18.325 Today: hyperbolic equations (linear)
09/15/09 acoustic, elastic, electromagnetic waves (from physics to wave eq.)
conservation laws (mass, momentum en)

Thurs: plane wave sol^n, dispersion rel^n
traveling wave sol^n, etc.

1) Acoustic waves (air, water)

v = velocity (molecules)
ρ = density
p = pressure

Gas dynamics: no viscosity

$$\rho\left(\underbrace{\frac{\partial v}{\partial t} + v \cdot \nabla v}_{Dv/Dt}\right) = -\nabla p \qquad \text{(momentum)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \qquad \text{(mass)}$$

$$p \sim \rho^\gamma \qquad \text{(constitutive rel - ideal adiabatic revers.)}$$

where $\nabla \cdot v = \sum_i \partial_i v_i$

$(\nabla v)_{ij} = \partial_i v_j$

⚠ Small disturbances.

(real) $p_1 = p_0 + p$
$\rho_1 = \rho_0 + \rho$
$v_1 = v_0 + v$ neglect $O(p^2 + \rho^2 + v^2)$

⚠ Medium at rest.

p_0, ρ_0 independent of t.
$v_0 = 0$

Momentum: $(\rho_0 + \cancel{\rho})\left(\frac{\partial v}{\partial t} + \cancel{v_0 \cdot \nabla v_0} + \cancel{v \cdot \nabla v_0} + \cancel{v_0 \cdot \nabla v}\right) = -\nabla p_0 - \nabla p$

Order 0: $\nabla p_0 = 0$ (no static stress)

Order 1:

$$\rho_0 \frac{\partial v}{\partial t} = -\nabla p \qquad (1)$$

Mass: $\frac{\partial \rho}{\partial t} + \cancel{\rho_0 \nabla \cdot v_0} + \rho_0 \nabla \cdot v + \cancel{\rho \nabla \cdot v_0} = 0$ (2)

Constitutive: "$p \sim \rho^\gamma$" $\gamma = 1.4$ (air)

"$\frac{dp}{d\rho} = \gamma \frac{p}{\rho}$" $\Rightarrow$ $\frac{p}{\rho} = \gamma(x) \frac{p_0}{\rho_0(x)}$

$= c_0^2(x)$ (def.)

$$\rho = \frac{1}{c_0^2} p$$

$$\frac{\partial p}{\partial t} = -\underbrace{\rho_0(x) c_0^2(x)}_{\kappa(x) \text{ bulk modulus}} \nabla \cdot v$$

$$\boxed{\begin{aligned} \frac{\partial v}{\partial t} &= \frac{-1}{\rho_0(x)} \nabla p \\ \frac{\partial p}{\partial t} &= -\kappa(x) \nabla \cdot v \end{aligned}}$$

acoustics

I.C. on p, v.

Wave equation: $\frac{\partial^2 p}{\partial t^2} = \kappa(x) \nabla \cdot \frac{1}{\rho_0(x)} \nabla p$

I.C. on $p, \frac{\partial p}{\partial t}$

Rmk: $\kappa, \rho_0 > 0$. Rmk: $\frac{\partial^2 p}{\partial t^2} = c_0^2 \Delta p$ if ρ_0 const.

Conservation of energy.

$w = \begin{pmatrix} v \\ p \end{pmatrix}$ $\quad \frac{\partial}{\partial t}\begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa \nabla \cdot & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$

$$\frac{\partial w}{\partial t} = L w$$

$L \to$ generator.

Def. $\frac{\partial w}{\partial t} = Lw$ is said to be a hyperbolic system of eqn. if

- L is 1^{st}-order differential
- there exists an inner product $\langle w, w' \rangle$ with respect to which $L^* = -L$ (skew-Hermitian), where L^* is defined as

$$\forall w, w', \quad \langle Lw, w' \rangle = \langle w, L^* w' \rangle$$

Ex. $L = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa \nabla\cdot & 0 \end{pmatrix}$

$$\langle w, w' \rangle = \frac{1}{2}\int \left[\rho_0\, \underline{v}\cdot\underline{v}' + \frac{1}{\kappa} p p'\right] dx.$$

$$\begin{aligned}
\langle Lw, w' \rangle &= \frac{1}{2}\int \left[\rho_0\left(-\frac{1}{\rho_0}\nabla p\right)\cdot v' + \frac{1}{\kappa}(-\kappa \nabla\cdot v)\, p'\right] dx \\
&= -\frac{1}{2}\int \left[\nabla p\cdot v' + \nabla\cdot v\, p'\right] dx \\
&= \frac{1}{2}\int \left[p\, \nabla\cdot v' + v\cdot\nabla p'\right] dx \\
&= \frac{1}{2}\int -\frac{1}{\kappa} p\left(-\kappa\nabla\cdot v'\right) - \rho_0 v\left(-\frac{1}{\rho_0}\nabla p'\right) dx \\
&= -\langle w, Lw' \rangle \\
&= \langle w, L^* w' \rangle \quad \Rightarrow L^* = -L.
\end{aligned}$$

Thm. If $\frac{\partial w}{\partial t} = Lw$ is hyperbolic, then $E = \langle w, w \rangle$ is conserved.

Pf. $\frac{d}{dt}(w,w) = \left(\frac{\partial w}{\partial t}, w\right) + \left(w, \frac{\partial w}{\partial t}\right)$

$= 2\left(\frac{\partial w}{\partial t}, w\right)$

$= 2(Lw, w)$

$= 2(w, L^* w)$ $= 0$

$= 2(w, (-L)w)$

$= 0$. $\square$.

Ex. $E = (w,w) = \frac{1}{2}\int \rho_0 w^2 + \frac{p^2}{K_0}$ total energy

kinetic potential

2) Elastic waves (solid materials).

u = displacement

$\underline{w} = \frac{\partial u}{\partial t}$ velocity

$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla u + (\nabla u)^T)$ strain tensor

$\underline{\underline{\sigma}}$ = stress tensor.

$\rho \frac{\partial w}{\partial t} = \nabla \cdot \underline{\sigma}$ $(+f)$ ($F = ma$, momentum)

$\underline{\underline{\sigma}} = \underline{\underline{\underline{C}}} : \underline{\underline{\varepsilon}}$ (constitutive, Hooke) stress-strain

C = elastic tensor

$\frac{\partial \varepsilon}{\partial t} = \frac{1}{2}(\nabla w + \nabla w^T)$ (def.)

⚠ ρ indep. of t. $(\rho(x,0) = \rho(x+u(x,t), t))$

or $\rho \frac{\partial v_i}{\partial t} = \sum_j \partial_j \sigma_{ij}$

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2}\partial_i v_j + \partial_j v_i$$

Hyperbolic system: $w = \begin{pmatrix} v \\ \varepsilon \end{pmatrix}$

$$\frac{\partial w}{\partial t} = \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix} w = L w \qquad L^* = \begin{pmatrix} 0 & L_1^* \\ L_2^* & 0 \end{pmatrix}$$

where $L_1 v = \frac{1}{2}(\nabla v + \nabla v^T)$

$$L_2 \varepsilon = \frac{1}{\rho} \nabla \cdot (C : \varepsilon)$$

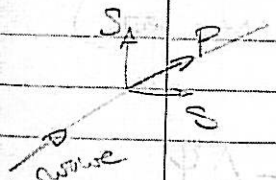
$L_1^* = -L_2 \Rightarrow L^* = -L$ provided $(w, w') = \frac{1}{2}\int \rho\, v \cdot v' + \varepsilon^T : C : \varepsilon'$

$E = \langle w, w \rangle$ conserved

In general, C has $81 \to 21$ components

<u>Isotropic elasticity</u>

compressional longitudinal (P) transverse shear (S).



