10/01

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2(x)} \Delta u = f(x,t;\overset{\text{source}}{s}) \rightarrow u(x,t;s) \\ \text{zero I.C., no B.C.} \end{cases}$$

Inv. prob:

Known: $f(x,t;s)$

"background" part of $c(x) \rightarrow c_0(x)$

data $u(r,t;s)$ at $x=r$.

Unknown: $c(x)$

Rmk: map $c(x) \rightarrow u(r,t;s)$ nonlinear!
although $f \rightarrow u$ is linear

Ex 1) Radar (SAR) ⚠ Maxwell

$u = E$, elec. fied (some component)

$f = j$, current at antenna.

⚠ Assume $f(x,t;s) = \delta(x-s)\underset{\text{pulse waveform}}{p(t)}$

point-like antenna

⚠ Assume $c(x)$ indep. of t:
Stationary targets relative to c_0 / plane motion

Perturbation about c_0 (dry air).

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2} - V(x)$$ with c_0 indep. x

↳ reflectivity function
(measure of radar reflectivity
picked up by antenna)

Ex 2) Seismology (reflection) ⚠ Elastic

u = stress or strain disturbance,
or their potentials
f = explosion / noise / earthquake

⚠ Assume $f(x,t;s) = \delta(x-s)\, f(t)$
point source

source ~~signature~~
time function

Perturbation about a known $c_0(x)$:

$$c(x) = c_0(x) + \delta c(x)$$

smooth background
model velocity

oscillatory or rough
reflectors

Scattering series: • use $\frac{1}{c^2(x)} = \frac{1}{c_0^2} - V(x)$

• Fix source s
(drop in notation)

Split $u = u_{inc} + u_{sc}$

incident
field

scattered field

such that

$$\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t) \qquad (1)$$

$$\frac{1}{c_0^2}\frac{\partial^2 u_{inc}}{\partial t^2} - \Delta u_{inc} = f(x,t) \qquad (2)$$

$$(1) \Rightarrow \quad \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t) + V(x)\frac{\partial^2 u}{\partial t^2}$$

$$U_{sc}\,(2) \Rightarrow \quad \frac{1}{c_0^2}\frac{\partial^2 u_{sc}}{\partial t^2} - \Delta u_{sc} = V(x)\frac{\partial^2 u}{\partial t^2}$$

Solve with Green's function:

$$u_{sc}(x,t) = \int_0^t \int_{\mathbb{R}^n} g(x-y, t-\tau) \cdot V(y)\, \frac{\partial^2 u}{\partial t^2}(y,\tau)\, dy\, d\tau.$$

Notational abuse: denote

- G the operator of conv. by $g(x,t)$
- V ______ mult. by $V(x)$

then $u_{sc} = G V \frac{\partial^2 u}{\partial t^2}$

$$\Rightarrow \boxed{u = u_{inc} + GV\frac{\partial^2 u}{\partial t^2}}$$

implicit relation for u.
Called a Lippmann - Schwinger eqn.

Solution: $\left(I - GV\frac{\partial^2}{\partial t^2}\right) u = u_{inc}$

$$\Rightarrow u = \left(I - GV\frac{\partial^2}{\partial t^2}\right)^{-1} u_{inc}.$$

Recall $\frac{1}{1-a} = 1 + a + a^2 + \dots$

△1. Neumann series : provided

$$\| G V \frac{\partial^2}{\partial t^2} \| < 1 \qquad \text{in some norm (homework)}$$

$$u = u_{inc} + \underbrace{GV \frac{\partial^2 u_{inc}}{\partial t^2}}_{\text{linear in } V} + \underbrace{GV\frac{\partial^2}{\partial t^2}\left(GV \frac{\partial^2 u_{inc}}{\partial t^2}\right)}_{\text{quadratic in } V} + \dots$$

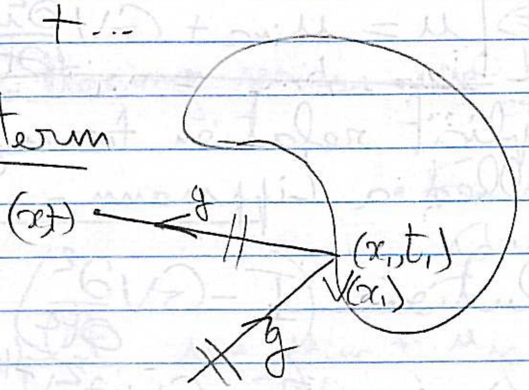
Rmk: $\| G V \frac{\partial^2}{\partial t^2} \| < 1$ is a weak scattering assumption

Explicitly,

$$u(x,t) = u_{inc}(x,t)$$
$$+ \int g(x-x_1, t-t_1) V(x_1) \frac{\partial^2 u_{inc}}{\partial t_1^2}(x_1,t_1)\, dx_1 dt_1$$
$$+ \int g(x-x_1,t-t_1) V(x_1) \frac{\partial^2}{\partial t_1^2} \int g(x_1-x_2, t_1-t_2)$$
$$V(x_2) \frac{\partial^2 u_{inc}}{\partial t^2}(x_2,t_2)\, dx_2\, dt_2$$
$$+ \dots$$

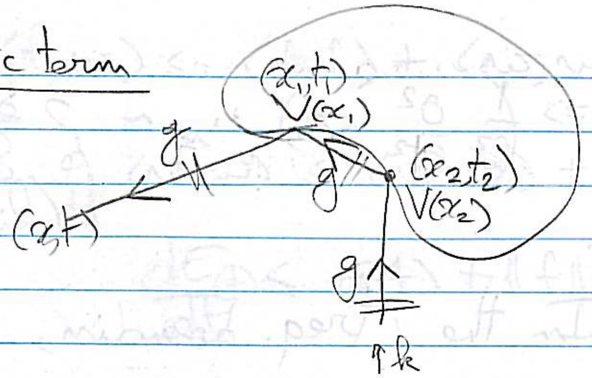
linear term

g = propagator

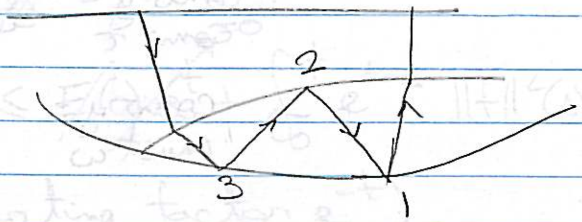


Single scattering

quadratic term



cubic term



→ multiple scattering.

