- Radiation field (far from antenna near the origin)

$$u_{inc}(x,\omega) \simeq \int \frac{e^{i\omega\|x\|}}{4\pi\|x\|} e^{-i\omega\hat{x}\cdot y} j(y,\omega)\,dy$$

j ↳ current density (vector) $J(y)\,p(\omega)$ × pulse p.

$$= \frac{e^{i\omega\|x\|}}{4\pi\|x\|} \hat{j}(\omega\hat{x},\omega)$$

↳ called radiation vector, $= J(\hat{x},\omega)$ or its scalar analogue.

when antenna at x_0:

$$u_{inc}(x,\omega) = \frac{e^{i\omega\|x-x_0\|}}{4\pi\|x-x_0\|} J(\widehat{x-x_0},\omega)$$

↳ radiation beam pattern & transmitted waveform

- Reception: (supp V far from antenna x_0)

$$S_{rec,B}(\omega) = \int_{\text{antenna near } x_0} u_{sc,B}(z,\omega)\,\underset{\text{window (vec→scal)}}{w(z,\omega)}\,dz$$

$$= \iint \frac{e^{i\omega\|z-y\|}}{4\pi\|z-y\|}(-\omega^2)\,V(y)\,u_{inc}(y,\omega)\,w(z,\omega)\,dz\,dy$$

$$\simeq \int \frac{e^{i\omega\|x_0-y\|}}{4\pi\|x_0-y\|}(-\omega^2)\,V(y)\,u_{inc}(y,\omega)\,\underbrace{\hat{w}(\omega(\widehat{y-x_0}),\omega)}_{W(\widehat{y-x_0},\omega)}\,dy$$

reception beam pattern (usually, same as radiation beam pattern disregarding pulse ($p(\omega)=1$))

$$\simeq \int e^{2i\omega\|x_0-y\|} A(\omega,x_0,y)\,V(y)\,dy$$

where
$$A(\omega,x_0,y) = -\omega^2 \frac{J(\omega,\widehat{y-x_0})\,W(\omega,\widehat{y-x_0})}{(4\pi\|x_0-y\|)^2}$$

⚠ $\begin{pmatrix}\text{speed}\\ \text{targets}\\ V_{\text{target}}\end{pmatrix} \ll \overset{\text{speed}}{\text{aircraft}} \ll \begin{pmatrix}\text{speed}\\ \text{light}\\ (x_0\ \text{rad} = x_0\ \text{rec.})\end{pmatrix}$

Continuum model for slow time, $x_0 = \gamma(s)$

$$d(s,\omega) = S_{rec,B}(\gamma(s),\omega)$$

$$\boxed{d(s,\omega) = \int e^{2i\omega\|\gamma(s)-y\|} A(\omega,s,y)\, V(y)\, dy} = FV$$

$$\Rightarrow d(t,s) = \iint e^{-i\omega[t - 2\|\gamma(s)-y\|/c]} A(\omega,s,y)\, V(y)\, dy\, d\omega$$

$$= \int \hat{A}\left(t - 2\frac{\|\gamma(s)-y\|}{c}, s, y\right) V(y)\, dy$$

Formal adjoint:

$$\boxed{F^* d = \int e^{-2i\omega\|\gamma(s)-x\|}\, \overline{A(\omega,s,x)}\, d(s,\omega)\, ds}$$

$\rightarrow$ Backprojection.

10/20 Compl.: $F: X \to \mathbb{R}$ functional.

$f \to F(f)$

Begin appendix

F is Fréchet-differentiable at $f_0 \in X$ if

$$\lim_{h\to 0} \frac{\|F(f_0+h) - F(f_0) - A_{f_0}(h)\|}{\|h\|} = 0$$

for some bounded lin. op. A_{f_0}:
(functional)

Call $A_{f_0} = \frac{\delta F}{\delta f}[f_0]$ the derivative of F at f_0.

also called 1^{st} variation of F at f_0.

e.g. if $F: \mathbb{R}^n \to \mathbb{R}$,

$$\frac{\delta F}{\delta f}[f_0](h) = (\nabla F)(f_0) \cdot h$$

↳ inner product

Formula for $\frac{\delta F}{\delta f}$: Gâteaux derivative

$$\frac{\delta F}{\delta f}[f_0](h) = \lim_{t \to 0} \frac{F(f_0 + th) - F(f_0)}{t}$$

(directional derivative in the direction h).

Ex. $F(f) = \langle f, g \rangle$

Then $\frac{\delta F}{\delta f}[f_0] = g$ independent of f_0

$$\frac{\delta F}{\delta f}[f_0](h) = \langle h, g \rangle$$

[Lag. mult -) $\min_{\max}$ $f(\underline{x})$ s.t. $g(\underline{x}) = c$

$$\Rightarrow \mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \underline{\lambda} \cdot (g(\underline{x}) - c)$$

then $\nabla_{\underline{x}} \mathcal{L} = 0$

$\nabla_{\underline{\lambda}} \mathcal{L} = 0$.]

Ex. of functionals and their derivatives

$$F(f) = \langle f, g \rangle \Rightarrow \frac{\delta F}{\delta f} = g(x)$$

$$F(f) = f(0) \Rightarrow \frac{\delta F}{\delta f} = \delta(x)$$

$$F(f) = f'(0) \Rightarrow \frac{\delta F}{\delta f} = -\delta'(x)$$

Nonlinear functional: $F(f) = \langle f^2, g \rangle$

$$\Rightarrow \frac{\delta F}{\delta f} = 2fg$$

Rmk: any bounded linear functional can be represented as the inner product with a distribution (Riesz represent$^{\text{n}}$ thm)

end appendix

1) Time formul^{n}:

10/22/09 Intro. Linearized field $u_{sc,B}(x,t)$
(end of Day 3) Compare to (processed) data
$$d(x,t) = rec(x,t) - u_{inc}(x,t)$$
$\hookrightarrow$ actual recorded

Linearized inverse problem:

$$J[u_{sc,B}, V] = \frac{1}{2}\| u_{sc,B} - d \|_2^2$$

$$\text{s.t.} \quad u_{sc,B} = F V$$
$\hookrightarrow$ linearized forward map

a) Least-squares:
$$\Rightarrow -\frac{\delta J}{\delta V} = F^*(d - u_{sc,B})$$

b) Variations of $\mathcal{L} = 0$:

$$\left(\frac{\delta L}{\delta V} = 0\right) \longrightarrow -\frac{\delta J}{\delta V} = \int_0^T q(x,t)\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt \quad \text{(RTM)}$$

$$\left(\frac{\delta L}{\delta u_{sc,B}(x,t)} = 0\right) \text{ with } \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) q = d_{ext}(x,t) - u_{sc,B,ext}(x,t)$$

$$\left(\frac{\delta L}{\delta u_{sc,B}(x,T)} = 0\right) \longrightarrow \frac{\partial q}{\partial t}(x,T) = 0$$

$$\left(\frac{\delta L}{\delta \frac{\partial u_{sc,B}}{\partial t}(x,T)} = 0\right) \longrightarrow q(x,T) = 0$$

q = Lagrange multiplier
= adjoint state.

Comp. (a) & (b):

$$F^*(d - u_{sc,B}) = \int_0^T q(x,t)\,\frac{\partial^2 u_{inc}}{\partial t^2}(x,t)\,dt$$
(RTM)

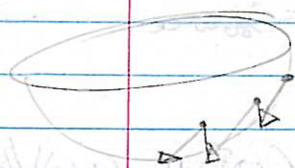
Imaging $= \tilde{V}(x)$ approx. to $V(x)$

Go further? $d = rec - u_{inc}$

a) Fix c_0, obtain the min of J:

$F^{-1} = (F^*F)^{-1}F^*$

Could be obtained by iteration:



① $d \to V_0 = \alpha_0 F^* d$
↳ small number

② $V_0 \to u_{sc,B,0} = F V_0$

$\to V_1 = V_0 + \alpha_1 F^*(d - u_{sc,B,0})$

③ $V_1 \to u_{sc,B,1} = F V_1$

$\to V_2 = V_1 + \alpha_2 F^*(d - u_{sc,B,1})$

or linear solver (GMRES) or...

b) Update $c(x)$ at each step: $d_0 = rec - u_{inc,0}$

① $d_0 \to V_0 = \alpha_0 F^* d_0$

② $V_0 \to \frac{1}{c_1^2} = \frac{1}{c_0^2} - V_0$

$\to u_{inc,1}$ simulated in c_1^2

$\to d_1 = rec - u_{inc,1}$

$\to V_1 = \alpha_1 F^* d_1$

③ $V_1 \to \frac{1}{c_2^2} = \frac{1}{c_1^2} - V_1$ etc.