$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \quad (\ast)$$

$$\rho \frac{\partial v}{\partial t} = \nabla(\lambda \nabla \cdot u) + \nabla \cdot (\mu (\nabla u + \nabla u^T))$$

$$\rho \frac{\partial v_i}{\partial t} = \partial_i (\lambda \partial_j u_j) + \partial_j (\mu (\partial_i u_j + \partial_j u_i))$$

free: i

dummy: j

$$L_2 \varepsilon = \frac{1}{\rho}[\nabla(\lambda\, \mathrm{tr}\, \varepsilon) + 2 \nabla \cdot (\mu \varepsilon)]$$

$$\mathrm{tr}\, \varepsilon = \varepsilon_{ii}$$

$$\langle v, v' \rangle = \frac{1}{2}\int \Big(\rho\, v^* v' + 2\mu\, \mathrm{tr}(\varepsilon^T \varepsilon') + \lambda (\mathrm{tr}\,\varepsilon)(\mathrm{tr}\,\varepsilon') \Big)\, dx$$

Link to the wave equation.

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda \nabla\cdot u) + \nabla\cdot\big(\mu(\nabla u + (\nabla u)^T)\big) + f$$

⚠ Assume λ, μ constant

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot u) + \mu \Delta u$$

$$\Delta u = \nabla(\nabla\cdot u) - \nabla\times\nabla\times u$$

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla\cdot u) - \mu \nabla\times\nabla\times u$$

Helmholtz decomposition:

$$u = \nabla \underbrace{\phi}_{\text{scalar f.}} + \nabla\times \underbrace{\psi}_{\text{vector f.}} \ (+h) \qquad \Delta h = 0 \Rightarrow h = 0.$$

$\nabla\cdot\nabla\times v = 0$
$\nabla\times\nabla f = 0$

Gauge choices $\phi_{new} = \phi_{old} + C$
$\psi_{new} = \psi_{old} + \nabla f$

Choose f such that

$$\nabla\cdot\psi_{new} = 0$$
$$0 = \nabla\cdot\psi_{old} + \Delta f$$
$$\Delta f = -\nabla\cdot\psi_{old} \quad \text{(Poisson eq.)}$$

Then $\nabla\cdot u = 0$

$$\nabla\times u = \nabla\times\nabla\times\psi = -\Delta\psi$$

$$\nabla\left[\rho\frac{\partial^2\phi}{\partial t^2} - (\lambda+2\mu)\Delta\phi\right] + \nabla\times\left[\rho\frac{\partial^2\psi}{\partial t^2} - \mu\Delta\psi\right] = 0$$

$$\Rightarrow \rho\frac{\partial^2\phi}{\partial t^2} - (\lambda+2\mu)\Delta\phi = \text{harmonic} = 0.$$
$$\rho\frac{\partial^2\psi}{\partial t^2} - \mu\Delta\psi = \nabla(\text{something}) = 0.$$

ϕ: compression wave
ψ: shear wave
⚠ Lose mode conversion at interfaces
Limit $\mu \to 0$: $\lambda = K$, bulk modulus
⚠ Lose shear wave.

09/17/09 3) Electromagnetic waves.

E = electric field
H = magnetic field
ε = el. permittivity $\varepsilon(x)$
μ = mag permeability $\mu(x)$
j = current
ρ = charge

$$\nabla \times E = -\frac{\partial(\mu H)}{\partial t} \quad \text{(Faraday)}$$

$$\nabla \times H = \frac{\partial(\varepsilon E)}{\partial t} + j \quad \text{(Ampère-Maxwell)}$$

$$\left\{\begin{array}{ll} \nabla \cdot (\varepsilon E) = \rho & \text{(Gauss)} \\ \nabla \cdot (\mu H) = 0. & \text{(Gauss)} \end{array}\right.$$

$\rightarrow$ constraints on I.C. (!)

1861: Maxwell
1880: vector calculus (Gibbs)

Constitutive: $D = \varepsilon E$
$B = \mu H$

Dry air: ε_0, μ_0.
Total charge: $\int \rho \, dV$
Total current: $\int j \cdot dS$.

System form $v = \begin{pmatrix} E \\ H \end{pmatrix}$

$$\frac{\partial v}{\partial t} = \overbrace{\begin{pmatrix} 0 & \frac{1}{\varepsilon}\nabla\times \\ -\frac{1}{\mu}\nabla\times & 0 \end{pmatrix}}^{=L} v + \begin{pmatrix} -j/\varepsilon \\ 0 \end{pmatrix}$$

$$(v, v') = \frac{1}{2}\int (\varepsilon E E' + \mu H H')\, dx$$

$$L = -L^* \Rightarrow E = \frac{1}{2}\int (\varepsilon E^2 + \mu H^2)\, dx \text{ \underline{conserved}}$$

<u>Continuity eq.</u> $\nabla\cdot\frac{\partial \varepsilon E}{\partial t} = \frac{\partial \rho}{\partial t} = -\nabla\cdot j$

<u>Wave equation</u> $(j=0)$

$$\nabla\cdot(F\times G) = G\cdot(\nabla\times F) - F\cdot(\nabla\times G)$$

$$\frac{1}{\varepsilon}\nabla\times\left(\frac{1}{\mu}\nabla\times E\right) = -\frac{\partial^2 E}{\partial t^2}$$

$$\frac{1}{\mu}\nabla\times\left(\frac{1}{\varepsilon}\nabla\times H\right) = -\frac{\partial^2 H}{\partial t^2}$$

Either : potentials

or : $\nabla\times\nabla\times v = \nabla(\nabla\cdot v) - \Delta v$

$$\Delta E - \varepsilon\mu\frac{\partial^2 E}{\partial t^2} + \underbrace{\left(\frac{\nabla\mu}{\mu}\right)\times(\nabla\times E) + \nabla\left(E\cdot\left(\frac{\nabla\varepsilon}{\varepsilon}\right)\right)}_{\substack{1^{st}\text{ order junk} \\ =0 \text{ when } \varepsilon,\mu \text{ const.}}} = 0$$

$$c = \frac{1}{\sqrt{\varepsilon\mu}} \quad \text{speed of light}$$

Insist: waves when • 1^{st} order system in time, 1^{st} order diff. L, $L^* = -L$

or • 2^{nd} order scalar (in t), 2^{nd} order diff. in x

Announce: register! ref on webpage

1) Equations

2) Plane wave solutions – dispersion relations

Assume a time-harmonic dependence

$$w(x,t) = e^{-i\omega t} f_\omega(x)$$

$$\frac{\partial w}{\partial t} = Lw \Rightarrow \quad -i\omega f_\omega = L f_\omega$$

eigenvalue problem.