Born approximation : linearization of the map $V \to u_{sc}$.

$$u \simeq u_{inc} + GV \frac{\partial^2 u_{inc}}{\partial t^2}$$

$$u_{sc} \simeq GV \frac{\partial^2 u_{inc}}{\partial t^2}.$$

Back to PDE :

$$\frac{1}{c_0^2}\frac{\partial^2 u_{sc}}{\partial t^2} - \Delta u_{sc} \simeq V(x)\frac{\partial^2 u_{inc}}{\partial t^2}.$$

Seismology : $c(x) = c_0(x) + \delta c(x)$
↳ small

$$\frac{1}{c^2} = \frac{1}{c_0^2}\left(1+\frac{\delta c}{c_0}\right)^{-2} \overset{⚠}{\simeq} \frac{1}{c_0^2}\left(1-2\frac{\delta c}{c_0}\right)$$

$$= \frac{1}{c_0^2} - 2\frac{\delta c}{c_0^3}.$$

$$\Rightarrow V(x) = -2\frac{\delta c(x)}{c_0^3(x)}$$

$$\Rightarrow \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) u_{sc} \simeq 2\,\frac{\delta c}{c_0^3}\,\frac{\partial^2 u_{inc}}{\partial t^2}.$$

(linearized fwd model)

In the freq. domain, $\frac{\partial^2}{\partial t^2} \to -\omega^2$

conv y, conv τ $\iint g(x-y, t-\tau)\, F(y,\tau)\, dy\, d\tau$

→ conv y, mult ω $\int \phi(x-y,\omega)\, \hat{F}^2(y,\omega)\, dy$

F.T. in τ variable.

Accuracy of linearization

① energy estimates

$w = \begin{pmatrix} \partial u/\partial t \\ \nabla u \end{pmatrix}$ $\quad \frac{\partial w}{\partial t} = Lw + f$

$L^* = -L$ for some $\langle w, w\rangle$.

$E(t) = \langle w(t), w(t)\rangle_x = \|w\|^2(t)$

See next lecture for a different version

Thm $E(t) \leq \int_0^t e^{t-s}\, \|f\|^2(s)\, ds$

Pf. $\frac{d}{dt}\langle w, w\rangle = 2\left\langle \frac{\partial w}{\partial t}, w\right\rangle$

$= 2\langle Lw + f, w\rangle$

with $\langle Lw, w\rangle = -\langle w, Lw\rangle = 0$

$\frac{d}{dt}\langle w, w\rangle = 2\langle f, \omega\rangle$

$$2\langle f,w\rangle \leq \langle f,f\rangle + \langle w,w\rangle$$

$$\Rightarrow \frac{d}{dt}\langle w,w\rangle \leq \langle w,w\rangle + \langle f,f\rangle$$

$$\frac{dE(t)}{dt} \leq E(t) + \|f\|^2(t)$$

Gronwall $\Rightarrow$

$$E(t) \leq \overset{=0}{\cancel{E(0)e^t}} + \int_0^t e^{t-s}\|f\|^2(s)\,ds.$$

(Integrating factor e^{-t}).

10/06/09

Ⓐ

Now $\left(\frac{\partial}{\partial t} - L\right) w_{sc} = V w$ (1)

Born: $\left(\frac{\partial}{\partial t} - L\right) w_{sc,B} = V w_{inc}$ (2)

(3) where $\left(\frac{\partial}{\partial t} - L\right) w_{inc} = 0$

(4) and $\left(\frac{\partial}{\partial t} - \tilde{L}\right) w = 0$ $\left(\begin{matrix} L \Leftrightarrow c_0 \\ \tilde{L} \Leftrightarrow c(x)\end{matrix}\right)$

(1), (2) $\Rightarrow \left(\frac{\partial}{\partial t} - L\right)(w_{sc} - w_{sc,B}) = V w_{sc}$

$$\Rightarrow \|w_{sc} - w_{sc,B}\|^2 \leq \int_0^t e^{t-s}\|V w_{sc}\|^2(s)\,ds$$

(1) $\Rightarrow \|w_{sc}\|^2 \leq \int_0^t e^{t-s}\|V w\|^2(s)\,ds$

Say $\|w\|^2 = \int |w(x,t)|^2\,dx$ for simplicity

$$\Rightarrow \|Vw\|^2 \leq \max_x |V(x)|^2 \|w\|^2.$$

$$= \|V\|_\infty^2 \|w\|^2$$

$$\|w_{sc} - w_{sc,B}\|^2 \leq \int_0^t e^{t-s} \|V\|_\infty^2 \|w_{sc}\|^2(s)\,ds$$

$$\leq \int_0^t e^{t-s} \|V\|_\infty^2 \left[\int_0^s e^{s-\tau} \|V\|_\infty^2 \|w\|^2(\tau)\,d\tau\right] ds$$

$$= \|V\|_\infty^4 \int_0^t \left(\int_0^s e^{t-\tau} \|w\|^2(\tau)\,d\tau\right) ds$$

correct but not ideal.

$$\Rightarrow \|w_{sc} - w_{sc,B}\| \leq \underbrace{\|V\|_\infty^2}_{\text{quadratic error}} \sqrt{\int_0^t\int_0^s e^{t-\tau} \|w\|^2(\tau)\,d\tau\,ds}$$

and in particular $\|w_{sc} - w_{sc,B}\| \leq \|V\|_\infty \sqrt{\int_0^t e^{t-s} \|w_{sc}\|(s)\,ds}$

NB: absent on 10/5

Ⓐ Recall $\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) u_{sc} = V \frac{\partial^2 u}{\partial t^2}$; zero IC

where $u = u_{sc} + u_{inc}$

Born approximation; models primary refl.

