18.325 - Waves and Imaging
Fall 2012 - Class notes

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Preface

In this text we use the symbol ($) to draw attention every time a physical assumption or simplification is made.

Chapter 1

Wave equations

1.1 Acoustic waves

Acoustic waves are propagating pressure disturbances in a gas or liquid. With $p(x,t)$ the pressure fluctuation (a time-dependent scalar field) and $v(x,t)$ the particle velocity (a time-dependent vector field), the acoustic wave equations read

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\nabla p, \tag{1.1}$$

$$\frac{\partial p}{\partial t} = -\kappa_0 \nabla \cdot v. \tag{1.2}$$

The two quantities ρ_0 and κ_0 are the mass density and the bulk modulus, respectively. They are linked to the wave speed c through $\kappa_0 = \rho_0 c^2$. Initial conditions on p and v must be supplied.

Let us now explain how these equations are obtained from a linearization of Euler's gas dynamics equations in a uniform background medium (\$). Consider the mass density ρ as a scalar field. In the inviscid case (\$), conservation of momentum and mass respectively read

$$\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

An additional equation, called constitutive relation, must be added to close the system. It typically relates the pressure and the density in an algebraic way, and encodes a thermodynamic assumption about compression and dilation. For instance if the gas is assumed to be ideal, and if the compression-dilation process occurring in the wave is adiabatic reversible (no heat transfer), then $p \sim \rho^\gamma$, $\gamma = 1.4$, where $\sim$ indicates equality up to a dimensional constant. More generally, assume for the moment that the constitutive relation takes the form

$$p = f(\rho)$$

for some scalar function f, which we assume differentiable and strictly increasing ($f'(\rho) > 0$ for all $\rho > 0$).

Consider small disturbances off of an equilibrium state:

$$p = p_0 + p_1, \qquad \rho = \rho_0 + \rho_1, \qquad v = v_0 + v_1.$$

In what follows, neglect quadratic quantities of p_1, ρ_1, v_1. Consider a medium at rest ($): p_0, ρ_0 independent of t, and $v_0 = 0$. After some algebraic simplification the conservation of momentum becomes

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_0 - \nabla p_1.$$

To zero-th order (i.e., at equilibrium, $p_1 = \rho_1 = v_1 = 0$,) we have

$$\nabla p_0 = 0 \qquad \Rightarrow \qquad p_0 \text{ constant in } x.$$

To first order, we get

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1,$$

which is exactly (1.1) after renaming $v_1 \to v$, $p_1 \to p$. The constitutive relation must hold at equilibrium, hence p_0 constant in x implies that ρ_0 is also constant in x (uniform). Conservation of mass becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot v_1 = 0.$$

Differentiate the constitutive relation to obtain $p_1 = f'(\rho_0)\rho_1$. Call $f'(\rho_0) = c^2$, a number that we assume positive. Then we can eliminate ρ_1 to get

$$\frac{\partial p_1}{\partial t} + \rho_0 c^2 \nabla \cdot v_1 = 0.$$

This is exactly (1.2) with $\kappa_0 = \rho_0 c^2$.

Conveniently, the equations for acoustic waves in a variable medium $\rho_0(x)$, $\kappa_0(x)$ are obvious modifications of (1.1), (1.2):

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0(x)}\nabla p, \tag{1.3}$$

$$\frac{\partial p}{\partial t} = -\kappa_0(x)\nabla \cdot v. \tag{1.4}$$

A different argument is needed to justify these equations, however. The previous reasoning does not leave room for variable $\rho_0(x)$ or $\kappa_0(x)$. Instead, it is necessary to introduce a more realistic constitutive relation

$$p = f(\rho, s),$$

where s is the entropy. An additional equation for conservation of entropy needs to be considered. The new constitutive relation allows ρ_0 and s_0 to be functions of x in tandem, although p_0 is still (necessarily) uniform in x. The reasoning leading to (1.3), (1.4) is the subject of an exercise in section 1.4.

Acoustic waves can take the form of a first-order system of equations, or else a second-order scalar equation. Combining (1.3), (1.4), we get

$$\frac{\partial^2 p}{\partial t^2} = \kappa_0(x)\nabla \cdot (\frac{1}{\rho_0(x)}\nabla p).$$

Initial conditions on both p and $\partial p/\partial t$ must be supplied. This equation may come with a right-hand side $f(x,t)$ that indicates forcing. When ρ_0 and κ_0 are constant, the scalar wave equation reduces to

$$\frac{\partial^2 p}{\partial t^2} = c_0^2\,\Delta p.$$

Waves governed by (1.3), (1.4) belong in the category of hyperbolic waves because they obey conservation of energy. Define

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa_0\nabla\cdot & 0 \end{pmatrix}.$$

Then the acoustic system simply reads

$$\frac{\partial w}{\partial t} = Lw.$$

L is called the generator of the evolution.