2.1) $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ ("one-way" wave eq.) $c =$ const.

$$L = -c\frac{\partial}{\partial x}$$

$$L f_\omega = \lambda_\omega f_\omega \qquad f_\omega(x) = e^{ikx} \text{ (oscill. in } x\text{)}$$

$$\lambda_\omega = -ikc$$

$$\lambda_\omega = -ikc = -i\omega \Rightarrow \boxed{\omega = kc}$$

(f_ω is a generalised eigenfunction)

$\omega =$ angular frequency

$k =$ wave number

$\omega = kc$: dispersion relation.

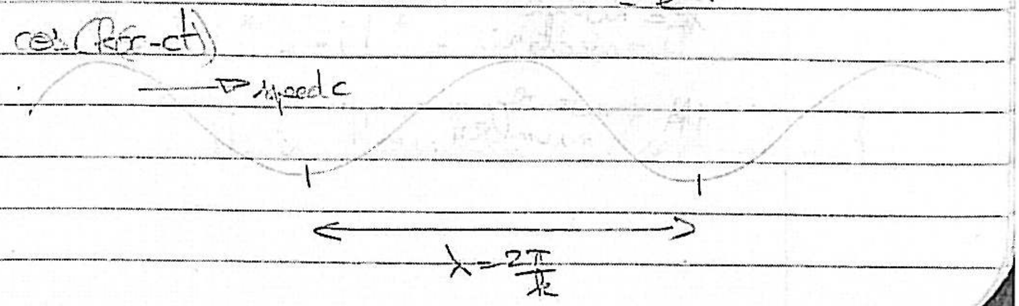
$\omega = \frac{2\pi}{T}$ ($T =$ period)

$k = \frac{2\pi}{\lambda}$ ($\lambda =$ wavelength)

$$\Rightarrow \frac{2\pi}{T} = \frac{2\pi}{\lambda} c, \qquad \lambda = cT$$

Solution: $w(x,t) = e^{i(kx-\omega t)}$ with $\omega = kc$

$\to$ plane wave solution $= e^{ik(x-ct)}$



$$\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x} = 0 \Rightarrow w = e^{ik(x+ct)}$$

left-traveling wave

2.2) $\frac{\partial^2 u}{\partial t^2} = \kappa \nabla \cdot \frac{1}{\rho} \nabla u$ κ, ρ constant

$$= c^2 \Delta u \qquad c = \sqrt{\frac{\kappa}{\rho}}$$

$$u(x,t) = e^{-i\omega t} f_\omega(x)$$

$$(-i\omega)^2 f_\omega = c^2 \Delta f_\omega$$
$$-\omega^2 f_\omega = c^2 \Delta f_\omega$$

Guess: $f_\omega(x) = e^{ik\cdot x}$ $\quad k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

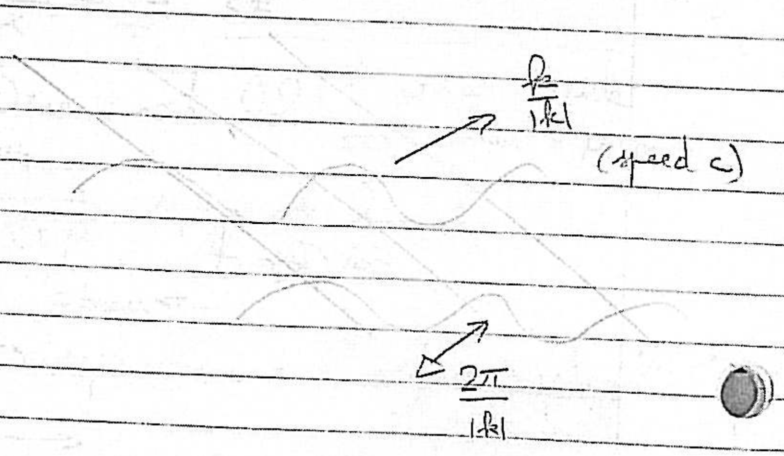
$$-\omega^2 = c^2 (ik \cdot ik)$$

$\boxed{\omega^2 = c^2 |k|^2}$ characteristic equation

$\boxed{\omega = \pm |k| c}$ dispersion relations.

$$u(x,t) = e^{i(k\cdot x - \omega t)} \quad \text{with } \omega = \pm |k| c$$

$$= e^{ik\cdot(x \pm \frac{k}{|k|} c t)}$$



k = wave vector

$|k|$ = wave number.

2.31 $\frac{\partial w}{\partial t} = Lw, \qquad L^* = -L$

L given in pseudodifferential form: (multiplication in the Fourier domain)

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, d^n x$$

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k)\, dk.$$

$$\nabla f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ik\cdot x} (ik)\, \hat{f}(k)\, dk.$$

$$\Delta f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ik\cdot x} (-|k|^2)\, \hat{f}(k)\, dk$$

Translation invariant diff. op.:

$$Lf(x) = \int_{\mathbb{R}^n} e^{ik\cdot x} P(k)\, \hat{f}(k)\, dk$$

$\hookrightarrow$ matrix with polynomial entries = dispersion matrix.

Non-transl. inv. diff op.: $P(x,k)$

$$w(x,t) = e^{-i\omega t} f_\omega(x)$$

$$-i\omega f_\omega = L f_\omega$$

$$\omega f_\omega = iL f_\omega$$

(ω, f_ω) is an eigenvalue / eigenfunction pair for iL

$$(iL)^* = \bar{i} L^* = -iL^* = -i(-L) = iL$$

iL is Hermitian / self-adjoint.

$\Rightarrow \omega$ real-valued
(not discrete)

f_ω orthogonal: $(f_\omega, f_{\omega'}) = 0$.

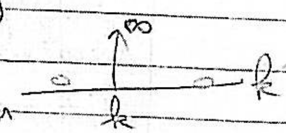
$\Rightarrow$ ΨDO form.

e.g.
$$L = \begin{pmatrix} 0 & 0 & 0 & -\partial/\partial x_1 \\ 0 & 0 & 0 & -\partial/\partial x_2 \\ 0 & 0 & 0 & -\partial/\partial x_3 \\ -\partial/\partial x_1 & -\partial/\partial x_2 & -\partial/\partial x_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \\ -\nabla\cdot & 0 \end{pmatrix}$$

$$\Rightarrow P(k) = \begin{pmatrix} 0 & 0 & 0 & -ik_1 \\ 0 & 0 & 0 & -ik_2 \\ 0 & 0 & 0 & -ik_3 \\ -ik_1 & -ik_2 & -ik_3 & 0 \end{pmatrix} \qquad P^*(k) = -P(k)$$

Try
$$L\, e^{ik\cdot x}\, \underline{r} = \frac{1}{(2\pi)^n} \int e^{ik'\cdot x}\, P(k')\, \underline{r}\, F(e^{ik\cdot x})(k')\, dk'$$

$$F(e^{ik\cdot x})(k') = \int e^{-ik'\cdot x}\, e^{ik\cdot x}\, dx = \delta(k-k')\,(2\pi)^n$$
(approximate identity)



$$L e^{ik\cdot x}\, \underline{r} = \int e^{ik'\cdot x}\, \underbrace{P(k')}_{\text{matrix}}\, \underbrace{\underline{r}}_{\text{vector}}\, \delta(k-k')\, dk'$$
$$= e^{ik\cdot x}\, P(k)\, \underline{r}$$