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) u_{sc,B} = V \frac{\partial^2 u_{inc}}{\partial t^2}, \text{ zero I.C.}$$

Scattering series $\frac{1}{c^2} = \frac{1}{c_0^2} - V$

Call $u(x,t) = \mathcal{F}\left(\frac{1}{c_0^2} - V\right)$ forward map.

Put $m = \frac{1}{c_0^2} + V$ model

Then $u(x,t) = \underbrace{\mathcal{F}\left(\frac{1}{c_0^2}\right)}_{u_{inc}} + \underbrace{\frac{\delta \mathcal{F}}{\delta m}\left(\frac{1}{c_0^2}\right) V}_{u_{sc,B}} + \underbrace{\ldots}_{\text{h.o.t.}}$

Call $F(m) = \frac{\delta \mathcal{F}}{\delta m}\left(\frac{1}{c_0^2}\right)$ the linearized forward map

(Lin. op. mapping V to $u_{sc,B}$)

1) Want estimate on u_{sc}, $(u_{sc,B} - u_{sc})$

Put $w = \begin{pmatrix} \partial u/\partial t \\ \nabla u \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

then $\dfrac{\partial w_{sc}}{\partial t} = \begin{pmatrix} 0 & c_0^2(x)\nabla\cdot \\ \nabla & 0 \end{pmatrix} w_{sc} + \underbrace{\begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \dfrac{\partial w}{\partial t}}_{= \begin{pmatrix} V \frac{\partial w_1}{\partial t} \\ 0 \end{pmatrix}}$

$= L w_{sc} + f$

↳ contains w_{sc} implicitly.

$\dfrac{\partial w_{sc,B}}{\partial t} = L w_{sc,B} + \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \dfrac{\partial w_{inc}}{\partial t}$

$= L w_{sc,B} + f_B$

↳ does not contain $w_{sc,B}$.

Note : $L^* = -L$ for

$$\langle w, w' \rangle = \int \left[\frac{1}{c_0^2(x)} w_1(x,t) w_1'(x,t) + w_2(x,t) w_2'(x,t) \right] dx$$

$E(t) = \langle w, w \rangle (t)$ is conserved when $f = 0$

$= \|w\|^2(t)$

Thm (slightly different) : Let $\dfrac{\partial w}{\partial t} = Lw + f(x,t)$, $L^* = -L$

↳ contains w or not.

and $w(0) = 0$,

then $\boxed{E(t) \leq \left(\int_0^t \|f\|(s)\, ds \right)^2}$

$\|w\|(t) \leq \int_0^t \|f\|(s)\, ds$.

Pf $\dfrac{d}{dt} \langle w, w \rangle = 2 \left\langle \dfrac{\partial w}{\partial t}, w \right\rangle$

$= 2 \langle Lw + f, w \rangle$

$= 2 \langle f, w \rangle$ because $L^* = -L$.

Cauchy - Schwarz:

$$\frac{d}{dt}\|w\|^2(t) \le 2\,\|f\|(t)\,\|w\|(t)$$

$$2\,\cancel{\|w\|}\,\frac{d}{dt}\|w\| \le 2\,\|f\|\,\cancel{\|w\|}$$

$$\|w\|(t) \le \int_0^t \|f\|(s)\,ds$$

$$E(t) \le \left(\int_0^t \|f\|(s)\,ds\right)^2 \quad \square.$$

Consequences:

(1) $\|w_x\|(t) \le \int_0^t \|f\|(s)\,ds$

$$\le \int_0^t \left\|\begin{pmatrix} V(x) \\ 0 \end{pmatrix}\frac{\partial w}{\partial t}\right\|(s)\,ds$$

and $\|g\,h\|^2 = \int\left(\frac{1}{c^2}g_1^2h_1^2 + g_2^2h_2^2\right)dx$

$$\le \overset{\sup}{\cancel{\max_x}}\ \max\{g_1^2(x,t), g_2^2(x,t)\}\int\left(\frac{1}{c^2}h_1^2 + h_2^2\right)dx$$

$$\le \|g\|_\infty^2\ \|h\|^2$$

$$\|gh\| \le \|g\|_\infty\,\|h\|$$

$$\Rightarrow \|w_x(t)\| \le \|V\|_\infty \int_0^t \left\|\frac{\partial w}{\partial t}\right\|(s)\,ds \qquad (1)$$

small if V small.

(2) $\|w_{x,E}\| \le \int_0^t \|f_B\|(s)\,ds$

$$\le \ldots \le \|V\|_\infty \int_0^t \left\|\frac{\partial w_{inc}}{\partial t}\right\|(s)\,ds.$$

Equation for $u_{sc} - u_{sc,B}$:

$$\left(\frac{\partial}{\partial t} - L\right)(u_{sc} - u_{sc,B}) = \begin{pmatrix} V \\ 0 \end{pmatrix} \frac{\partial}{\partial t} u_{sc}$$

$$\| u_{sc} - u_{sc,B} \| \le \int_0^t \left\| \begin{pmatrix} V \\ 0 \end{pmatrix} \frac{\partial u_{sc}}{\partial t} \right\| (s)\, ds.$$

Remark that $\left\| \frac{\partial u_{sc}}{\partial t} \right\|(t) \le \|V\|_\infty \int_0^t \left\| \frac{\partial^2 u}{\partial t^2} \right\|(s)\, ds$

(same reasoning as before).

$$\Rightarrow \| u_{sc} - u_{sc,B} \| \le \|V\|_\infty \int_0^t \left\| \frac{\partial u_{sc}}{\partial t} \right\|(s)\, ds$$

smaller than $\left\| \frac{\partial u_{sc}}{\partial t} \right\|$ if V small

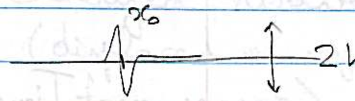
$$\boxed{\| u_{sc} - u_{sc,B} \| \le \|V\|_\infty^2 \int_0^t \int_0^s \left\| \frac{\partial^2 u}{\partial t^2} \right\|(\tau)\, d\tau\, ds}$$

quadratic in V.