Called full waveform inversion

*Here: talked about Tikhonov regularization.

(Chap 3,4) 2) freq. formulation.

10/27/09 Kinematics of F, F^* in radar imaging: Backprojection (Stripmap SAR)

- Radiation field

$$u_{inc}(x,\omega) = \int \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} \underbrace{j(y,\omega)}_{\text{source}}\, dy.$$

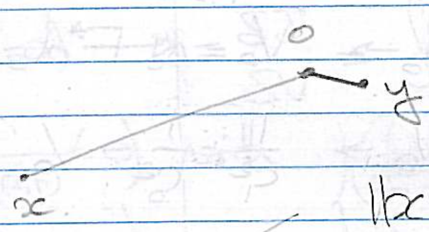
- Scattered field

$$u_{sc,B}(x,\omega) = \int \frac{e^{i\omega\|x-y\|}}{4\pi\|x-y\|} (-\omega^2)\, u_{inc}(y,\omega) V(y)\, dy$$

- Far field expansion:

Rad field: antenna at $y=0$.

$\Rightarrow \|x\| \gg \|y\|$ ⚠



$$\|x-y\| = \ldots = \|x\| - \hat{x}\cdot y + O\left(\frac{\|y\|^2}{\|x\|}\right)$$

geometry: spherical wavefronts look planar from a distance

$$e^{i\omega\|x-y\|} = e^{i\omega\|x\|}\, e^{-i\omega\hat{x}\cdot y} \times \text{remainder}$$

with $|\text{remainder}| \le \left| e^{i\omega O(\|y\|^2/\|x\|)} \right|$

$$\le C\, \frac{\omega\|y\|^2}{\|x\|}$$

$$= O\left(\omega\, \frac{\|y\|^2}{\|x\|}\right)$$

$$\frac{1}{\|x-y\|} = \frac{1}{\|x\|}\left(\frac{1}{1 - \frac{\hat{x}\cdot y}{\|x\|} + O\left(\frac{\|y\|^2}{\|x\|^2}\right)}\right)$$

$$= \frac{1}{\|x\|}\left(1 + O\left(\frac{\|y\|}{\|x\|}\right)\right)$$

$$\Rightarrow G(x,y,\omega) \simeq \frac{e^{i\omega\|x\|}}{4\pi\|x\|} e^{-i\omega\hat{x}\cdot y}$$

when $\begin{cases} \|y\| \ll \|x\| \\ \omega\|y\|^2 \ll \|x\| \end{cases}$

· Radiation field far from antenna:

$$u_{inc}(x,\omega) \simeq \int_{antenna} \frac{e^{i\omega\|x\|}}{4\pi\|x\|} e^{-i\omega\hat{x}\cdot y} \, j(y,\omega)\,dy$$

$$= \frac{e^{i\omega\|x\|}}{4\pi\|x\|} \hat{j}'(\omega\hat{x},\omega)$$

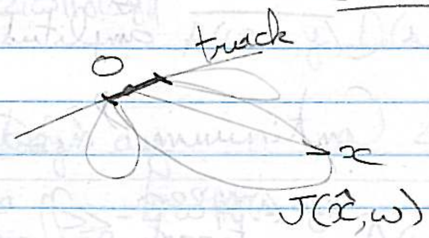
$j(y,\omega) = c\,d\sigma(y)\,P(\omega)$

$c\,d\sigma(y)$ ↳ "current density vector" (antenna shape) (scalar analogue)

$P(\omega)$ → pulse (signal sent) (in freq.)

$\hat{j}'(\omega\hat{x},\omega) = J(\hat{x},\omega)$

↳ "radiation vector" (scalar analogue) gives the radiation beam pattern.



Change coord: antenna at x_0

$$\Rightarrow u_{inc}(x,\omega) \simeq \frac{e^{i\omega\|x-x_0\|}}{4\pi\|x-x_0\|} J(\widehat{x-x_0},\omega).$$

• Scattered field / reception:

Stripmap SAR: antenna fixed along track.

$$S_{rec,B}(\omega) = \int_{\substack{\text{antenna} \\ \text{near } x_0}} u_{sc,B}(z,\omega)\, w(z,\omega)\, dz$$

window (scalar)

$$= \int_a \int_{\text{supp} V} \frac{e^{i\omega\|z-y\|}}{4\pi\|z-y\|} (-\omega^2) u_{inc}(y,\omega) V(y)\, w(z,\omega)\, dy\, dz$$

⚠ far field

$$\simeq \int_{\text{supp} V} \frac{e^{i\omega\|y-x_0\|}}{4\pi\|y-x_0\|} (-\omega^2) u_{inc}(y,\omega) V(y)\, \hat{w}'(\omega \widehat{(y-x_0)}, \omega)\, dy$$

Call $\hat{w}'(\omega\widehat{y-x_0}, \omega) = W(\widehat{y-x_0}, \omega)$ the reception beam pattern

Same as J, usually, up to pulse =

$$J(\widehat{y-x_0}, \omega) = W(\widehat{y-x_0}, \omega) P(\omega)$$

Plug u_{inc} (same x_0 for rad/rec)

$$S_{rec,B}(\omega) \overset{\triangle}{\simeq} \int_{\text{supp} V} e^{2i\omega\|y-x_0\|} A(y,x_0,\omega) V(y)\, dy$$

with $A(y,x_0,\omega) = -\omega^2 \dfrac{J(\widehat{y-x_0},\omega) W(\widehat{y-x_0},\omega)}{4\pi\|x_0-y\|^2}$

amplitude

⚠ Continuum model for slow time: $x_0 = \gamma(s)$

speed targets $\ll$ speed aircraft $\ll$ speed light

($V \perp t$) (x_0 rec $= x_0$ rad)

$$D(s,\omega) = S_{rec,B}(\gamma(s), \omega)$$

in this approx,

slight notational abuse.

$$\boxed{D(s,\omega)=\int e^{2i\omega\|\gamma(s)-y\|}A(\omega,s,y)\,V(y)\,dy}$$
$$=(FV)(s,\omega)$$

Back to t:

$$\mathfrak{F}D(s,\omega) = d(t,s)=\iint e^{-i\omega[t-2\|\gamma(s)-y\|]}A(\omega,s,y)\,V(y)\,dy\,d\omega$$
$$=\int \hat{A}'\Big(t-2\underbrace{\frac{\|\gamma(s)-y\|}{c}}_{\to c=1},s,y\Big)V(y)\,dy$$
$$\simeq\int p''\Big(t-\frac{2\|\gamma(s)-y\|}{c}\Big)\cdot\frac{V(y)}{(4\pi\|\gamma(s)-y\|)^2}\,dy$$