In addition, let $\underline{r}$ be a right eigenvector of $P(k)$: $\quad P(k)\,\underline{r}_k = \lambda_k\, \underline{r}_k$

$$\Rightarrow L\left(e^{ik\cdot x}\, \underline{r}_k\right) = \lambda_k \left(e^{ik\cdot x}\, \underline{r}_k\right)$$

$$L\, f_\omega = -i\omega\, f_\omega$$

$\Rightarrow f_\omega(x) = e^{ik\cdot x}\, r_k$ and
ω is det so that $-i\omega$ is an eigenvalue of $P(k)$:

$$\boxed{\det\left(i\omega I + P(k)\right) = 0}$$

characteristic equation

$Av = \lambda v$
$v \in \mathrm{null}(A-\lambda I)$
$\det(A-\lambda I)=0$

solutions $\omega = \omega_j(k)$
dispersion relations
(all skew-Hermitian matrices are diagonalizable)

Concl.

$$\begin{array}{ccc} & F & \text{dual var.} \\ \text{time } t & \longrightarrow & \text{frequency } \omega \\ \text{space } x & \longleftarrow & \text{wave number/vector } k \\ & F^{-1} & \end{array}$$

waves like $f(x \pm ct) \longleftrightarrow$ dispersion relation
$\omega = \pm|k|c$

Characteristic eq.: restriction on ω and k such that $e^{i(k\cdot x - \omega t)}$ is a solution

L not x-inv.:

$$Lf(x) = \frac{1}{(2\pi)^n}\int e^{ik\cdot x}\, P(x,k)\, \hat{f}(k)\, dk$$

(microlocal analysis)

09/22

3) Traveling wave solutions

3.1.1
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = u_0(x)$$

$\xi = x + ct$
$\eta = x - ct$

$$x = \frac{\xi+\eta}{2}$$
$$t = \frac{1}{c}\left(\frac{\xi-\eta}{2}\right)$$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}$$
$$= \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial t} = \frac{1}{2c}\frac{\partial U}{\partial \xi} = 0 \qquad U(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta))$$

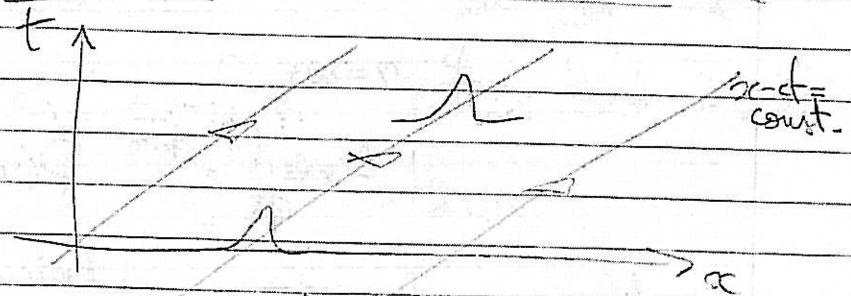
$$\Rightarrow \quad U(\xi,\eta) = F(\eta)$$

$$u(x,t) = F(x-ct)$$
$$\boxed{u(x,t) = u_0(x-ct)}$$

general solution to the eq.

$\boxed{\eta}$ is called a characteristic coordinate
→ along which total derivatives appear

$\boxed{\eta = \text{const.}}$ is a characteristic curve (and the PDE turns into an ODE.)



right-traveling wave.

u_0 = initial data

other choices of boundary data:
cannot be specified on the characteristic curve.

Rmk $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \Rightarrow u = G(x+ct)$
left-moving waves.

3.2) $\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{cases}$

$$\xi = x+ct \qquad 2c\frac{\partial}{\partial \xi} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$$
$$\eta = x-ct \qquad -2c\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

(only when $x \in \mathbb{R}$)

$$= -4c^2\frac{\partial^2}{\partial\xi\partial\eta}$$

$$\Rightarrow \frac{\partial^2 U}{\partial\xi\partial\eta} = 0$$

$$\Rightarrow \frac{\partial U}{\partial \eta} = f(\eta)$$

$$U = \int^{\eta} f(\eta')d\eta' + G(\xi)$$

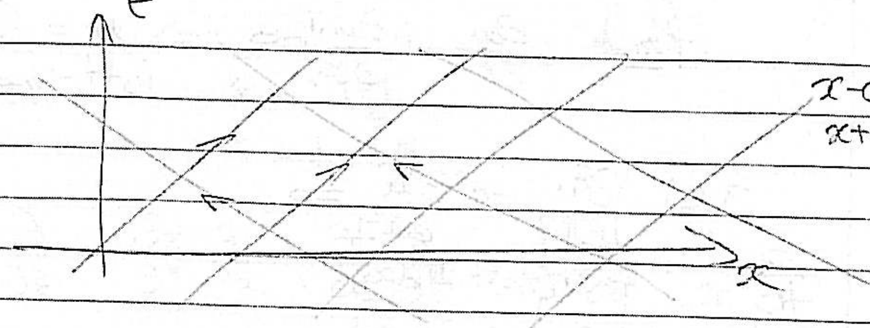
$$= F(\eta) + G(\xi)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

Fit I.C. $\Rightarrow$ $u(x,t) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy.$

d'Alembert's formula, 1746

③ ⑩



Double family of characteristics
$\xi=$ const. ; $\eta=$ const.
Cannot prescribe data on either characteristics.

1/22 $A^*=A$ $\qquad A^*=\bar{A}^T$

$Av=\lambda v \qquad u\cdot v = u^* v$
$= \sum \bar{u}_i v_i$

$\forall u, v \quad Au\cdot v = (Au)^* v$
$= u^* A^* v$
$= u\cdot A^* v$

Real eigenvalues
If $Av=\lambda v$, $A^*=A$: $\quad v^* A v = v^* A^* v$

$v^*Av = \lambda v^* v$

$v^*A^*v = (Av)^* v = (\lambda v)^* v = \bar{\lambda} v^* v$

$\lambda=\bar{\lambda} \Rightarrow \lambda$ real.

If $A^*=-A \Rightarrow \ldots \Rightarrow \lambda=-\bar{\lambda}$
$\Rightarrow \lambda$ imag.

Orthog. eigenvectors

If $A^* = A$, $\quad A v = \lambda v$

$A w = \mu w \qquad \mu \neq \lambda$

$$w^* A v = w^* A^* v$$

$$\lambda w^* v \qquad (A w)^* v = \bar{\mu} w^* v = \mu w^* v$$

$$(\lambda - \mu) w^* v = 0$$

$$\Rightarrow w^* v = 0 \quad \text{orthog.} \qquad (\mu \neq \lambda)$$

Plane wave solutions. $\quad \left.\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u\right|_{u_0, u_1}$

$$u(x,t) = \sum_{\pm} \int g_{\pm}(k)\, e^{i(k\cdot x - \omega_{\pm} t)}\, dk \qquad \omega_{\pm} = \pm |k| c$$

is also a solution (arb. $g_{\pm}$)

$$u(x,0) = u_0(x) = \int [g_+(k) + g_-(k)]\, e^{ik\cdot x}\, dk$$

$$\Rightarrow g_+(k) + g_-(k) = \hat{u}_0(k)\,(2\pi)^{-n}$$

$$\frac{\partial u}{\partial t}(x,0) = u_1(x) = \int [g_+(k) - g_-(k)]\, |k| c\, e^{ik\cdot x}\, dk$$

$$\Rightarrow g_+(k) - g_-(k) = \frac{\hat{u}_1(k)}{|k| c}(2\pi)^{-n}$$

$$\Rightarrow g_{\pm}(k) = \frac{1}{2}(2\pi)^{-n}\left[\hat{u}_0(k) \pm \frac{\hat{u}_1(k)}{|k| c}\right]$$

Can accommodate u_0, u_1

$\Rightarrow$ general solution of WE

Only works when $c = 1$, $x \in R^n$.
Not always useful.