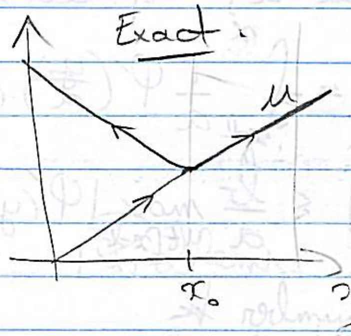
2) Heuristics (seismology): Born approx. works well when

- Background $\frac{1}{c_0^2(x)}$ is smooth
- Perturbation $V(x)$ is oscillatory and/or localized

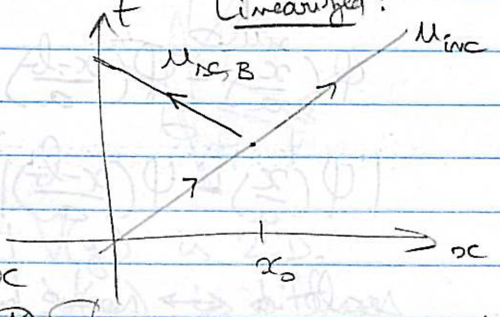
Ex 1 (Best case) $\frac{1}{c_0^2(x)} = 1$ $V(x) =$



Exact:



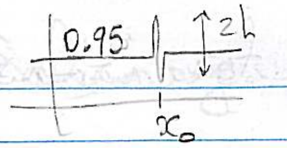
Linearized:



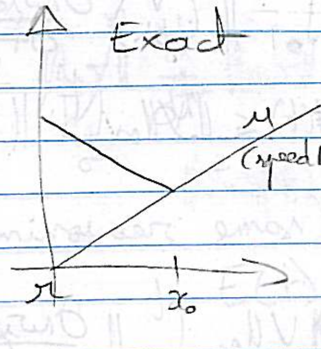
approx. primary refl. very well

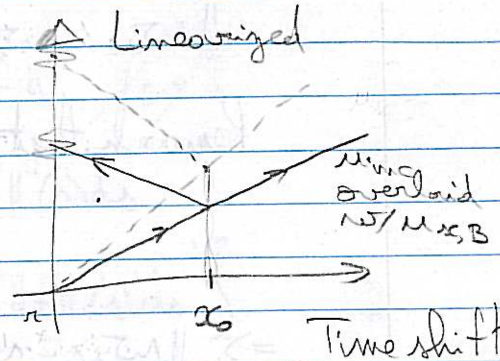
Ⓐ Show error in predicting multiples

Ex 3. (Worst case) $\frac{1}{c_0^2(x)} =$



$V(x) = -0.05$ $\qquad \frac{1}{c^2} = \frac{1}{c^2} - V$



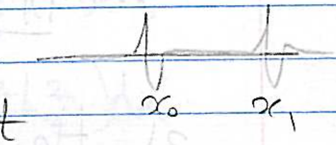


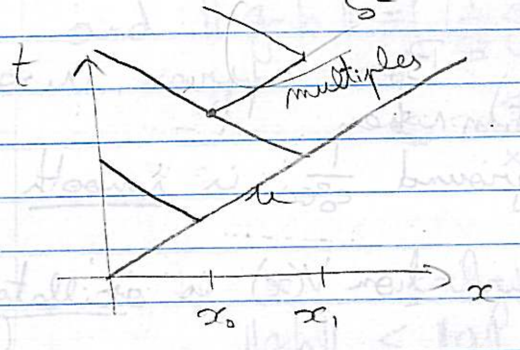
Time shifts!

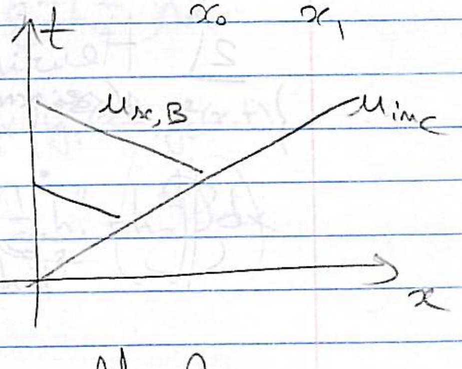
Show Bill's num. exp.

Bib ref (project): Lewis & Symes 1991

Ⓐ **Ex2.** $\frac{1}{c_0^2} = 1$, $V =$







does not capture multiples

Time shifts: pulse at scale a, spaced by b

$$\varphi\left(\frac{x}{a}\right) - \varphi\left(\frac{x-b}{a}\right) = -\int_{x-b}^{x} \frac{1}{a} \varphi'\left(\frac{y}{a}\right) dy$$

$$\left|\varphi\left(\frac{x}{a}\right) - \varphi\left(\frac{x-b}{a}\right)\right| \le \frac{b}{a} \max_{y \in [x-b, x]} |\varphi'(y)|$$

scale a ⟷ wave number k

⇒ big error for high-freq. waves!

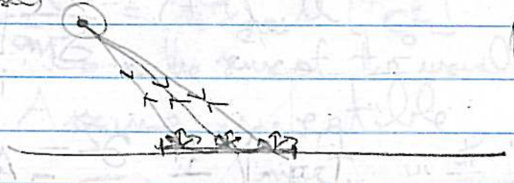
3) Heuristics (radar)

Frequency formulation: $u = E$,

$$u_{sc}(x,\omega) = -\int \frac{e^{ik\|x-y\|}}{4\pi\|x-y\|} V(y)\,\omega^2 u(y,\omega)\,dy$$

$$u_{sc,B}(x,\omega) = -\int \frac{e^{ik\|x-y\|}}{4\pi\|x-y\|} V(y)\,\omega^2 u_{inc}(y,\omega)\,dy.$$

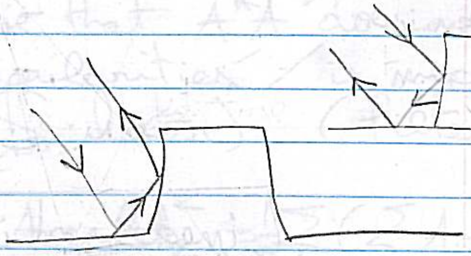
primary reflections (diffuse) captured by B.A.:

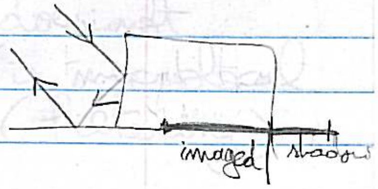


horizontal objects ✓

multiple reflections not captured by BA

e.g. specular reflections: corner





makes certain objects appear brighter.
displaced away from sensor

e.g. internal refl ⇒ delayed wave (displaced away from sensor).