⚠ $\int_{-\omega}^{+\omega}$

p located near $t=0$

Geometry: time for signal to come back:
$$t=2\frac{\|\gamma(s)-y\|}{c}.$$

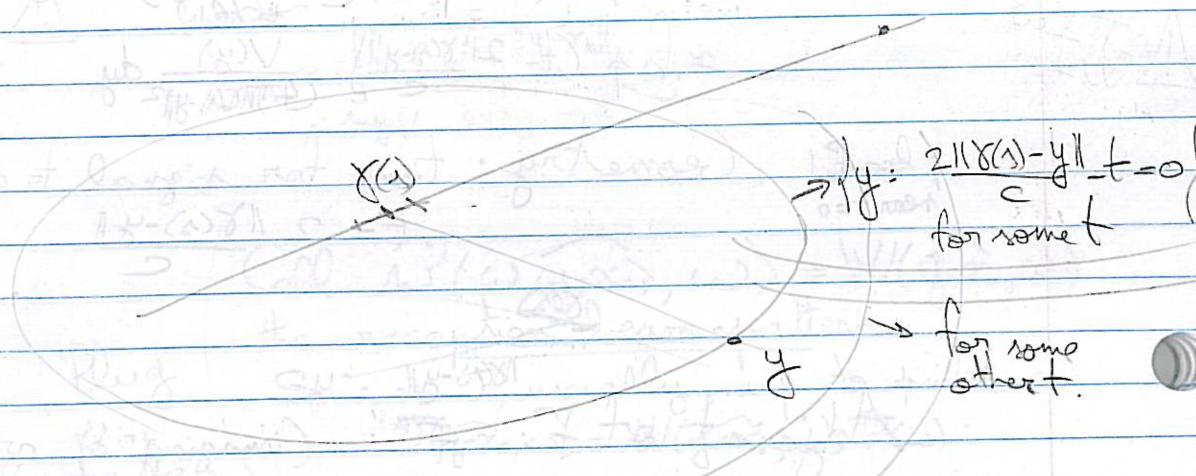
Adjoint of F: F^* (imaging op. w. o.o)

$$\langle D, FV\rangle_{(s,\omega)}=\langle F^*D, V\rangle_y$$
$$\|$$
$$\iint D(s,\omega)\overline{\int e^{2i\omega\|\gamma(s)-y\|}A(\omega,s,y)V(y)\,dy}\,ds\,d\omega$$
$$\|$$
$$\int V(y)\iint e^{-2i\omega\|\gamma(s)-y\|}\,\overline{A(\omega,s,y)}\,D(s,\omega)\,ds\,d\omega.$$
$$\|$$
$$\int V(y)\;F^*D(y)\,dy$$

$$\Rightarrow\boxed{F^*D(y)=\iint e^{-2i\omega\|\gamma(s)-y\|}\,\overline{A(\omega,s,y)}\,D(s,\omega)\,ds\,d\omega.}$$
$$=\iint e^{-2i\omega\|\gamma(s)-y\|}\,\overline{A(\omega,s,y)}\int e^{+i\omega t}d(s,t)\,dt\,ds\,d\omega$$
$$=\iint \hat{A}'\Big(\frac{2\|\gamma(s)-y\|}{c}-t,s,y\Big)d(s,t)\,ds\,dt$$

Because
$$\begin{aligned} f(\omega) &\to \hat{f}(t) = \int e^{-i\omega t} f(\omega)\,d\omega \\ \overline{f}(\omega) &\to g(t) = \int e^{-i\omega t}\, \overline{f}(\omega)\,d\omega \\ &= \overline{\int e^{i\omega t} f(\omega)\,d\omega} \\ &= \overline{\hat{f}(-t)}. \end{aligned}$$

$$\simeq \iint p''\left(\frac{2\|\gamma(s)-y\|}{c} - t\right) \frac{d(s,t)}{(4\pi)^2\|\gamma(s)-y\|^2}\, ds\, dt.$$



Consider circles in y, $y = (y_1, 0)$
weighted by data $d(s,t)$,
corresp. to geom. locus $\frac{2\|\gamma(s)-y\|}{c} = t$,
then add up.

$\Rightarrow$ <u>backprojection</u>.

Similar to inverting a Radon trf.

<u>Rmk</u>: possible project.

$y_T = \binom{y}{0}$ | Recall $D(s,\omega) = FV(s,\omega) = \int e^{2i\omega\|\gamma(s)-y_T\|} A(\omega,s,y_T)\,V(y)\,dy$ (1) $y \in \mathbb{R}^2$

$F^*D(y) = \iint e^{-2i\omega\|\gamma(s)-y_T\|} \overline{A}(\omega,s,y_T)\,D(s,\omega)\,ds\,d\omega$

10/29/09

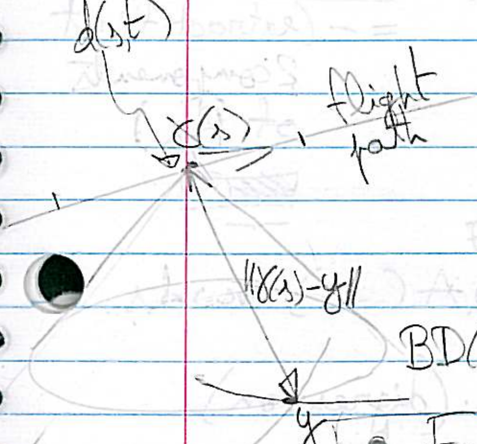
Imaging operator v. 1.0:

Find better amplitude $Q(\omega, s, y_T)$ in

$$F^*D \to BD(y) = \iint e^{-2i\omega\|\gamma(s)-y_T\|} Q(\omega,s,y_T)\,D(s,\omega)\,ds\,d\omega$$

and again, $y_T = (y, 0)$,
$y = P_2 y_T$ (extract first 2 compon.)

- If $Q = 1$, then pure backprojection.
because (idealize $\omega \in \mathbb{R}$).



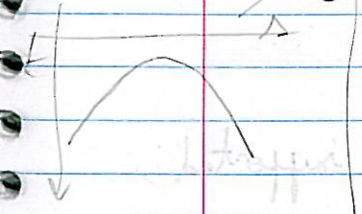
$$\iint e^{-2i\omega\|\gamma(s)-y_T\|} D(s,\omega)\,ds\,d\omega$$

$$= \iint e^{-2i\omega\|\gamma(s)-y_T\|} \int e^{i\omega t} d(s,t)\,dt\,ds\,d\omega$$

$$= \iint \delta\left(t - 2\frac{\|\gamma(s)-y_T\|}{c}\right) d(s,t)\,ds\,dt.$$

$$BD(y) = \int d\left(s, \frac{2\|\gamma(s)-y_T\|}{c}\right) ds$$

- For inversion,



$$BD(y) = \iiint_x \underbrace{e^{-2i\omega\|\gamma(s)-y_T\| + 2i\omega\|\gamma(s)-x_T\|} Q(\omega,s,y_T)\,A(\omega,s,x_T)\,d\omega\,ds}_{K(y,x)} \times V(x)\,dx$$

would like $\simeq V(y)$

$\to$ choose Q s.t. $K(y,x) \simeq \delta(y-x)$ $\quad x, y \in \mathbb{R}^2$

by reducing $K(y,x)$ to $\int_{\mathbb{R}^2} e^{i(y-x)\cdot\xi}\,d\xi$ for some ξ.

Here $\iint d\omega\,ds$ over a data collection manifold M
$\to$ idealize over $\mathbb{R}^2$. ⚠

$$\text{Call } \overbrace{R_{y,s}}^{\text{range vector}} = \gamma(s) - y_T$$

For ref, $\frac{\partial R_{ys}}{\partial s} = \dot\gamma(s)$ $\quad \frac{\partial R_{ys}}{\partial y} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= -P_2$

$\frac{\partial}{\partial x}\|x\| = \frac{x}{\|x\|}$

$$\frac{\partial}{\partial s}\|R_{ys}\| = \left(\frac{R_{ys}}{\|R_{ys}\|}\right)^T \frac{\partial R_{ys}}{\partial s} = \boxed{\underbrace{\hat R_{ys}}_{\text{unit vector}} \cdot \dot\gamma(s)}$$

$$\frac{\partial}{\partial y}\|R_{ys}\| = \left(\frac{R_{ys}}{\|R_{ys}\|}\right) \frac{\partial R_{ys}}{\partial y} = \boxed{-\hat R_{ys} \cdot P_2}$$

$= -$ (extract first 2 components of $\hat R_{ys}$)

$$K(y,x) = \iint e^{2i\omega(\|R_{ys}\| - \|R_{xs}\|)} Q A(\ldots)\, d\omega\, ds$$

Stationary phase: (digression)

a) $\phi(x)$ has no crit. point:

$\nabla\phi(x^*) \neq 0$,

$a(x)$ smooth, compactly supported,

Thm $\int_{\mathbb{R}^n} e^{i\beta\phi(x)} a(x)\,dx = I(\beta)$, $\beta \to \infty$.

then $|I(\beta)| \leq C.\beta^{-N}$ as $\beta \to \infty$. $\forall N \geq 0$.