Back to characteristics

3.3)
$C^2(x) = \frac{\kappa}{\rho}$

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \kappa(x) \frac{\partial}{\partial x} \frac{1}{\rho(x)} \frac{\partial u}{\partial x} & x \in R \\ u(x,0) = u_0 \quad \frac{\partial u}{\partial t}(x,0) = u_1 \end{cases}$$

Choose $\xi(x,t)$, $\eta(x,t)$ such that U solves

$$\frac{\partial^2 U}{\partial \xi \partial \eta} + \underbrace{p(x) \frac{\partial U}{\partial \xi} + q(x) \frac{\partial U}{\partial \eta} + r(x) U}_{\text{l.o.t.}} = 0$$

(First canonical form of the PDE)

$\Rightarrow$ " $U(\xi,\eta) \simeq F(\xi) + G(\eta)$ "
Singularities of U propagate along $\xi = $ const or $\eta = $ const.

(...)

$$\alpha(x) \frac{\partial^2 U}{\partial \xi^2} + \beta(x) \frac{\partial^2 U}{\partial \xi \partial \eta} + \gamma(x) \frac{\partial^2 U}{\partial \eta^2} + \dots = 0,$$

$C^2(x) = \frac{\kappa}{\rho}$

$$\boxed{\begin{aligned} \alpha(x) &= \left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x) \left(\frac{\partial \xi}{\partial x}\right)^2 = 0 \\ \gamma(x) &= \left(\frac{\partial \eta}{\partial t}\right)^2 - c^2(x) \left(\frac{\partial \eta}{\partial x}\right)^2 = 0 \end{aligned}}$$

Characteristic eqn.

only approximate.

solution: $\xi = x + c(x)t$
$\eta = x - c(x)t$

$\frac{dX}{dt} = c(X(t))$

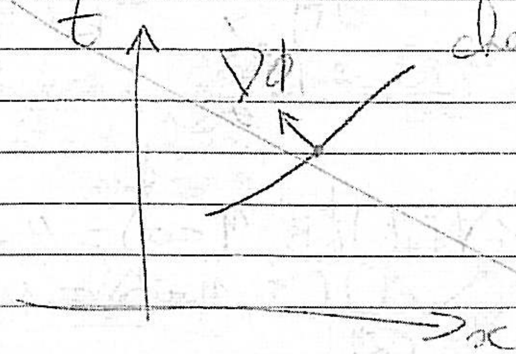
$\int^{X(t)} \frac{1}{c(x')} dx' = t + \text{Const.}$

$\int_{X(0)}^{X(t)} \frac{1}{c(x')} dx' = t$

$\Rightarrow \xi(x,t) = \int^{x} \frac{1}{c(x')} dx' - t$

$\phi = \xi$ or η

Char eqn for $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}$	dual var (disp. rel.)	Char. coord.
	$\omega^2 - c^2 k^2 = 0$	$\left(\frac{\partial\phi}{\partial t}\right)^2 - c^2\left(\frac{\partial\phi}{\partial x}\right)^2 = 0$
	(c const only)	(c variable ok)



$\nabla\phi_{(x,t)} = \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial t \end{pmatrix}$

constrained by $\frac{\partial\phi}{\partial t} = \pm c \frac{\partial\phi}{\partial x}$

$\Rightarrow \nabla\phi \overset{\text{same direction}}{\sim} \begin{pmatrix} 1 \\ \pm c \end{pmatrix}$

Also $\nabla\phi \sim \begin{pmatrix} k \\ \omega \end{pmatrix}$ when $\omega = \pm kc$

$\Rightarrow$ dual var. point $\perp$ to char. curve

[Do not make too much of $\omega = \frac{\partial \phi}{\partial t}$ $k = \frac{\partial \phi}{\partial x}$]

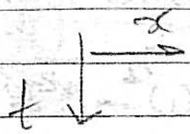
Specify Cauchy data: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

a) $u(x,0) = u_0$

$\frac{\partial u}{\partial t}(x,0) = u_1$

At time 0 determine all the P.D. of u;

(Cauchy-Kowal)



$u = u_0$ $u_x = (u_0)_x$ $u_{xx} = (u_0)_{xx}$

$u_t = u_1$

$u_{tt} = c^2 u_{00}$

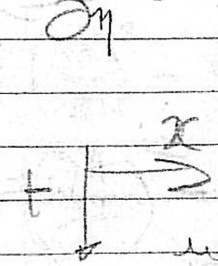
Here $\frac{\partial}{\partial t}$ is not parallel to the char.

it is normal to the Cauchy surface.

b) $U(\xi, \eta = 0) = u_0$

$\frac{\partial U}{\partial \eta}(\xi, \eta = 0) = u_1$

eq: $\frac{\partial^2 U}{\partial \xi \partial \eta} = 0$



u_0 $(u_0)_\xi$ $(u_0)_{\xi\xi}$

$(u_0)_\eta$ $(u_0)_{\xi\eta} = 0$

$(u_0)_{\eta\eta}$

?

~~8~~ 1

→ Can't solve eq. when Cauchy surface is characteristic.

→ Need to have term with 2 derivatives normal to S in the (2^{nd} order) eqn.

09/24/07 **Characteristics: wave/rays duality**

1D $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = K(x)\dfrac{\partial}{\partial x}\dfrac{1}{\rho(x)}\dfrac{\partial u}{\partial x}, \quad x \in \mathbb{R} \\ \text{I.C.} \end{cases}$ $\qquad c^2(x) = \dfrac{K(x)}{\rho(x)}$

Characteristic variables $\begin{cases} \xi(x,t) \\ \eta(x,t) \end{cases}$

such that

- equation has no $\dfrac{\partial^2 U}{\partial \xi^2}$ or $\dfrac{\partial^2 U}{\partial \eta^2}$ terms

  → first canonical form $\dfrac{\partial^2 U}{\partial \xi \partial \eta} + \text{stuff} = 0$.

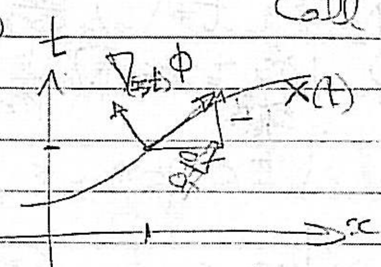
- Cauchy data cannot be posed on $\{\eta = C\}$ or $\{\xi = C\}$: charact. curves

Get $\left(\dfrac{\partial \xi}{\partial t}\right)^2 - c^2(x)\left(\dfrac{\partial \xi}{\partial x}\right)^2 = 0$: characteristic eqn

(same for η) $\qquad$ Call $\phi = \xi$ or η

Say $\dfrac{\partial \phi}{\partial t} = c(x)\dfrac{\partial \phi}{\partial x}$ (choice +)

Solution:



normal vec: $\underline{n} = \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial t \end{pmatrix}$

tang. vec: $\underline{t} = \begin{pmatrix} \partial\phi/\partial t \\ -\partial\phi/\partial x \end{pmatrix} \propto \begin{pmatrix} dX/dt \\ 1 \end{pmatrix}$

going directions

$$\Rightarrow \frac{dX(t)}{dt} = \frac{-\partial\phi/\partial t\,(X(t),t)}{\partial\phi/\partial x\,(X(t),t)} = -c(X(t))$$

Sep. var.: $\displaystyle -\int^{X(t)} \frac{dx}{c(x)} = t + C$

For each C, the locus of $\displaystyle \int^{x} \frac{dx'}{c(x')} + t = C$ is a characteristic curve.