Problems:
- $V(y)$ is 2D ⚠
- $V(y)$ is diffuse ⚠
- Born does not do multiples ⚠

10/08 **Chap. 3** Adjoint-state methods.

Forward map: $u(x,t) = \mathcal{F}\left[\frac{1}{c^2(x)}\right]$

from $\left(\frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = f$.

Linearized forward map: $u = u_{inc} + u_{sc,B}$

$$\frac{1}{c^2} = \frac{1}{c_0^2} + V$$

script

$$u_{sc,B}(x,t) = \frac{\delta \mathcal{F}}{\delta m}\left[\frac{1}{c_0^2}\right] V = F\left[\frac{1}{c_0^2}\right] V$$

print

from $\left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right) u_{sc,B} = V \frac{\partial^2 u_{inc}}{\partial t^2}$

zero IC, $x \in \mathbb{R}^n$

⚠ Assume c_0 is known

- radar: $c_0 = 1$
- seismology: c_0 = model velocity

$\Rightarrow u_{inc}$ is known

Linearized inverse problem for $u(x,t) \to \frac{1}{c^2(x)}$

Solution:

① Get $u_{sc,B}(x,t) = u(x,t) - u_{inc}(x,t)$

② Solve for $V(x)$ in $u_{sc,B} = F\left[\frac{1}{c_0^2}\right] V$

③ Get $\frac{1}{c^2} = \frac{1}{c_0^2} - V$

Concl: How to solve a linear system $u_{sc,B} = F\left[\frac{1}{c_0^2}\right] V$

(after discretization)

However, calculating the action of $F[\frac{1}{c_0^2}]$ requires solving a wave equation

$\Rightarrow$ cannot represent $(F[\frac{1}{c_0^2}])^{-1}$ directly

Seismic/radar : find decent approximations to the inverse w/ fast algorithms

<u>Claim</u> (adjoint-state methods) :

Adjoint $F[\frac{1}{c_0^2}]^* \approx F[\frac{1}{c_0^2}]^{-1}$ (poorman's inverse

$\hookrightarrow$ in the sense of the usual inner prod. in L^2

Why? Assume invertible $\hookrightarrow u_x = F V$ functi

$\hookrightarrow b = Ax$ vector

① $A^{-1} = (A^*A)^{-1}A^*$

and $A^*A \approx I$ (sometimes)

in the sense that A^*A does not move singularities / is microlocal

(\$, later) (filtering)

② $J[x] = \frac{1}{2}\|Ax-b\|_2^2 = \frac{1}{2}\sum_i(\sum_j A_{ij}x_j - b_i)^2$

$$\frac{\partial J}{\partial x_k} = \frac{1}{2}\sum_i \frac{\partial}{\partial x_k}(\sum_j A_{ij}x_j - b_i)^2$$

$$= \sum_i (\sum_j A_{ij}x_j - b_i)(\sum_j A_{ij}\frac{\partial x_j}{\partial x_k})$$

$$= \sum_i A_{ik}\sum_j A_{ij}x_j - b_i$$

$$= \sum_i (A^T)_{ki}\sum_j A_{ij}x_j - b_i$$

$$= (A^*(Ax-b))_k$$

$$D_x J = A^*(Ax-b)$$

Of course, $\nabla_x J = 0$

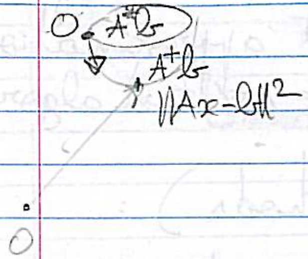
when $A^* A x = A^* b$

$\Rightarrow x = (A^* A)^{-1} A^* b$

$= A^+ b$ (pseudo inverse)

Iterative method:

- current guess $x_0 = 0$



direction of descent:

$$d = -\nabla_x J = -A^*(A x_0 - b)$$

$$= A^* b = x_1$$

- current guess $x_1 \neq 0$

$$-\nabla_x J = A^* \tilde{b} \text{ where}$$

$$\tilde{b} = b - A x_1 = \text{residual}$$

involves the adjoint.

Structure of $\left(F\left[\frac{1}{c^2}\right]\right)^*$: time formulation (seismology)

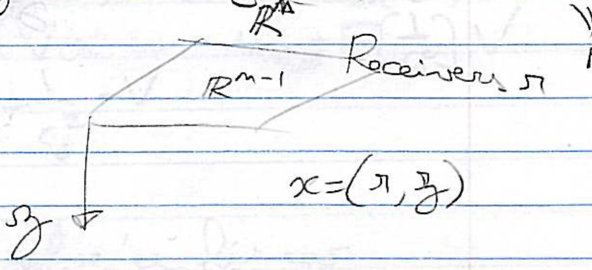
$$u_{x,B} = F\,V$$

$\downarrow$ fn. of r, t $\quad$ $\searrow$ fn of x

Take $d(r,t)$: (will want to take $d = u_{x,B}$ later)

$$\langle d, FV \rangle_{(r,t)} = \langle F^* d, V \rangle_x$$

$$\text{LHS} = \int_0^T \sum_r d(r,t)\, u_{x,B}(r,t)\, dr\, dt = \underbrace{\int_{\mathbb{R}^n} (F^* d)(x)\, V(x)\, dx}_{\text{RHS}}$$



