

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's law)} \tag{1.6}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + j \qquad \text{(Ampère's law with Maxwell's correction)} \tag{1.7}$$

$$\nabla \cdot D = \rho \qquad \text{(Gauss's law for the electric field)} \tag{1.8}$$

$$\nabla \cdot B = 0 \qquad \text{(Gauss's law for the magnetic field)} \tag{1.9}$$

The integral forms of these equations are obtained by a volume integral, followed by a reduction to surface equations by Stokes's theorem for (1.6), (1.7) and the divergence (Gauss's) theorem for (1.8), (1.9). The integral equations are valid when ϵ and μ are discontinuous, whereas the differential equations strictly speaking are not.

The total charge in a volume V is $\int_V \rho dV$, while the total current through a surface S is $\int_S j \cdot dS$. Conservation of charge follows by taking the divergence of (1.7) and using (1.8):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

In vacuum, or dry air, the parameters are constant and denoted $\epsilon = \epsilon_0$, $\mu = \mu_0$. They have specific numerical values in adequate units.

We now take the viewpoint that (1.6) and (1.7) are evolution equations for E and H (or D and B) that fully determine the fields when they are solved forward (or backward) in time. In that setting, the other two equations (1.8) and (1.9) are simply constraints on the initial (or final) condition at $t = 0$. As previously, we may write Maxwell's equations in the more concise hyperbolic form

$$\frac{\partial w}{\partial t} = Lw + \begin{pmatrix} -j/\epsilon \\ 0 \end{pmatrix}, \qquad \text{with } w = \begin{pmatrix} E \\ H \end{pmatrix},$$

provided

$$L = \begin{pmatrix} 0 & \frac{1}{\epsilon}\nabla\times \\ -\frac{1}{\mu}\nabla\times & 0 \end{pmatrix}.$$

The "physical" inner product that makes $L^* = -L$ is

$$\langle w, w' \rangle = \frac{1}{2}\int (\epsilon E E' + \mu H H')\, dx.$$

The electromagnetic energy $E = \langle w, w \rangle$ is conserved when $j = 0$.

It is the balanced coupling of E and H through (1.6) and (1.7) that creates wave-like solutions to Maxwell's equations (and prompts calling the physical phenomenon electromagnetism rather than just electricity and magnetism.) Combining both equations, we obtain

$$\frac{\partial^2 E}{\partial t^2} = -\frac{1}{\epsilon}\nabla \times (\frac{1}{\mu}\nabla \times E),$$

$$\frac{\partial^2 H}{\partial t^2} = -\frac{1}{\mu}\nabla \times (\frac{1}{\epsilon}\nabla \times H).$$

These wave equations may be stand-alone but E and H are still subject to essential couplings.

A bit of algebra[4] reveals the more familiar form

$$\Delta E - \epsilon\mu \frac{\partial^2 E}{\partial t^2} + \frac{\nabla \mu}{\mu} \times (\nabla \times E) + \nabla(E \cdot \frac{\nabla \epsilon}{\epsilon}) = 0.$$

We now see that in a uniform medium, ϵ and μ are constant and the last two terms drop, revealing a wave equation with speed

$$c = \frac{1}{\sqrt{\epsilon\mu}}.$$

[4] Using the relations $\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - \Delta F$ again, as well as $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$.

The speed of light is $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. Even when ϵ and μ vary in x, the last two terms are kinematically much less important than the first two because they involve lower-order derivatives of E. They would not, for instance, change the path of the "light rays", a concept that we'll make clear later.

For reference, we now list the jump conditions that the electric and magnetic fields obey at a dielectric interface. These relations can be obtained from the integral form of Maxwell's equations, posed over a thin volume straddling the interface. Let n be the vector normal to a dielectric interface.

$$n \times E_1 = n \times E_2 \qquad \text{(continuous tangential components)}$$

$$n \times H_1 = n \times H_2 + j_S$$

$$n \cdot D_1 = n \cdot D_2 + \rho_S$$

$$n \cdot H_1 = n \cdot H_2 \qquad \text{(continuous normal component)}$$

We have used j_S and ρ_S for surface currents and surface charges respectively. If the two dielectrics correspond to finite parameters ϵ_1, ϵ_2 and μ_1, μ_2, then these currents are zero. If material 2 is a perfect electric conductor however, then these currents are not zero, but the fields E_2, H_2, D_2 and H_2 are zero. This results in the conditions $n \times E = 0$ (E perpendicular to the interface) and $n \times H = 0$ (H parallel to the interface) in the vicinity of a perfect conductor.