Also, $\phi(x,t) = C$

$$\Rightarrow \boxed{\phi(x,t) = \int^{x} \frac{dx'}{c(x')} + t}$$

Rmk: $\displaystyle \frac{dX}{dt} \neq \frac{\partial\phi}{\partial t}\Big/\frac{\partial\phi}{\partial x}$

instead, negative sign!

Comes from $\phi(X(t),t) = C$

$$\Rightarrow \frac{dX}{dt}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t} = 0 \Rightarrow \frac{dX}{dt} = -\frac{\partial\phi}{\partial t}\Big/\frac{\partial\phi}{\partial x}$$

Rmk: $\displaystyle t = \int_{X(0)}^{X(t)} \frac{dx'}{c(x')}$ is travel time to go from $X(0)$ to $X(t)$

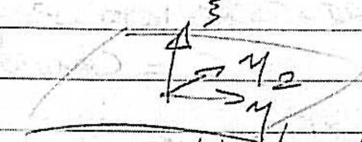
nD $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = K(x)\nabla\cdot\dfrac{1}{\rho(x)}\nabla u & x \in \mathbb{R}^n, \quad c^2 = \dfrac{K}{\rho} \\ \text{I.C.} \end{cases}$

Char. coord.: $(\xi, \eta_1, \ldots, \eta_{n-1})$, functions of x, t

Surface onto which we'll try to specify Cauchy data:

$$S = \{(x,t) : \xi(x,t) = 0\}$$

Coordinate locally perpendicular to S : ξ



Transform $u(x,t) \to U(\xi, \eta)$

Char. surface when $\frac{\partial^2 U}{\partial \xi^2}$ is absent from the transformed eq.

$$\alpha(x) \frac{\partial^2 U}{\partial \xi^2} + \cdots = 0$$

with $\boxed{\alpha(x) = \left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x) \, \|\nabla_x \xi\|^2 = 0}$

$\left(\|v\| = \sqrt{v^t v} = \sqrt{\sum v_i^2} \right)$ Characteristic eqn. Hamilton-Jacobi eqn.

Ex of solution when $c(x) = c$ const.

1) $\xi(x,t) = x \cdot k - \omega t$

when $\omega = \pm \|k\| c$

$\nabla_x \xi = k$, $\|\nabla_x \xi\|^2 = \|k\|^2$

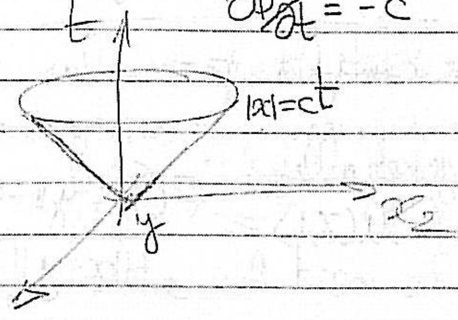
$\partial \xi / \partial t = \omega$, $\omega^2 - \|k\|^2 c^2 = 0$

(1) Gives rise to travel. waves $F(x \cdot k - \omega t)$ (exact sol.)

2) $\xi(x,t) = \|x - y\| - ct$

$\nabla_x \xi = \frac{x-y}{\|x-y\|}$, $\|\nabla_x \xi\| = \left\| \frac{x-y}{\|x-y\|} \right\| = 1$

$\partial \xi / \partial t = -c$, $c^2 - 1 \cdot c^2 = 0$



also, $\xi(x,t) = \|x-y\| - ct$.

4| Spherical wave solutions + Green's fn.

$(c = \text{const}, \; x \in \mathbb{R}^3)$

$n=3$: important

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \rightarrow \text{seek } u \text{ radially symmetric}$$

Spherical coordinates :

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \underbrace{\text{angular}}_{=0}$$

when u is radial

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}(ru) = \frac{\partial^2}{\partial r^2}(ru) \qquad \text{1D!}$$

$$\Rightarrow \quad r\,u(r,t) = F(r-ct) + G(r+ct)$$

$$u(r,t) = \underbrace{\frac{F(r-ct)}{r}}_{\text{outgoing}} + \underbrace{\frac{G(r+ct)}{r}}_{\text{incoming}}$$

Outgoing spherical wave :

$$\boxed{u(r,t) = \frac{F(r-ct)}{r}} \text{ for any (smooth) } F$$

(More complex formulas for cyl. waves)

More generally,

$$u(x,t) = \frac{F(\|x-y\| - ct)}{\|x-y\|} \quad \text{is a sol.}$$

Specialize $F=\delta$, $\dfrac{\delta(\|x-y\|-ct)}{\|x-y\|}$

4.1) Initial conditions $u(x,0)=u_0(x)$

$\frac{\partial u}{\partial t}(x,0)=u_1(x)$

Superpositions of spherical waves:

Try to find ψ such that

$$u(x,t)=\int_{\mathbb{R}^3}\psi(y)\,\frac{\delta(\|x-y\|-ct)}{\|x-y\|}\,dy$$

$$=\int\psi(y)\,\frac{\delta(\|x-y\|-ct)}{ct}\,dy$$

$$=\frac{1}{ct}\int_{\partial B_x(ct)}\psi(y)\,dy$$

$B_x(ct)=$ ball with center x, radius ct

($) t small, ψ smooth: $\int_{\partial B_x(ct)}\psi(y)\,dy \to 4\pi c^2t^2\psi(x)$

$$\Rightarrow u(x,t)\to 4\pi ct\,\psi(x)$$

$$\to 0 \text{ as } t\to 0.$$

$$\frac{\partial u}{\partial t}=\frac{-1}{ct^2}\int_{\partial B_x(ct)}\psi(y)\,dy+\frac{1}{ct}\frac{d}{dt}\int_{\partial B_x(ct)}\psi(y)\,dy$$

$$\xrightarrow{t\to0}\ \frac{-1}{ct^2}4\pi c^2t^2\psi(x)+\frac{1}{ct}\frac{d}{dt}\left(4\pi c^2t^2\psi(x)\right)$$ (#)

$$=-4\pi c\,\psi(x)+2\cdot4\pi c\,\psi(x)$$

$$=4\pi c\,\psi(x)$$

- If $u(x,0)=0$

$\frac{\partial u}{\partial t}(x,0)=u_1(x)$,

put $\psi(x)=\frac{1}{4\pi c}u_1(x)$,

then $u(x,t)=\frac{1}{ct}\int_{\partial B_x(ct)}\frac{1}{4\pi c}u_1(y)\,dy$

is the solution of WE.