Radar $n=2$

Seismo $n=3$

$x = (r, z)$

View receivers as $\delta(x-r)$

$$d(r,t) \to \sum_r \delta(x-r)\, d(r,t) = d_{ext}(x,t)$$

$$LHS = \int_0^T \sum_r d(r,t)\, u_{sc,B}(r,t)\, dr\, dt$$

$$= \int_0^T \int_{R^n} d_{ext}(x,t)\, u_{sc,B}(x,t)\, dx\, dt$$

Define an $\underline{\text{adjoint wavefield}}$ $q(x,t)$ such that

$$\begin{cases} \left(\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta\right) q(x,t) = d_{ext}(x,t) \\ x \in R^n, \text{ other B.C. in time TBD} \end{cases}$$

$$LHS = \int_0^T \int_{R^n} \left[\left(\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta\right) q(x,t)\right] \cdot u_{sc,B}(x,t)\, dx\, dt$$

2 integrations by parts in t

2 ———————— in x

Bdy terms: $\int_{R^n} \frac{1}{c^2} \frac{\partial q}{\partial t} u \Big|_0^T dx$, $\int_{R^n} \frac{1}{c^2} q \frac{\partial u}{\partial t}\Big|_0^T dx$,

$\int_0^T \int_S \frac{\partial q}{\partial n} u \, dS_x dt$, $\int_0^T \int_S q \frac{\partial u}{\partial n} dS_x dt$

then let $S \to \infty$

Vanish if: $u_{sc,B}(x,t) = 0$, $t \leq 0$ ✓

or $\|x\| \to \infty$ at fixed t ✓

$\underline{\text{and}}$ $q(x,t) = 0$, $t \geq T$

→ impose the final conditions $\begin{cases} \frac{\partial q}{\partial t}(x,T)=0 \\ q(x,T)=0 \end{cases}$

then solve $\left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2}-\Delta\right) q = d_{ext}(x,t)$

backward in t.

$$LHS = \int_0^T\int_{\mathbb{R}^n} q(x,t)\left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2}-\Delta\right) u_{sc,B}(x,t)\,dx\,dt$$

$$= \int_0^T\int_{\mathbb{R}^n} q(x,t)\, V(x)\frac{\partial^2 u_{inc}}{\partial t^2}\,dx\,dt$$

$$= \int_{\mathbb{R}^n} V(x)\int_0^T q(x,t)\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt$$

$$= RHS = \langle V, F^* d\rangle$$

$$= \int_{\mathbb{R}^n} V(x)\, F^*d\,(x)$$

$$\Rightarrow \boxed{F^*d\,(x) = \int_0^T q(x,t)\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt}$$

Computation of F^*d : ⓪ Form $d_{ext} = \sum_r \delta(x-r)\, d(r,t)$

(e.g. $d = u_{sc,B}$)

① Solve adjoint wave equation

$$\begin{cases} \left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2}-\Delta\right) q = d_{ext}(x,s) \\ \text{final cond. } q(x,T)=0. \\ \qquad\qquad \frac{\partial q}{\partial t}(x,T)=0 \end{cases}$$

② Form time integral

$$\int_0^T q(x,t)\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt.$$

for each x

q = adjoint state (not phys., wavefield run back in time !)

Seismology:

Called reverse-time migration

F^*: imaging operator/migration op. (F=demigration)

~~10/13/09~~
10/20/09

Motivation for q: it is a Lagrange multiplier

Consider misfit functional

$$J[u_{sc,B}, V] = \frac{1}{2}\|u_{sc,B} - d\|_2^2$$

where d = recorded data minus u_{inc}

$u_{sc,B}$ = predicted data (linearized)

$$\min J[u_{sc,B}, V]$$

$$\text{s.t.} \left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right)u_{sc,B} = V(x)\frac{\partial^2 u_{inc}}{\partial t^2}$$

$$(u(t<0)=0)$$

Minimize J under constraints:

Form Lagrangian

$$\mathcal{L}[u_{sc,B}, V, q] = J[u_{sc,B}, V] - \int_0^T\int_{\mathbb{R}^n} q(x,t)\left[\left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right)u_{sc,B} - V(x)\frac{\partial^2 u_{inc}}{\partial t^2}\right]dx\,dt$$

$$+ \int \mu_0(x)\, u_{sc,B}(x,0)\,dx + \int \mu_1(x)\frac{\partial u_{sc,B}}{\partial t}(x,0)\,dx$$

(int. by parts)

$$= J[u_{sc,B}, V] - \int_0^T\int_{\mathbb{R}^n}\left(\frac{1}{c_0^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right)q \cdot u_{sc,B}\,dx\,dt$$

$$+ \int_{\mathbb{R}^n}\frac{1}{c_0^2(x)}\, q(x,T)\frac{\partial u_{sc,B}}{\partial t}(x,T)\,dx$$

$$- \int_{\mathbb{R}^n}\frac{1}{c_0^2(x)}\frac{\partial q}{\partial t}(x,T)\, u_{sc,B}(x,T)\,dx$$

$$+ \int\left[\mu_1(x) - \frac{1}{c_0^2(x)}\, q(x,0)\right]\frac{\partial u_{sc,B}}{\partial t}(x,0)\,dx$$

$$+ \int\left[\mu_0(x) - \frac{1}{c_0^2(x)}\frac{\partial q}{\partial t}(x,0)\right]u_{sc,B}(x,0)\,dx$$

$$+ \int_0^T\int_{\mathbb{R}^n} q(x,t)\, V(x)\frac{\partial^2 u_{inc}}{\partial t^2}\,dx\,dt$$

with $J[u_{x_s,B}, V]$

$$= \frac{1}{2}\int_0^T \sum_r \left(d(r,t) - u_{x_s,B}(r,t)\right)^2 dt$$

$$= \frac{1}{2}\int_0^T \sum_r \left(d(r,t)^2 - 2d(r,t)\,u_{x_s,B}(r,t) + u^2_{x_s,B}(r,t)\right) dt$$