Pf. $\mathcal{L} = \frac{1}{i\beta\phi'(x)}\frac{d}{dx}$

$\mathcal{L} e^{i\beta\phi(x)} = e^{i\beta\phi(x)}$

$|\mathcal{L}^N a(x)| \leq \beta^{-N}$. $\square$

b) $\phi(x)$ has a crit. point : x^*

$\nabla\phi(x^*) = 0$

$D^2\phi(x^*) \neq 0.$

$a(x)$ smooth, comp. supp.

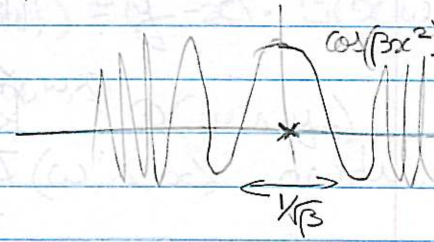
Thm. $\int_{\mathbb{R}^n} e^{i\beta\phi(x)}\, a(x)\,dx$

$$= \left(\frac{2\pi}{\beta}\right)^{n/2} a(x^*)\, e^{i\beta\phi(x^*)} \frac{e^{i\frac{\pi}{4}\operatorname{sgn} D^2\phi(x^*)}}{\sqrt{|\det D^2\phi(x^*)|}}$$

$$+\, O\left(\beta^{-n/2-1}\right) \quad \text{as } \beta\to\infty.$$

$\operatorname{sgn} M = \#\,\text{pos eigv}(M) - \#\,\text{neg eigv}(M).$

Ex. $\int_{\mathbb{R}} e^{i\beta x^2}\,dx$



$\sim \frac{1}{\sqrt{\beta}}$

Extension: $\begin{cases} \partial^k\phi/\partial x^k\,(x^*) = 0 & k = 1,\dots,l \\ \partial^l\phi/\partial x^l\,(x^*) \neq 0. \end{cases}$

$\longrightarrow O(\beta^{-1/l})$

Ex. $\iint e^{i(y-x)\cdot\xi}\,d\xi\, f(x)\,dx \overset{?}{\simeq} f(y)$ $\quad$ X

$\iint e^{i(y-x)\cdot\xi}\,d\xi\left(\int e^{ix\cdot\eta}F(\eta)\,d\eta\right)dx$

$\to$ stat. phase in pair (x,ξ) of variables!

$\phi(x,\xi) = (y-x)\cdot\xi + x\cdot\eta$ $\qquad$ y,η fixed param

$\partial\phi/\partial x = \eta - \xi = 0 \Rightarrow \boxed{\xi = \eta}$

$\partial\phi/\partial\xi = y - x = 0 \Rightarrow \boxed{x = y}$ $\quad$ canonical relation.

$\partial^2\phi/\partial\xi\partial x = -\mathrm{Id}$ nonsing

$\phi(y,\eta) = (y-y)\cdot\eta + y\cdot\eta = y\cdot\eta$

$\beta = 1$

$\to$ expect $\sim \int e^{iy\cdot\eta}F(\eta)\,d\eta = f(y)$

Back to $K(y,x)$: $e^{2i\omega(|R_{ys}|-|R_{xs}|)}$

Large parameter : $\omega = \beta\omega'$

$d\omega = \beta d\omega'$

Phase: $\phi(\omega', s) = 2\omega'(|R_{ys}|-|R_{xs}|)$ x, y fixed.

$$\frac{\partial\phi}{\partial\omega'} = 2(|R_{ys}|-|R_{xs}|).$$

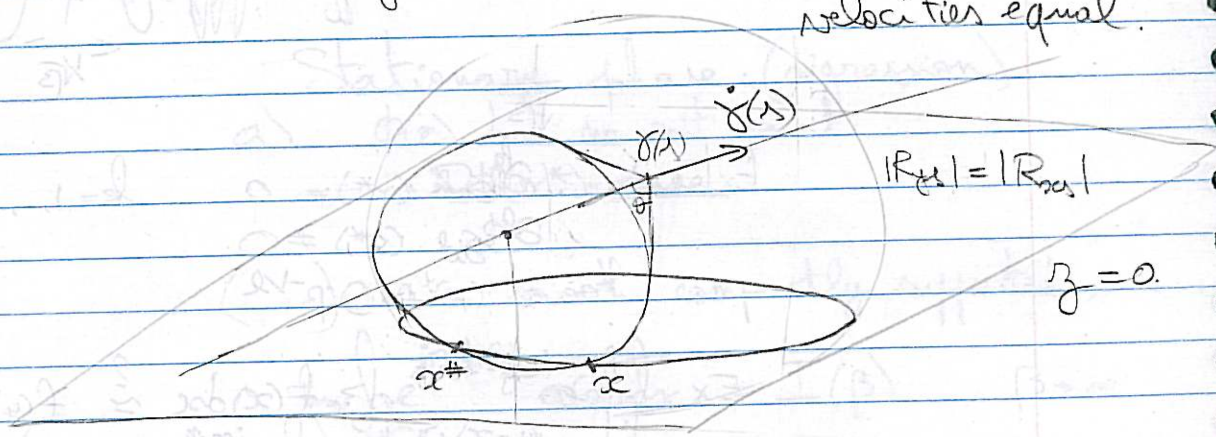
$$\frac{\partial\phi}{\partial s} = 2\omega'(\hat{R}_{ys}\cdot\dot{\gamma}(s) - \hat{R}_{xs}\cdot\dot{\gamma}(s))$$

$\frac{\partial^2\phi}{\partial\omega\partial s}$ singular but OK.

Stat. phase points:

$|R_{ys}| = |R_{xs}|$ ranges equal.

$\hat{R}_{ys}\cdot\dot{\gamma}(s) = \hat{R}_{xs}\cdot\dot{\gamma}(s)$ down-range velocities equal.



$\Rightarrow$ 2 solutions : x and its mirror w.r.t. flight plane

Side(ways) looking antenna : only stat phase point.

$\boxed{y = x.}$

Transform $2\omega |R_{ys}| - |R_{xs}| = (y-x)\cdot \xi$ ← something

Trick: $f(y) - f(x) = \int_0^1 \frac{d}{d\mu} f(x+\mu(y-x))\, d\mu$

$$= (y-x)\cdot \int_0^1 \nabla f\,(x+\mu(y-x))\, d\mu .$$

Here: $f(x) = 2\omega \| R_{xs} \|$

$\nabla f(x) = -2\omega\ \hat{R}_{xs}\cdot P_2$

⚠ $\int_0^1 (\nabla f)(x+\mu(y-x))\, d\mu \simeq -2\omega\ \hat{R}_{xs}\cdot P_2$ when $y \simeq x$.

$\Rightarrow$ call this $\Xi(x,\omega,s)$. (capital xi)

$\Rightarrow$

$$2\omega(|R_{ys}| - |R_{xs}|) \simeq (y-x)\cdot \Xi(x,\omega,s)$$

$$K(y,x) = \int e^{i(y-x)\cdot \Xi(x,\omega,s)}\, Q(\omega,s,y_T) \times \bar{A}(\omega,s,x_T)\ d\omega\, ds .$$

Change var $(\omega,s) \to \xi = \Xi(x,\omega,s) \quad \in \mathbb{R}^2$.

$Q(\omega,s,y_T) \to Q(\xi,y_T)$ (slight

$\bar{A}(\omega,s,x_T) \to \bar{A}(\xi,x_T)$. (abuse

$d\omega\, ds = \underbrace{\left|\frac{\partial(s,\omega)}{\partial \xi}\right|}_{\text{Beylkin determinant}} d\xi$

$$K(y,x) = \int e^{i(y-x)\cdot\xi}\ \underbrace{Q(\xi,y_T)\,\bar{A}(\xi,x_T)}_{y_T \simeq x_T} \left|\frac{\partial(s,\omega)}{\partial\xi}\right| d\xi .$$

Want $\boxed{Q(\xi,x_T) = \frac{1}{A(\xi,x_T)\left|\frac{\partial(s,\omega)}{\partial\xi}\right|}\cdot \chi(\xi,x)}$ prevent division by zero.

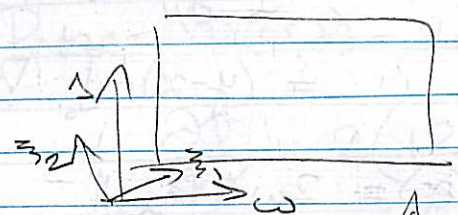
⚠ intro χ.