Also write

$$u(x,t) = t \cdot ⨍_{\partial B_x(ct)} u_1(y)\,dy$$

where ⨍ is average:

$$⨍_{\partial B_x(ct)} u_1(y)\,dy = \frac{1}{4\pi c^2 t^2}\int_{\partial B_x(ct)} u_1(y)\,dy$$

$$u(x,t) = \int u_1(y) \underbrace{\boxed{\frac{\delta(\|x-y\|-ct)}{4\pi c^2 t}}}_{G(x,y;t)}\,dy$$

Green's function

"Convolution w/ Green's function = t × average over ball of radius ct."

- For $u_0 \neq 0$, $u_1 = 0$, use a trick:
  if u is solution (same as before), then so is $\frac{\partial u}{\partial t} = v$

$$\left[\frac{\partial^2}{\partial t^2} - c^2\Delta\right]\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2} - c^2\Delta\right]u = 0$$

Use $u = \int \psi(y)\,\delta(\|x-y\|-ct)/\|x-y\|\,dy$.

Have already seen $\lim_{t\to 0} v(x,t) = 4\pi c\,\psi(x)$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$\lim_{t\to 0}\frac{\partial v}{\partial t} = c^2 \lim_{t\to 0}\Delta u$$

$$= c^2 \Delta \lim_{t\to 0} u \quad \text{when } u \text{ is smooth}$$

$$= 0$$

$$\Rightarrow \quad v(x,t) \xrightarrow{t\to 0} 4\pi c\,\psi(x) = v_0(x)$$
$$\frac{\partial v}{\partial t}(x,t) \xrightarrow{t\to 0} 0 \qquad = v_1(x)$$

So $\displaystyle \frac{\partial}{\partial t}\int \frac{v_0(y)}{4\pi c}\,\frac{\delta(\|x-y\|-ct)}{ct}\,dy$

solves W.-E. with $v_0 \neq 0$, $v_1 = 0$

$$= \frac{\partial}{\partial t}\int v_0(y)\, G(x,y;t)\,dy$$
$$= \int v_0(y)\,\frac{\partial G}{\partial t}(x,y;t)\,dy.$$

Complete solution to W.-E.:

$$\boxed{\begin{aligned} u(x,t) &= \int u_0(y)\frac{\partial G}{\partial t}(x,y;t)\,dy \\ &\quad + \int u_1(y)\, G(x,y;t)\,dy \\ \text{where } G(x,y;t) &= \frac{\delta(\|x-y\|-ct)}{4\pi c^2 t} \end{aligned}}$$

(\$): check obeys W.-E.

$$x\in\mathbb{R}^2:\ G(x,y;t) = \frac{1/(2\pi c)}{\sqrt{c^2t^2 - \|x-y\|^2}} \qquad (\$)$$

Homework: check (Whitham, Folland)

Relate to Huygens's principle?

Next: Inhomog. eq., Helmholtz eq. + its charact. eq., Reflections

+ Pick up later w/ geom. optics

01/29/09 Folland Whitham

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, & x \in \mathbb{R}^n,\ t>0. \\ u(x,0) = u_0(x) \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) \end{cases}$$

$\Box u = 0$

$\Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta$

d'Alembertian

$$u(x,t) = \int u_0(y) \frac{\partial G}{\partial t}(x,y;t)\, dy + \int u_1(y)\, G(x,y;t)\, dy$$

where $G(x,y;t) = \dfrac{\delta(\|x-y\| - ct)}{4\pi c^2 t}$

Can specify $G(x,y;t) = 0$ for $t<0$

Thm (Duhamel principle, fundam. solution)

Inhomogeneous problem (IWE) $\begin{cases} \Box u = f(x,t) & x \in \mathbb{R}^n,\ t>0 \\ u_0 = u_1 = 0 \end{cases}$

Then $u(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,y;t-s)\, f(y,s)\, dy\, ds$ (*)

Pf. Check (*) solves IWE.

For each s, consider

$$\begin{cases} \Box v_s = 0 \\ v_s(x,0) = 0 \quad \frac{\partial v_s}{\partial t}(x,0) = f(x,s) \end{cases}$$

Then $v_s(x,t) = \int G(x,y;t)\, f(y,s)\, dy$

$$u(x,t) = \int_0^t v_s(x, t-s)\, ds$$

$$u(x,0) = \int_0^0 = 0$$

$$\frac{\partial u}{\partial t}(x,t) = \cancel{v_s(x,t-s)\Big|_{s=t}}^{\,0} + \int_0^t \frac{\partial v_s}{\partial t}(x,t-s)\,ds$$

$$\frac{\partial u}{\partial t}(x,0) = \int_0^0 = 0$$

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial v_s}{\partial t}(x,t-s)\Big|_{s=t} + \int_0^t \frac{\partial^2 v_s}{\partial t^2}(x,t-s)\,ds$$

$$= f(x,t) + \int_0^t c^2 \Delta v_s(x,t-s)\,ds$$

$$= f(x,t) + c^2 \Delta \int_0^t v_s(x,t-s)\,ds$$

$$= f(x,t) + c^2 \Delta u$$

$$\Box u = f \qquad \Box$$

Rmk: G is called · fundam sol of $\Box u = f$,
· (retarded) propagator

$$G(x,y;t-s) = 0 \quad \text{for } s > t$$

$$f(y,s) = 0 \quad \text{for } s < 0.$$

$$G(x,y;t-s) = g(x-y;t-s)$$

where $g(x) = \dfrac{\delta(\|x\| - ct)}{4\pi c^2 t}$.

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^m} g(x-y;t-s)\, f(y,s)\,dy\,ds$$

$$= g \underset{(x,t)}{\star} f \qquad \text{(convolution)}$$

Specialize to $f(y,s) = \delta(y - x_0)\,\delta(s)$

Then $u(x,t) = g(x-x_0;t) = G(x,x_0;t)$

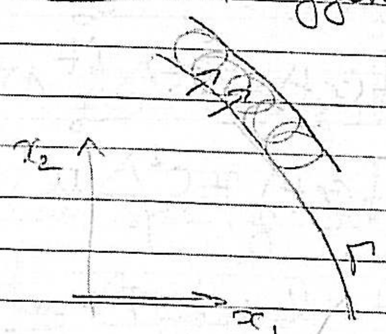
$\Rightarrow$ $G(x,x_0;t)$ solves

$$\Box_{(x,t)} G = \delta(x-x_0)\,\delta(t). \qquad \text{(IVE)}$$

Recap.

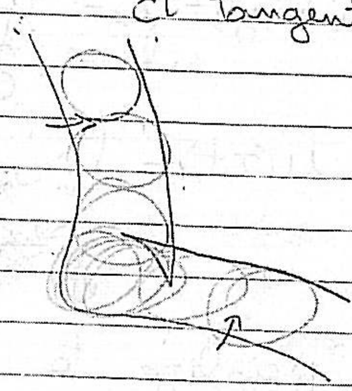
$\Box u = f(x,t)$	$Ax = b$
$u = g * f$	$x = A^{-1}b$
$\Box G = \delta(x-x_0)\delta(t)$	$AA^{-1} = I$
$(\Box g = \delta(x)\delta(t))$	

Rmk: Huygens's principle



Γ locus of | support (u_0, u_1) / discontinuity / singularity of u_0, u_1 / forcing f

Solution at a later time:
supported / singular on the envelope of circles of radius ct tangent to Γ.



May give rise to swallowtail patterns (linked to caustics)

4) Spherical sol. / Green's fn.