$$= 2\int_0^T \sum_r d(r,t)^2 - \int_0^T\int_{\mathbb{R}^n} d_{ext}(x,t)\,u_{x,B}(x,t)\,dx\,dt + \frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\left(\sum_r \delta(x-r)\,u^2_{x_s,B}(x,t)\right) dx\,dt$$

$$\frac{\delta J}{\delta u_{x_s,B}(x,t)} = -d_{ext}(x,t) + \underbrace{\sum_r \delta(x-r)\,u_{x_s,B}(x,t)}_{u_{x,B,ext}(x,t)}$$

$$\frac{\delta L}{\delta u_{x_s,B}(x,t)} = \frac{\delta J}{\delta u_{x_s,B}(x,t)} - \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) q = 0$$

$$\Rightarrow \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) q = d_{ext}(x,t) - u_{x_s,B,ext}(x,t)$$

$$\frac{\delta L}{\delta u_{x_s,B}(x,T)} = \frac{1}{c_0^2(x)}\frac{\partial q}{\partial t}(x,T) = 0$$

$$\frac{\delta L}{\delta \frac{\partial u_{x_s,B}}{\partial t}(x,T)} = \frac{1}{c_0^2}\,q(x,T) = 0$$

$$\frac{\delta L}{\delta V} = \frac{\delta J}{\delta V} + \int_0^T q(x,t)\,\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt = 0$$

$$\Rightarrow \boxed{\frac{\delta J}{\delta V} = -\int_0^T q(x,t)\,\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt}$$

$A^*(b - A x_1)$ $\cancel{⑧}$ ③

= "adjoint applied to residual"

$= F^*\left(d(r,t) - u_{x,B}(r,t)\right) = F^*\left(\tilde{d}(r,t)\right)$

⓪ Form $d_{ext}(x,t) - u_{x,B,ext}(x,t) = \tilde{d}_{ext}(x,t)$

① Solve adj. state eq. w/ $\tilde{d}_{ext}$

② Form time integral

$\frac{\delta L}{\delta q} = 0 \Rightarrow$ eq. for $u_{x,B}$

$\frac{\delta L}{\delta \mu_{0,1}} = 0 \Rightarrow$ init cond for $u_{x,B}$

$V_m = \alpha \frac{\delta J}{\delta V} V_{m-1}$

$\frac{1}{c_m^2} = \frac{1}{c_{m-1}^2} - V_m$

Convergence of gradient descent: slow

If update $c(x)$ at each step: full waveform inversion (convergence: bad)

Next: frequency formulation, radar, backprojection

Intro: Recall $J = \frac{1}{2}\|Ax - b\|_2^2$

Steepest descent direction: $-\nabla J = A^*(b - Ax)$
$= A^* r$

Interested in $u_{x,B} = F\,V$
$b = A\,x$

1) Time formul$^{\underline{o}}$: We saw $F^*\,d\,(x) = \int_0^T q(x,t)\,\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt$

where $\begin{cases} \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) q = d_{ext}(x,t) \\ \qquad = \sum \delta(x-r)\,d(r,t) \\ q(x,T) = \frac{\partial q}{\partial t}(x,T) = 0. \end{cases}$

q = adjoint state

Interpretation: Lagrange multiplier

2) Frequency formulation (radar)

$(c = \text{const} = 1)$

Take a Fourier transform in $t \to \omega$

$$G(x,\omega) = \frac{e^{ik\|x\|}}{4\pi\|x\|} \qquad k = \frac{\omega}{c} = \omega$$

obeys $(-\omega^2 - \Delta)\, G(x,\omega) = \delta(x)$

- Incident field:

$$(-\omega^2 - \Delta)\, u_{inc}(x,\omega) = f(x,\omega)$$

$$\Rightarrow u_{inc}(x,\omega) = \int \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} f(y,\omega)\, dy$$

f contains: • pulse function
• antenna beam pattern.

- Scattered field (Born)

$$(-\omega^2 - \Delta)\, u_{sc,B}(x,\omega) = -\omega^2 V(x)\, u_{inc}(x,\omega)$$

$$\Rightarrow u_{sc,B}(x,\omega) = \int \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} \times (-k^2)\, V(y) \int \frac{e^{i\omega\|y-z\|}}{4\pi\|y-z\|} f(z,\omega)\, dz\, dy$$

$$= F\, V$$

↳ lin. forw. map

$$F V(x,\omega) = \int \left[(-k^2) \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} \int \frac{e^{i\omega\|y-z\|}}{4\pi\|y-z\|} f(z,\omega)\, dz \right] V(y)\, dy$$

Imaging operator :

$$\text{LHS} = \langle FV, d \rangle_{(x,\omega)} = \langle V, F^* d \rangle_{(x)} = \text{RHS}$$

$$= \int \sum_{x} (FV)(x,\omega)\,\overline{d(x,\omega)}\,d\omega \qquad = \int V(x)\,F^*d(x)\,dx$$

$$\text{LHS} = \underset{\overset{\downarrow}{\omega}\;\overset{\downarrow}{x}}{\iint} \overline{d}_{ext}(x,\omega) \underset{\overset{\downarrow}{y}}{\int} V(y) \underset{\overset{\downarrow}{z}}{\int} (-\omega^2) \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} \frac{e^{i\omega\|y-z\|}}{4\pi\|y-z\|} f(z,\omega)\,dz \; dy\,dx\,d\omega$$

$$= \underset{\overset{\downarrow}{y}}{\int} V(y) \underset{\overset{\downarrow}{\omega}}{\int} (-\omega^2) \left[\underset{\overset{\downarrow}{x}}{\int} \overline{d}_{ext}(x,\omega) \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|}\,dx \right]$$

$$\times \left[\underset{\overset{\downarrow}{z}}{\int} f(z,\omega) \frac{e^{i\omega\|y-z\|}}{4\pi\|y-z\|}\,dz \right] d\omega\,dy$$

$$= \underset{\overset{\downarrow}{y}}{\int} V(y) \underset{\overset{\downarrow}{\omega}}{\int} (-k^2) \;\overline{q}\,(y,\omega)\, u_{inc}(y,\omega)\,d\omega$$

where $q(x,\omega) = \int \frac{e^{-i\omega\|x-y\|}}{4\pi\|x-y\|} d_{ext}(y,\omega)\,dy$

adjoint state: $(-\omega^2 - \Delta)\,q(x,\omega) = d_{ext}(x,\omega)$
(no I.C. or F.C.)