$1/A = \bar{A}/|A|^2$

$$K(y,x) \simeq \int e^{i(y-x)\cdot\xi}\, d\xi \simeq \delta(y-x)$$

Rmk about matched filtering: see next page.

$$\text{Resolution} = K(y,x) \simeq \int_{\substack{\text{data}\\ \text{manifold}\\ M}} e^{i(y-x)\cdot\xi}\, d\xi$$



⚠ Still $M \simeq M_1 \times M_2$.

$$\int_M e^{i(y-x)\cdot\xi}\, d\xi \simeq \int_{M_1} e^{i(y_1-x_1)\xi_1}\, d\xi_1 \times \int_{M_2} e^{i(y_2-x_2)\xi_2}\, d\xi_2$$

and $$\int_{-b}^{b} e^{ixk}\, dk = 2\frac{\sin bx}{x} = 2b\,\mathrm{sinc}(bx)$$

main lobe: $\frac{2\pi}{b}$

$[-b,b]$ in k domain $\Rightarrow$ resolution $\frac{2\pi}{b}$ in x domain
(Shannon sampling theorem)

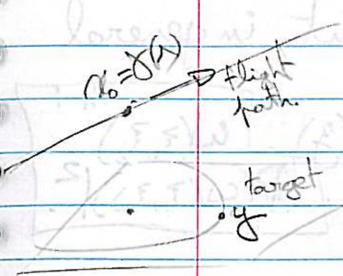
Rmk. matched filtering $\bar{A}$ includes $\bar{P}(\omega)$
$\rightarrow \int p(t-t')\, d(t',1)\, dt'$
$= p(-t) \star_t d(t,1)$

Today:
X-ray tomography

$$\frac{e^{i\omega\|x-y\|}}{\|x-y\|} \simeq \frac{e^{i\omega\|x\|}}{\|x\|} e^{-i\omega \hat{x}\cdot y} \quad \text{when } \|x\| \gg \|y\|$$

11/03/09 Far field approx $\Rightarrow$ $y_T = \binom{y}{0}$

$$D(s,\omega) = \int_{\mathbb{R}^2} e^{2i\omega\|x_0 - y_T\|} A(y,s,\omega)\, V(y)\, dy.$$



(have knocked off two integrals over the antenna $\to$ antenna beam patterns in A). and $x_0 = \gamma(s)$

⚠ ① Go further: $e^{2i\omega\|x_0 - y_T\|}$

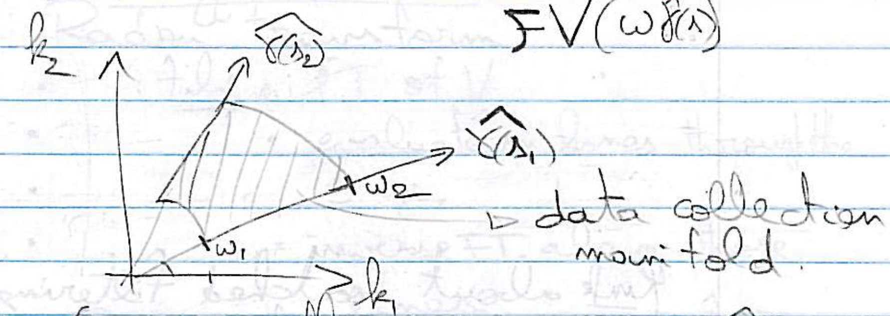
$$\simeq e^{2i\omega\|x_0\|} e^{-2i\omega\, \widehat{x_0}\cdot y_T}$$

also $A(y,s,\omega) \perp y$, say $A = P(\omega)$
(localized target, nondirectional beam)

$$D(s,\omega) \simeq \underbrace{e^{2i\omega\|\gamma(s)\|} P(\omega) \int_{\mathbb{R}^2} e^{-i\omega \widehat{\gamma(s)}\cdot y} V(y)\,dy}_{} = FV$$

Fapprox.

$$FV(\omega\hat{\gamma}(s))$$



Adjoint $(F^*D)(y) = \iint_M e^{-2i\omega\|\gamma(s)\|}\, \overline{P}(\omega)\, e^{i\omega\widehat{\gamma(s)}\cdot y} D(s,\omega)\, ds\,d\omega$

Composition: $F^*F\,V =$

$$\iint ds\,d\omega\, |P(\omega)|^2 \int e^{i\omega\hat{\gamma}(s)\cdot(y-z)}\, V(z)\, dz \overset{?}{\simeq} V(y)$$

let $\omega\hat{\gamma}(s) = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \Xi(\omega,s)$

$$d\xi_1 d\xi_2 = \left|\frac{\partial(s,\omega)}{\partial(\xi_1,\xi_2)}\right| ds\,d\omega$$

⚠ ex. $s = \theta$, and $\hat{\gamma}(s) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$. $\theta \in [0, 2\pi)$

$y \in \mathbb{R}^n$, here $n=2$.

then $\left|\dfrac{\partial(s,\omega)}{\partial(\xi_1,\xi_2)}\right| = \omega^{n-1}$ (polar to cartesian)

↳ Beylkin determinant in general

⚠ Data manifold $= \mathbb{R}^2$.

then $\displaystyle\int d\xi_1 d\xi_2 \, e^{i\xi\cdot(y-z)} V(z) \cdot \boxed{\dfrac{\omega(\xi_1,\xi_2)^{n-1}}{|P(\omega(\xi_1,\xi_2))|^2}}$

not too far from $V(y)$

→ instead of F^*, use

$$(BD(s,\omega))(y) = \iint e^{-2i\omega\|\gamma(s)\|}\, \overline{P(\omega)} \cdot \left(\frac{\omega^{1-n}}{|P(\omega)|^2}\right) \times e^{i\omega\hat{\gamma}(s)\cdot y} D(s,\omega)\, ds\, d\omega$$

then ($\hat{\gamma}(s) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$), get

$$BFV(y) = \iint d\xi_1 d\xi_2 \, e^{i\xi\cdot(y-z)} V(z) = V(y)$$

Simplif.:
- full data manif.
- far field, twice
- OK to divide by $|P(\omega)|^2$
- $\hat{\gamma}(s) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

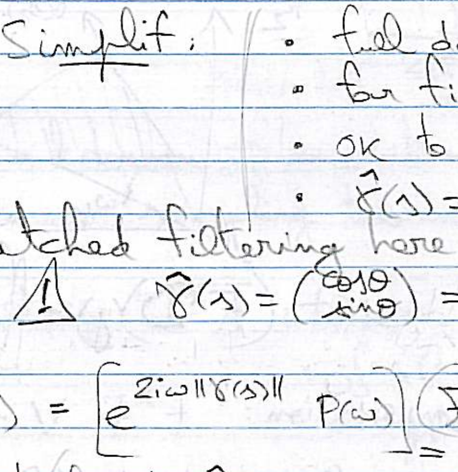
Rmk about matched filtering here (see p. 7 →)

Important special case ⚠ $\hat{\gamma}(s) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \hat{e}_\theta$, $e^{2i\omega\|\gamma(s)\|} P(\omega) = 1$

$$D(\theta,\omega) = (FV)(s,\omega) = \underbrace{\left[e^{2i\omega\|\gamma(s)\|} P(\omega)\right]}_{=1} (\mathcal{F}V)\left(\omega\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)$$

$$d(\theta,t) = \int e^{+i\omega t}\left(\int e^{-i\omega\hat{e}_\theta\cdot y} V(y)\, dy\right) d\omega$$

$$= \int \delta(t - \hat{e}_\theta\cdot y)\, V(y)\, dy$$



Integration along a line with
- angle θ
- offset t

→ Radon transform $(n=2)$

(or X-ray transform $(n=3)$)

$$d(\theta,t) = (R\,V(y))(\theta,t)$$

we have $\left(F_t\, d(\theta,t)\right)(\theta,\omega) = D(\theta,\omega) = F_y V\left(\omega\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)$

$$\Rightarrow (F_t\, R\, V(\theta,t))(\theta,\omega) = (F_y V)\left(\omega\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)$$

$$\boxed{R\,V(\theta,t) = \underbrace{F_t^*}_{=F_t^{-1}}\left[(F_y V)\left(\omega\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)\right](\theta,t)}$$