Definition 1. *The system* $\frac{\partial w}{\partial t} = Lw$ *is said to be hyperbolic if* L *is a matrix of first-order differential operators, and there exists an inner product* $\langle w, w' \rangle$ *with respect to which* $L^* = -L$*, i.e.,* L *is anti-self-adjoint.*

An adjoint operator such as L^* is defined through the equation[1]

$$\langle Lw, w' \rangle = \langle w, L^* w' \rangle, \qquad \text{for all } w, w'.$$

For instance, in the case of the acoustic system, the proper notion of inner product is (the factor 1/2 is optional)

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho_0 v \cdot v' + \frac{1}{\kappa_0} p p') \, dx.$$

It is an exercise in section 1.4 to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for that inner product, for all w, w'.

Theorem 1. *If* $\frac{\partial w}{\partial t} = Lw$ *is a hyperbolic system, then* $E = \langle w, w \rangle$ *is conserved in time.*

Proof.

$$\begin{aligned}
\frac{d}{dt} \langle w, w \rangle &= \langle \frac{\partial w}{\partial t}, w \rangle + \langle w, \frac{\partial w}{\partial t} \rangle \\
&= 2 \, \langle \frac{\partial w}{\partial t}, w \rangle \\
&= 2 \, \langle Lw, w \rangle \\
&= 2 \, \langle w, L^* w \rangle \\
&= 2 \, \langle w, (-L) w \rangle \\
&= -2 \, \langle Lw, w \rangle .
\end{aligned}$$

A quantity is equal to minus itself if and only if it is zero. □

In the case of acoustic waves,

$$E = \frac{1}{2} \int (\rho_0 v^2 + \frac{p^2}{\kappa}) \, dx,$$

[1] The existence of L^* can be traced back to the Riesz representation theorem once $\langle Lw, w' \rangle$ is shown to be a continuous functional of w in some adequate Hilbert space norm.

which can be understood as kinetic plus potential energy. We now see that the factor 1/2 was chosen to be consistent with the physicists' convention for energy.

For reference, common boundary conditions for acoustic waves include

- Sound soft boundary condition: Dirichlet for the pressure, $p = 0$.
- Sound-hard boundary condition: Neumann for the pressure, $\frac{\partial p}{\partial n} = 0$, or equivalently $v \cdot n = 0$.

Another important physical quantity is related to acoustic waves: the acoustic impedance $Z = \sqrt{\rho_0 \kappa_0}$. We will see later that impedance jumps determine reflection and transmission coefficients at medium discontinuities.

1.2 Elastic waves

Elastic waves are propagating pressure disturbances in solids. The interesting physical variables are

- The displacement $u(x,t)$, a time-dependent vector field. In terms of u, the particle velocity is $v = \frac{\partial u}{\partial t}$.
- The strain tensor
$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T),$$
a symmetric time-dependent tensor field.
- The stress tensor σ, also a symmetric time-dependent tensor field.

For elastic waves, the density ρ is very often assumed independent of t along particle trajectories, namely $\rho_0(x,0) = \rho_0(x + u(x,t), t)$.

The equation of elastic waves in an isotropic medium reads

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda \nabla \cdot u) + \nabla \cdot (\mu(\nabla u + (\nabla u)^T)). \tag{1.5}$$

where ρ, λ, and μ may possibly depend on x.

To justify this equation, start by considering the equation of conservation of momentum ("$F = ma$"),

$$\rho \frac{\partial v}{\partial t} = \nabla \cdot \sigma,$$

possibly with an additional term $f(x,t)$ modeling external forces. The notation $\nabla\cdot$ indicates tensor divergence, namely $(\nabla \cdot \sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$. Stress and strain are linked by a constitutive relation called Hooke's law,

$$\sigma = C : \epsilon,$$

where C is the 4-index elastic tensor. In three spatial dimensions, C has 81 components. The colon indicates tensor contraction, so that $(C : \epsilon)_{ij} = \sum_{k\ell} C_{ijk\ell}\epsilon_{k\ell}$.

These equations form a closed system when they are complemented by

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

which holds by definition of ϵ.

At this point we can check that the first-order system for v and ϵ defined by the equations above is hyperbolic. Define

$$w = \begin{pmatrix} v \\ \epsilon \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix},$$

with

$$L_1 v = \frac{1}{2}(\nabla v + (\nabla v)^T), \qquad L_2 \epsilon = \frac{1}{\rho_0} \nabla \cdot (C : \epsilon).$$

Then, as previously, $\frac{\partial w}{\partial t} = Lw$. An exercise in section 1.4 asks to show that the matrix operator L is anti-selfadjoint with respect to the inner product

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + \epsilon : C : \epsilon) \, dx.$$

The corresponding conserved elastic energy is $E = \langle w, w \rangle$.