$$\Rightarrow \boxed{F^* d(x) = -\omega^2 \int \overline{q}(x,\omega)\, u_{inc}(x,\omega)\,d\omega}$$
(real-valued)

$\left(= \int \check{q}(x,t) \frac{\partial^2 \check{u}_{inc}}{\partial t^2}(x,t)\,dt \right.$ by Plancherel $\left. \right)$

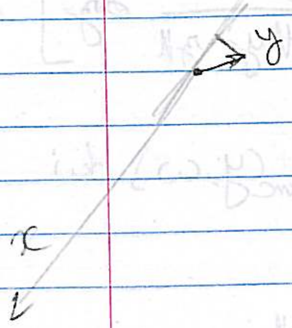
$\Rightarrow$

$$\mathbb{R} \ni F^*d(x) = \underset{\overset{\downarrow}{y}}{\int}\underset{\overset{\downarrow}{z}}{\int}\underset{\overset{\downarrow}{\omega}}{\int} d_{ext}(y\omega)\, \frac{e^{-i\omega\|x-y\|}}{4\pi\|x-y\|}$$

$$\times\ \overline{f(z,\omega)}\, \frac{e^{-i\omega\|x-z\|}}{4\pi\|x-z\|}\, dz\, dy\, d\omega.$$

Simplification commonly made in radar.

Far-field approx. $\|x\| >> \|y\|$



$$\|x-y\| = \sqrt{\|x\|^2 - 2x\cdot y + \|y\|^2}$$

$$= \|x\|\sqrt{1 - \frac{2\hat{x}\cdot y}{\|x\|} + \frac{\|y\|^2}{\|x\|^2}} \qquad \hat{x} = \frac{x}{\|x\|}$$

$$= \|x\|\left(1 - \frac{\hat{x}\cdot y}{\|x\|} + O\left(\frac{\|y\|^2}{\|x\|^2}\right)\right)$$

$$= \|x\| - \hat{x}\cdot y + O\left(\frac{\|y\|^2}{\|x\|}\right)$$

$$e^{i\omega\|x-y\|} = e^{i\omega\|x\|}\, e^{-i\omega\hat{x}\cdot y}\left(1 + O\left(\frac{\omega\|y\|^2}{\|x\|}\right)\right)$$

$$\frac{1}{\|x-y\|} = \frac{1}{\|x\|}\left(1 + O\left(\frac{\|y\|}{\|x\|}\right)\right)$$

$$G(x-y) \simeq \frac{e^{i\omega\|x\|}}{4\pi\|x\|}\, e^{-i\omega\hat{x}\cdot y}$$

provided $\|y\| \ll \|x\|$

$\omega\|y\|^2 \ll \|x\|$

- Radiation field (far from antenna near the origin)

$$u_{inc}(x,\omega) = \int \frac{e^{i\omega\|x\|}}{4\pi\|x\|} e^{-i\omega\hat{x}\cdot y} \, j(y,\omega)\, dy$$

↳ current density (vector) $J(y)\, p(\omega)$ × pulse p.

$$= \frac{e^{i\omega\|x\|}}{4\pi\|x\|} \, \hat{j}(\omega\hat{x}, \omega)$$

↳ called radiation vector, $= J(\hat{x}, \omega)$ or its scalar analogue.

when antenna at x_0:

$$u_{inc}(x,\omega) = \frac{e^{i\omega\|x-x_0\|}}{4\pi\|x-x_0\|} \, J(\widehat{x-x_0}, \omega)$$

↳ radiation beam pattern & transmitted waveform

- Reception: $S_{rec,B}(\omega) = \int_{\substack{\text{antenna} \\ \text{near } x_0}} u_{sc,B}(z,\omega)\, w(z,\omega)\, dz$

(supp V far from antenna x_0)

window (vec → scal)

$$= \iint \frac{e^{i\omega\|z-y\|}}{4\pi\|z-y\|} (-\omega^2) \, V(y)\, u_{inc}(y,\omega)\, w(z,\omega)\, dz\, dy$$

$$\simeq \int \frac{e^{i\omega\|x_0-y\|}}{4\pi\|x_0-y\|} (-\omega^2) \, V(y)\, u_{inc}(y,\omega) \, \underbrace{\hat{w}(\omega(\widehat{y-x_0}), \omega)}_{W(\widehat{y-x_0}, \omega)} \, dy$$

reception beam pattern

(usually same as radiation beam pattern, disregarding pulse ($\hat{p} = 1$))

$$\simeq \int e^{2i\omega\|x_0-y\|} A(\omega, x_0, y)\, V(y)\, dy$$

where $A(\omega, x_0, y) = -\omega^2 \dfrac{J(\omega, \widehat{y-x_0})\, W(\omega, \widehat{y-x_0})}{(4\pi\|x_0-y\|)^2}$

Continuum model for slow time, $x_0=\gamma(s)$

$$d(s,\omega) = S_{rec,B}(\gamma(s),\omega)$$

$$\boxed{d(s,\omega) = \int e^{2i\omega\|\gamma(s)-y\|} A(\omega,s,y)\, V(y)\, dy = FV}$$

$$\Rightarrow d(t,s) = \iint e^{-i\omega[t-2\|\gamma(s)-y\|/c]} A(\omega,s,y) V(y)\, dy\, d\omega$$

$$= \int \hat{A}\left(t-2\frac{\|\gamma(s)-y\|}{c}, s, y\right) V(y)\, dy\, d\omega.$$

Formal adjoint:

$$\boxed{F^* d = \int e^{-2i\omega\|\gamma(s)-x\|}\, \overline{A(\omega,s,x)}\, d(s,\omega)\, ds}$$

$\rightarrow$ Backprojection.