→ Fourier-slice theorem for the Radon transform
- FT of V
- evaluate on lines through the origin
- inverse FT along those lines

$$F^*D(y) = \iint \underbrace{e^{-2i\omega\|\gamma(s)\|}\,\bar{P}(\omega)}_{=1\ (\triangle)}\, e^{i\omega \overbrace{\gamma(s)}^{=\hat{e}_\theta}\cdot y}\, D(\theta,\omega)\, d\theta\, d\omega$$

$$= \iint e^{i\omega\hat{e}_\theta\cdot y}\left(\int e^{-i\omega t}\, d(\theta,t)\, dt\right) d\theta\, d\omega$$

⚠ full data

$$= \iint \delta(t-\hat{e}_\theta\cdot y)\, d(\theta,t)\, d\theta\, dt$$

→ Backprojection.

$$BD(y) = \iint e^{i\omega \hat{e}_\theta \cdot y} \cdot \omega^{-1} D(\theta,\omega)\, d\theta\, d\omega$$

$$= \iint \frac{e^{i\omega \hat{e}_\theta \cdot y}}{\omega} \int e^{-i\omega t}\, d(\theta,t)\, d\theta\, dt$$

$$= \iint F\left[\tfrac{1}{\omega}\right](t - \hat{e}_\theta \cdot y)\; d(\theta,t)\, d\theta\, dt$$

$$= V(y) \qquad \text{with } F\left(\tfrac{1}{\omega}\right)(t) = 2i\,\mathrm{sgn}(t)$$

ω^{-1} is a filter $\qquad \mathrm{sgn}(t) = -1 \;\; 1 \;\; t=0$

$\Rightarrow$ <u>Filtered backprojection</u>.

Since $d(\theta,t) = (RV(y))(\theta,t)$,
<u>filtered backprojection</u> is R^{-1}:

$$\left(R^{-1} d(\theta,t)\right)(y) = \int e^{i\omega \hat{e}_\theta \cdot y} \cdot \frac{1}{\omega} \int e^{-i\omega t}\, d(\theta,t)\, d\theta\, dt$$

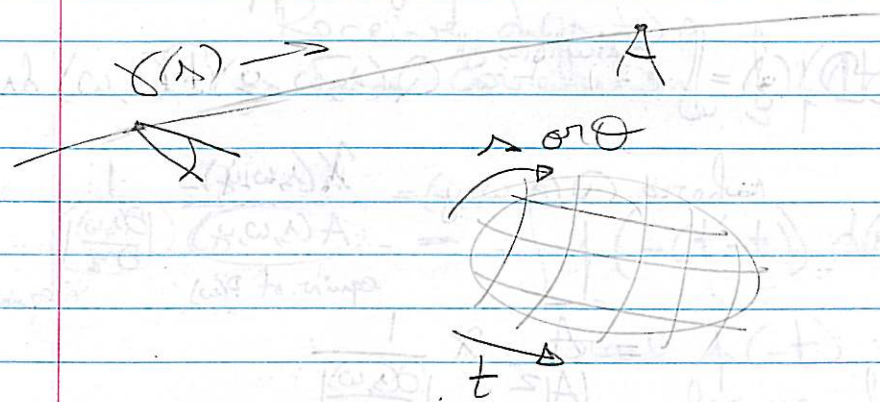
<u>Rmk</u>:
- R, R^{-1} basis for X-ray <u>tomography</u> (3D: $\omega^{1-m} = \omega^{-2}$)

  $\Rightarrow$ CT scans.
- $FV(\omega \hat{x}_0)$ like in <u>crystallography</u>
  also <u>quantum scattering</u>
- Coincidentally, similar equations in MRi although not same physics (absorption of radio waves).
- Radar:

$$e^{2i\omega\|\gamma(s) - y\|} \simeq e^{2i\omega\|\gamma(s)\|}\, e^{-2i\omega \widehat{\gamma(s)} \cdot y}$$

(todo: ultrasound and sonar)

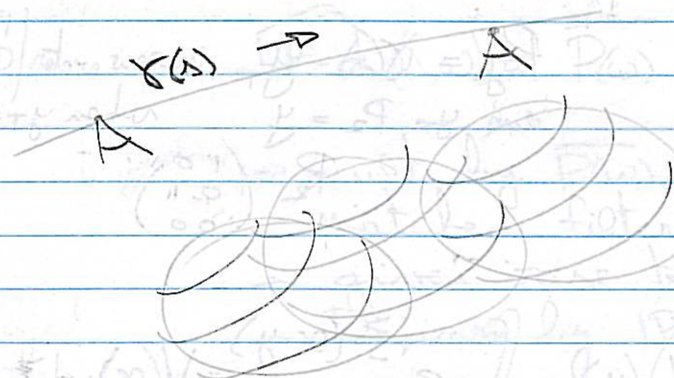
ok for <u>spotlight SAR</u>. (stares)



circles approx by lines

also ok for moving-targets SAR. (air traffic control, etc.)

not ok for <u>stripmap</u> SAR.



$\rightarrow$ back to formula $D(s,\omega) = \int_{\mathbb{R}^2} e^{2i\omega \|x_0 - y_T\|} A(y,s,\omega) V(y)\,dy$

then $d(s,t) = \iint p''(t - 2\|\gamma(s) - y\|)\, V(y)\,dy$

⚠ $A = P(\omega)$

$p(t) = \mathcal{F}^{-1} P(\omega)(t)$

integration along circles

$\longrightarrow$ <u>generalized</u> Radon transform

if $p'' = \delta$

still, F^* is backprojection
$\mathcal{B}$ is filtered backprojection
(good approx. to inverse)

$$(\mathcal{B}D)(y) = \iint e^{-2i\omega\|\gamma(s)-y\|} Q(s,\omega,y) D(s,\omega)\, ds\, d\omega$$

where $Q(s,\omega,y) = \dfrac{\chi(s,\omega,y)}{A(s,\omega,y)} \cdot \dfrac{1}{\left|\frac{\partial(s,\omega)}{\partial \xi}\right|}$

equiv. of $P(\omega)$ — equiv of ω^{n-1}

$$= \frac{\bar{A}}{|A|^2}\, \chi\, \frac{1}{\left|\frac{\partial(s,\omega)}{\partial \xi}\right|}$$

and here $\xi = \Xi(s,\omega,y) = -2\omega\, \hat{R}_{ys} \cdot P_2$
(slight approx.)

$R_{ys} = \gamma(s) - y_T$ in row vector | extract 1st two components
and $y_T P_2 = y$ when $y_T = (y \;\; 0)$

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$(\mathcal{B}FV)(y) = \iint_{\substack{\text{data} \\ \text{manifold } M}} e^{i\xi\cdot(x-y)} V(x)\, d\xi\, dx$$

restriction created by $\chi(s,\omega)$

$$= \int K(y,x) V(x) dx$$

where $K(y,x) = \int_M e^{i\xi\cdot(x-y)}\, d\xi$.

⚠ · OK to divide by $A(s,\omega,y)$,
ie. $P(\omega) \neq 0$, nondirectional beams
· (far-field)
· (Ξ approx.) · sampling of M.

Rmk about matched filtering (somewhere earlier)

Send $p(t)$
Receive $d(s,t)$
→ Form correlation $\int p(t-t')\, d(s,t')\, dt'$
$= (MF\,d)(s,t)$

$$= \int p(-(t'-t))\, d(s,t')\, dt' = (p(-t) *_t d)(s,t)$$

real $p(-t) \Longleftrightarrow \overline{P(\omega)}$

because $\int e^{-i\omega t} p(-t)\, dt$
$= \int e^{i\omega t} p(t)\, dt$
$= P(-\omega)$ or $\overline{P(\omega)}$

Convolution theorem ⇒

$$\widehat{MF\,d}(s,\omega) = \overline{P(\omega)}\, D(s,\omega)$$

→ mult by $\overline{P(\omega)}$ is the effect of matched filtering; present in F^*
→ gives rise to $|P(\omega)|^2$
Division by $|P(\omega)|^2$ present in B

11/05/09 Error in the last few pages!