5) the Helmholtz equation $\left[-\omega^2 - c^2\Delta\right] v(x,\omega) = f(x,\omega)$ (HE)

- If $u(x,t) = e^{-i\omega t} v(x,\omega)$, then

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = -\omega^2 e^{-i\omega t} v - c^2 e^{-i\omega t} \Delta v$$

$$\Rightarrow \left[-\omega^2 - c^2\Delta\right] v = 0.$$

- If $u(x,t) = \int e^{-i\omega t} v(x,\omega)\, d\omega$, (Fourier in time) then

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = \int (-\omega^2) e^{-i\omega t} v(x,\omega)\, d\omega$$

$$- c^2 \int e^{-i\omega t} \Delta v(x,\omega)\, d\omega$$

take inv. FT: $\left[-\omega^2 - c^2\Delta\right] v = 0.$

Compute FT in time of $\Box g(x,t) = \delta(x)\,\delta(t)$ with $g(x,t) = \dfrac{\delta(\|x\| - ct)}{4\pi c^2 t}$.

$$\mathcal{F}_t \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right) g = \delta(x)\, \mathcal{F}_t \delta$$

$$\left(-\omega^2 - c^2\Delta\right) \mathcal{F}_t g = \delta(x) \qquad \text{Call } \phi = \mathcal{F}_t g$$

$$\phi(x,\omega) = \int e^{-i\omega t} g(x,t)\, dt$$

$$= \int e^{-i\omega t} \frac{\delta(\|x\| - ct)}{4\pi c^2 t}\, dt$$

$$= \int e^{-i\omega t} \frac{\delta(\|x\| - ct)}{4\pi c \|x\|}\, dt \qquad \tau = ct$$

$$= \int e^{-i\frac{\omega}{c}\tau} \frac{\delta(\|x\| - \tau)}{4\pi \|x\|}\, d\tau$$

$$= \boxed{\frac{e^{-i\frac{\omega}{c}\|x\|}}{4\pi\|x\|} = \phi(x,\omega)}$$

Thm. $\left[-\omega^2-c^2\Delta\right]u = f(x,\omega)$ 1)

then $u(x,\omega) = \int_{\mathbb{R}^n} \phi(x-y,\omega)\, f(y,\omega)\,dy$.

Rmk. $\phi(x,\omega)$ is the fundamental solution of the HE. It obeys $\left[-\omega^2-c^2\Delta\right]\phi = \delta(x)$

Pf. $\left[-\omega^2-c^2\Delta\right]u(x,\omega) =$

($) $\int_{\mathbb{R}^n} \left[-\omega^2-c^2\Delta\right]\phi(x-y,\omega)\, f(y,\omega)\,dy$

$= \int_{\mathbb{R}^n} \delta(x-y)\, f(y,\omega)\,dy$

$= f(x,\omega)$ $\square$

Ex. $\omega=0$ $\quad \phi(x) = \frac{1}{4\pi\|x\|}$ is fundam. sol. of $-\Delta u = f$ (Poisson)

$\Rightarrow u(x) = \int \frac{1}{4\pi\|x-y\|} f(y)\,dy$

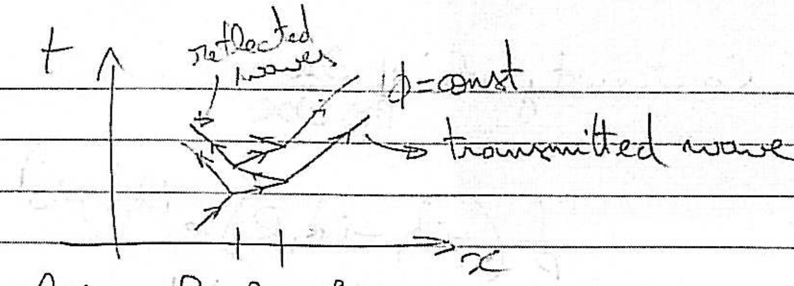
How: Fourier analysis of ϕ, g.

6) Reflection - transmission (1D)

$$\frac{\partial^2 p}{\partial t^2} = \kappa(x)\frac{\partial}{\partial x}\frac{1}{\rho(x)}\frac{\partial p}{\partial x} \qquad , \kappa, \rho \text{ discontinuous}$$

Char. eq: $\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{\kappa}{\rho}\left(\frac{\partial \phi}{\partial x}\right)^2 = 0$

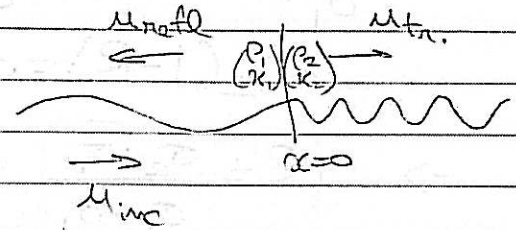
$$\phi(x,t) = \int^x \sqrt{\frac{\rho(x')}{\kappa(x')}}\,dx' + t.$$



slabs: ρ_1 ρ_2 ρ_3
K_1 K_2 K_3

At an interface (x,t) waves may break away and follow the other char-curve emanating from (x,t)
$\Rightarrow$ reflected wave(s)

Analysis for plane waves: single scattering event.



$c = \sqrt{\frac{K}{\rho}}$

$$u_{inc}(x,t) = e^{i(k_1 x - \omega t)} \qquad k_1 = \frac{\omega_1}{c_1}$$

$$u_{refl}(x,t) = R\, e^{i(-k_1 x - \omega t)}$$

$$u_{tr}(x,t) = T\, e^{i(k_2 x - \omega t)}$$

u continuous at $x=0$: $u_{inc}(x,0) + u_{refl}(x,0) = u_{tr}(x,0) \Rightarrow 1+R=T$.

At $x=0$ $\quad \frac{\partial^2 u}{\partial t^2} - K(x)\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial u}{\partial x} = 0$

dis.

co. dis. dis. co. dis.

Continuity of $\frac{1}{\rho(x)}\frac{\partial u}{\partial x}$:

$$\frac{1}{\rho_1}\left(+ik_1 - ik_1 R\right) e^{\cancel{ik_1 0 - i\omega t}}$$

$$= \frac{1}{\rho_2}\left(+ik_2 T\right) e^{\cancel{ik_2 0 - i\omega t}}$$

$$\frac{k_1}{\rho_1} = \frac{\omega}{c_1 \rho_1} = \frac{\omega}{\sqrt{\frac{K_1}{\rho_1}}\,\rho_1} = \frac{\omega}{\sqrt{K_1 \rho_1}}$$ (same for $1 \to 2$)

Call $\sqrt{K\rho} = \sigma$, the acoustic impedance (or Z)

$$\Rightarrow \frac{\cancel{\omega}}{\sigma_1}(1-R) = \frac{\cancel{\omega}}{\sigma_2} T$$

$$\sigma_2(1-R) = \sigma_1 T$$

$$\Rightarrow \boxed{R = \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}}, \quad \boxed{T = \frac{2\sigma_2}{\sigma_2 + \sigma_1}}$$

reflection coeff. transmission coeff.

Also, $\begin{cases} f(x - c_1 t) + R f(-x - c_1 t) & , x<0 \\ f\left(\frac{c_1}{c_2}(x - c_2 t)\right) & , x>0 \end{cases}$

is a solution $\forall f$

<u>Rmk</u> $|R|^2 + |T|^2 = 1$

<u>Rmk</u> Sign of R.