Materials conducting current are best described by a complex electric permittivity $\epsilon = \epsilon' + i\sigma/\omega$, where σ is called the conductivity. All these quantities could be frequency-dependent. It is the ratio σ/ϵ' that tends to infinity when the conductor is "perfect". Materials for which ϵ is real are called "perfect dielectrics": no conduction occurs and the material behaves like a capacitor. We will only consider perfect dielectrics in this class. When conduction is present, loss is also present, and electromagnetic waves tend to be inhibited. Notice that the imaginary part of the permittivity is σ/ω, and not just σ, because we want Ampère's law to reduce to $j = \sigma E$ (the differential version of Ohm's law) in the time-harmonic case and when $B = 0$.

1.4 Special solutions

1.4.1 Plane waves, dispersion relations

In this section we study special solutions of wave equations that depend on x like e^{ikx}. These solutions are obtained if we assume that the time dependence is harmonic, namely if the unknown is $w(x,t)$, then we assume ($)

$$w(x,t) = e^{-i\omega t} s f_\omega(x), \qquad \omega \in \mathbb{R}.$$

Choosing $e^{+i\omega t}$ instead makes no difference down the road. Under the time-harmonic assumption, the evolution problem $\frac{\partial w}{\partial t} = Lw$ becomes an eigenvalue problem:

$$-i\omega f_\omega = L f_\omega.$$

Consider the following examples

- The one-way, one-dimensional wave equation

  $$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0, \qquad x \in \mathbb{R}.$$

  Time harmonic solutions obey

  $$i\frac{\omega}{c} f_\omega = f_\omega', \qquad x \in \mathbb{R}.$$

  The solution to this equation is

  $$f_\omega(x) = e^{ikx}, \qquad k = \frac{\omega}{c} \in \mathbb{R}.$$

  Evanescent waves corresponding to decaying exponentials in x and t are also solutions over a half-line, say, but they are ruled out by our assumption ($) that $\omega \in \mathbb{R}$.

  While ω is the angular frequency (equal to $2\pi/T$ where T is the period), k is called the wave number (equal to $2\pi/\lambda$ where λ is the wavelength.) It is like a "spatial frequency", though it is prudent to reserve the word frequency for the variable dual to time. The quantity measured in Hertz (1/s) and also called frequency is $\nu = \omega/(2\pi)$.

  The full solution then takes the form

  $$w(x,t) = e^{i(kx-\omega t)} = e^{ik(x-ct)},$$

  manifestly a right-going wave at speed c. If the equation had been $\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x} = 0$ instead, the wave would have been left-going: $w(x,t) = e^{ik(x+ct)}$.

- The n-dimensional acoustic wave equation

1.4.2 Traveling waves, characteristic equations

1D, one-way

1D, two-way

1D, two-way variable medium (homework)

nD, charact eq

1.4.3 Spherical waves, Green's functions

1.4.4 Reflected waves

Impedance jump

nD version, angle of incidence

1.4.5 The Helmholtz equation

Green's function for Helmholtz

Sommerfeld RBC

1.5 Exercises

1. Continue the reasoning in section 1.1 with the entropy to justify the equations of variable-density acoustics. Conservation of entropy reads $\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$.

2. First show that $\int \nabla f \cdot g = -\int f \nabla \cdot g$, when f and g are smooth and decay fast at infinity, by invoking the divergence theorem. Second, use this result to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for variable-density acoustics (section 1.1).

3. Show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for elastic waves.

Chapter 2

Scattering series

Chapter 3

Adjoint-state methods

Chapter 4

Synthetic-aperture radar

Chapter 5

Computerized tomography

Chapter 6

Seismic imaging

Chapter 7

Optimization