Radon transform.

$$(F\,V)(\theta,\omega) = \int_{\mathbb{R}^2} e^{-i\omega \hat{e}_\theta \cdot y}\, V(y)\,dy$$

$$F^* D\,(y) = \int_{\mathbb{R}_+\times[0,2\pi]} e^{i\omega \hat{e}_\theta \cdot y}\, D(\theta,\omega)\,d\theta\,d\omega$$

$$F^* F V\,(y) = \int e^{i\omega \hat{e}_\theta\cdot(y-z)}\, V(z)\,d\theta\,d\omega\,dz$$

$$= \int \frac{1}{\omega(\xi)}\, e^{i\xi\cdot(y-z)}\, V(z)\,d\xi\,dz$$

$$\left(\left|\frac{\partial \xi}{\partial(\omega\theta)}\right| = \omega \quad \text{or} \quad \left|\frac{\partial(\omega\theta)}{\partial\xi}\right| = \frac{1}{\omega} \right) \text{ with } \omega(\xi) = \sqrt{\xi_1^2+\xi_2^2}$$

ω and not $1/\omega$ $\longrightarrow$

Def. $$(BD)(y) = \int_{\mathbb{R}_+\times[0,2\pi]} \omega\, e^{i\omega\hat{e}_\theta\cdot y}\, D(\theta,\omega)\,d\theta\,d\omega$$

then $$(BFV)(y) = \int \frac{\omega}{\omega}\, e^{i\xi\cdot(y-z)}\, V(z)\,d\xi\,dz$$

$$= V(y)$$

$$\Rightarrow B = F^{-1}$$

$$\left(RV\,(\theta,t) = \left[\mathcal{F}_{\omega\to t}(FV)\right](\theta,t) \qquad R = \mathcal{F}_{\omega\to t} F \right.$$

$$\left. R^{-1} = F^{-1}\,\mathcal{F}_{\omega\to t}^{-1} = \boxed{B\ \mathcal{F}^*_{\omega\to t}} \right)$$

filtered backprojection

ω = filter

Alternative $$R^{-1} = (R^*R)^{-1} R^*$$

with $$R^* = F^*\,\mathcal{F}^*_{\omega\to t}$$

$$R^*R = F^*\,\mathcal{F}^*_{\omega\to t}\mathcal{F}_{\omega\to t} F = F^*F$$

with $$F^*F = \mathcal{F}^*_{\xi\to y}\left(\frac{1}{\sqrt{\xi_1^2+\xi_2^2}}\right)\mathcal{F}_{y\to\xi}$$

$$= (-\Delta)^{-1/2} \qquad \text{because } (-\Delta) \Leftrightarrow \xi_1^2+\xi_2^2$$

$$F^*F = (-\Delta)^{1/2}, \quad R^{-1} = \boxed{\underbrace{(-\Delta)^{1/2}}_{\text{filter}}\ \underbrace{F^*\,\mathcal{F}^*_{\omega\to t}}_{\text{backp.}}}$$

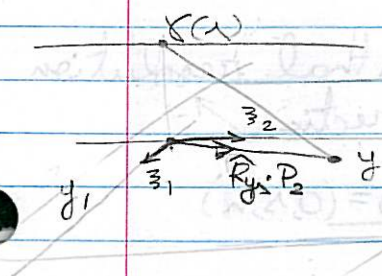
also, $$F^*Fu\,(y) = \int \frac{c}{\|x-y\|}\, u(y)\,dy$$

Resolution: Back to radar

$$(BFV)(y) = \int_{\substack{\text{data} \\ \text{manifold } M \\ \mathbb{R}^2}} e^{i\xi\cdot(y-x)}\, V(x)\, d\xi\, dx$$

$$= \int_{\mathbb{R}^2} K(y,x)\, V(x)\, dx$$

kernel $K(y,x) = \int_M e^{i\xi\cdot(y-x)}\, d\xi$.



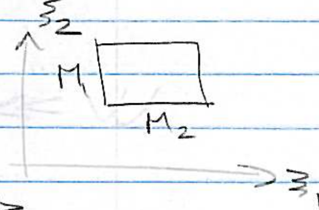
here $\xi = \Xi(s,\omega,y) = -2\omega\, \widehat{R_{y,s}} \cdot \underbrace{P_2}_{\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$

$-2\omega \times$ (first 2 components of $\widehat{R_{y,s}}$)

$$R_{y,s} = \gamma(s) - y_T \qquad y_T = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$\widehat{R_{y,s}} = R_{y,s} / \| R_{y,s} \| = \frac{\gamma(s) - y_T}{R}, \quad R = \text{range}$$

⚠ Assume $M = M_1 \times M_2$.



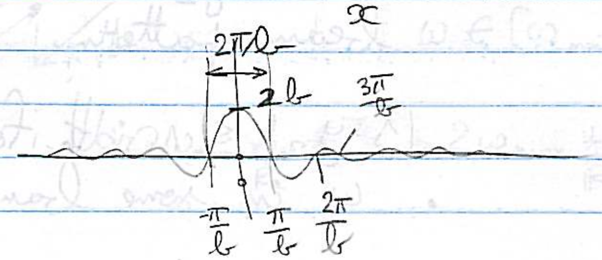
$\xi\cdot(y-x)$
$= \xi_1(y_1-x_1)$
$+ \xi_2(y_2-x_2)$

$$\int_M e^{i\xi\cdot(y-x)}\, d\xi = \int_{M_1} e^{i\xi_1(y_1-x_1)}\, d\xi_1 \times \int_{M_2} e^{i\xi_2(y_2-x_2)}\, d\xi_2$$

with $\displaystyle\int_{-b}^{b} e^{ixk}\, dk = \int_{-b}^{b} \cos(xk)\, dk = \frac{1}{x} \sin xk \Big|_{-b}^{b}$

$$= \frac{2 \sin xb}{x} = 2b\, \text{sinc}(bx)$$

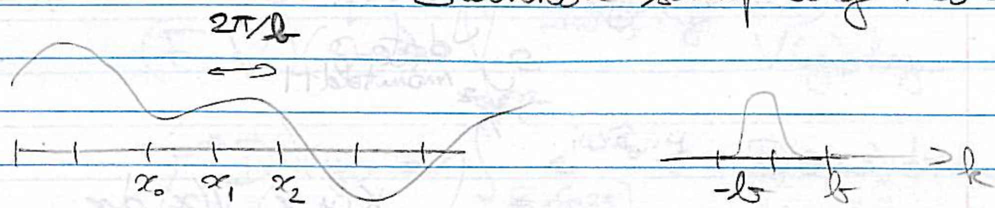
$\delta(x) \longrightarrow$



$bx = \pi$

→ $\boxed{\text{resolution of } \frac{2\pi}{b} \text{ when } \int_{-b}^{b}}$

(Nyquist rate - Shannon sampling theorem)

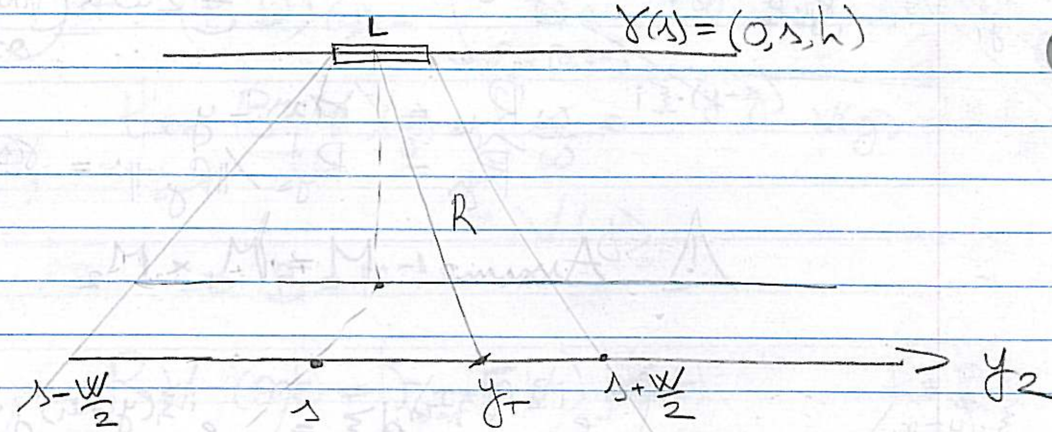


more generally, resolution of $\frac{4\pi}{|M_1|} \times \frac{4\pi}{|M_2|}$

Back to M: $\xi = -2\omega\, \widehat{R_{ys}} \cdot P_2$

① Along-track / azimuthal resolution
→ in the y_2 direction.