Isotropic elasticity is obtained where C takes a special form with 2 degrees of freedom rather than 81, namely

$$C_{ijk\ell} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}).$$

We are not delving into the justification of this equation. The two elastic parameters λ and μ are also called Lamé parameters:

- λ corresponds to longitudinal waves, also known as compressional, pressure waves (P).

- μ corresponds to transverse waves, also known as shear waves (S).

Originally, the denominations P and S come from "primary" and "secondary", as P waves tend to propagate faster, hence arrive earlier, than S waves.

With this parametrization of C, it is easy to check that the elastic system reduces to the single equation (1.5). In index notation, it reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_i(\lambda \partial_j u_j) + \partial_j(\mu(\partial_i u_j + \partial_j u_i)).$$

For reference, the hyperbolic propagator L_2 reduces to

$$L_2 \epsilon = \frac{1}{\rho}(\nabla(\lambda \operatorname{tr} \epsilon) + 2\,\nabla \cdot (\mu \epsilon)), \qquad \operatorname{tr} \epsilon = \sum_i \epsilon_{ii},$$

and the energy inner product is

$$\langle w, w \rangle = \frac{1}{2} \int (\rho v \cdot v' + 2\,\mu \operatorname{tr}(\epsilon^T \epsilon') + \lambda (\operatorname{tr} \epsilon)(\operatorname{tr} \epsilon'))\, dx.$$

The elastic wave equation looks like an acoustic wave equation with "2 terms, hence 2 waves". To make this observation more precise, assume that λ and μ are constant (\$). Use some vector identities[2] to reduce (1.5) to

$$\begin{aligned}\rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \nabla(\nabla \cdot u) + \mu \Delta u, \\ &= (\lambda + 2\mu) \nabla(\nabla \cdot u) - \mu \nabla \times \nabla \times u.\end{aligned}$$

Perform the Helmholtz (a.k.a. Hodge) decomposition of u in terms of potentials ϕ and ψ:

$$u = \nabla \phi + \nabla \times \psi,$$

where ϕ is a scalar field and ψ is a vector field[3]. These two potentials are determined up to a gauge choice, namely

$$\phi' = \phi + C, \qquad \psi' = \psi + \nabla f.$$

[2]In this section, we make use of $\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u$, $\nabla \cdot \nabla \times \psi = 0$, and $\nabla \times \nabla \psi = 0$.

[3]Normally the Helmholtz decomposition comes with a third term h which obeys $\Delta h = 0$, i.e., h is harmonic, but under suitable assumptions of decay at infinity the only solution to $\Delta h = 0$ is $h = 0$.

Choose f such that ψ' has zero divergence:

$$\nabla \cdot \psi' = 0 \qquad \Rightarrow \qquad \Delta f = -\nabla \cdot \psi.$$

This is a well-posed Poisson equation for f. With this choice of ψ', it holds that

$$\nabla \cdot u = \Delta \phi, \qquad \nabla \times u = \nabla \times \nabla \times u = -\Delta \psi.$$

The elastic wave equation can then be rewritten in terms of ϕ, ψ as

$$\nabla \left[\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi \right] + \nabla \times \left[\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi \right] = 0.$$

Take the gradient of this equation to conclude that (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \Delta \phi = \text{ harmonic } = 0.$$

Now that the first term is zero, we get (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi = \nabla(\text{something}) = 0.$$

Hence each potential ϕ and ψ solve their own scalar wave equation: one for the longitudinal waves (ϕ) and one for the transverse waves (ϕ). They obey a superposition principle. The two corresponding wave speeds are

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \qquad c_S = \sqrt{\frac{\mu}{\rho_0}}.$$

In the limit $\mu \to 0$, we see that only the longitudinal wave remains, and λ reduces to the bulk modulus. In all cases, since $\lambda \geq 0$ we always have $c_P \geq \sqrt{2} c_S$: the P waves are indeed always faster (by a factor at least $\sqrt{2}$) than the S waves.

The assumption that λ and μ are constant is a very strong one: there is a lot of physics in the coupling of ϕ and ψ that the reasoning above does not capture. Most important is mode conversion as a result of wave reflection at discontinuity interfaces of $\lambda(x)$ and/or $\mu(x)$.

1.3 Electromagnetic waves

1.4 Exercises

1. Continue the reasoning in section 1.1 with the entropy to justify the equations of variable-density acoustics. Conservation of entropy reads $\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$.

2. First show that $\int \nabla f \cdot g = \int f \nabla \cdot g$, when f and g are smooth and decay fast at infinity, by invoking the divergence theorem. Second, use this result to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for variable-density acoustics (section 1.1).

3. Show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for elastic waves.

Chapter 2

Scattering series

Chapter 3

Adjoint-state methods

Chapter 4

Synthetic-aperture radar

Chapter 5

Computerized tomography

Chapter 6

Seismic imaging

Chapter 7

Optimization