$\gamma(s) = (0, s, h)$



y_1 ∠

$$\xi_2 = -2\omega \frac{(\gamma(s)-y_T)_2}{\|\gamma(s)-y_T\|} = -2\omega \frac{(s-y_2)}{\underbrace{R}_{\text{range}}}$$

Data manifold in ξ_2:

- s and y_2 both in the antenna beam pattern.

$$|s-y_2| \leq \text{width footprint}/2$$

- ω in some bandwidth.

Width of an antenna footprint:

- wavelength $\lambda = 2\pi/k$ $\quad k=\omega$ $(c=1)$
- length L
- range R

$$W = R\left(\frac{2\lambda}{L}\right)$$

$$\Rightarrow |\xi_2| = 2\omega \frac{|s-y_2|}{R} \Big/ \cancel{2}$$

$$b = \max_{\xi_2 \in M_2} |\xi_2| = \omega \frac{W}{R} = \frac{\omega}{R} R\left(\frac{2\lambda}{L}\right)$$

$$= \frac{2\omega}{L}\frac{2\pi}{\omega} = \frac{4\pi}{L}$$

$$\Rightarrow \frac{2\pi}{b} = \boxed{\frac{L}{2}} \quad \text{azimuthal resolution.}$$

→ indep ω, R

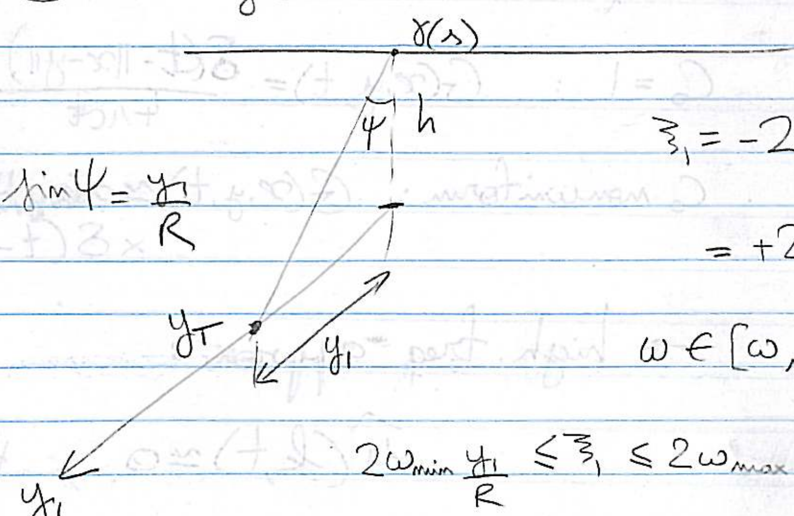
→ small antenna is better so that the beam pattern is wide

very counter-intuitive

Aperture $= W = R\frac{2\lambda}{L}$

Synthetic aperture: $L/2 \ll W$

(2) Range resolution



$$\sin\psi = \frac{y_1}{R}$$

$$\xi_1 = -2\omega \frac{(\gamma(s)-y_T)_1}{R}$$

$$= +2\omega \frac{y_1}{R}$$

$$\omega \in [\omega_{min}, \omega_{max}] = \operatorname{supp} P(\omega)$$

$$2\omega_{min}\frac{y_1}{R} \le \xi_1 \le 2\omega_{max}\frac{y_1}{R}$$

$$2\omega_{min} \sin\psi \le \xi_1 \le 2\,\omega_{max} \sin\psi$$

$$2b = \Delta\xi_1 = 2\,\Delta\omega \sin\psi$$

$$\frac{2\pi}{b} = \frac{2\pi}{(\Delta\omega)\sin\psi}$$

→ better when wider freq. band

→ better when far from antenna ($\psi \simeq \frac{\pi}{2}$)

(proj. of the range vector is longer)

(amplitude decreases though)

Chap. 5. Geom. optics & GRT.

Back to

$$u_{inc}(x,t) = \int G(x,y,t-s)\, f(y,s)\, dy\, ds$$

$$u_{sc,B}(x,t) = \int G(x,y,t-s)\, \frac{\partial^2 u_{inc}}{\partial t^2}(y,s)\, V(y)\, dy\, ds$$

$c_0 = 1$: $G(x,y,t) = \dfrac{\delta(t - \|x-y\|)}{4\pi t}$

c_0 nonuniform: $G(x,y,t) \simeq a(x,y,t) \times \delta(t - \tau(x,y))$

→ high freq. approx.:

$$\hat{u}(k,t) \simeq 0, \quad \forall |k| \le N$$

18.325. 11/10/09.

$$\left(\frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2}-\Delta\right)u = f$$

$$\Rightarrow \quad u_{inc}(x,t) = \int_0^t\int G(x,y,t-s)\, f(y,s)\,dy\,ds$$

$$u_{sc,B}(x,t) = \int_0^t\int G(x,y,t-s)\left(\frac{\partial^2 u_{inc}}{\partial t^2}\cdot V\right)(y,s)\,dy\,ds$$

Structure of G? with $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)G(x,y,t)=\delta(x-y)\delta(t)$

$c=1$: $G(x,y,t)=\dfrac{\delta(ct-|x-y|)}{4\pi c^2 t}$ $\quad (x\in\mathbb{R}^3)$ $\quad ct=|x-y|$

or $\dfrac{1/(2\pi)}{\sqrt{c^2t^2-|x-y|^2}}$ $\quad (x\in\mathbb{R}^2)$

Progressing wave expansion / geometrical optics (Courant & Hilbert).

$\tau(x,y)=\frac{|x-y|}{c}$ is the <u>traveltime</u> between x and y

Now $G(x,y,t) = a(x,y)\,S(t-\tau(x,y)) + R(x,y,t)$

smooth $\quad$ S $\quad$ traveltime $\quad$ remainder

<u>R</u>: <u>smoother</u> than G, <u>not small</u>

a,τ: <u>smooth</u> in an as-yet unspecified region.

$$\left(\frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right) G = \delta(x-y)\,\delta(t) \qquad \text{(def.)}$$
$$= 0 \qquad (t \neq 0)$$

$$\frac{\partial^2}{\partial t^2}\left(a\,S(t-\tau)\right) = a\,S''(t-\tau)$$

$\Delta = \nabla\cdot\nabla$
$$\nabla\left(a\,S(t-\tau)\right) = (\nabla a)\,S - a(\nabla\tau)\,S'$$
$$\nabla\cdot\left(\nabla\left(a\,S(t-\tau)\right)\right) = \Delta a\,S - (\nabla a)\cdot(\nabla\tau)\,S' - (\nabla a)\cdot(\nabla\tau)\,S' - a(\Delta\tau)\,S' + a|\nabla\tau|^2 S''$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) G = a\left(\frac{1}{c^2} - |\nabla\tau|^2\right) S'' + \left(2\nabla\tau\cdot\nabla a + a\,\Delta\tau\right) S' + \Delta a\,S + \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) R = 0.$$

In decreasing order of singularity,

(2) $\frac{1}{c^2} - |\nabla\tau|^2 = 0 \quad \rightarrow$ det. τ

(1) $2\nabla a\cdot\nabla\tau + a\,\Delta\tau = 0 \quad \rightarrow$ det. a

(0) choose R such that $\Delta a\,S + \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) R = 0$

p. 4 here Rmk: Geom. optics & Helmholtz equation. (*)

Eikonal equation of geometrical optics.

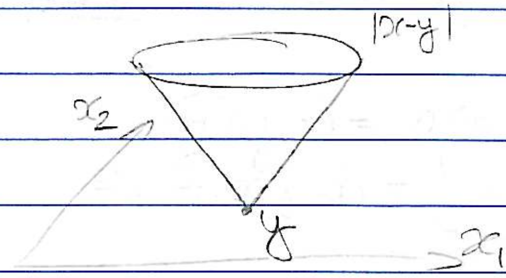
Fix y, PDE in x:

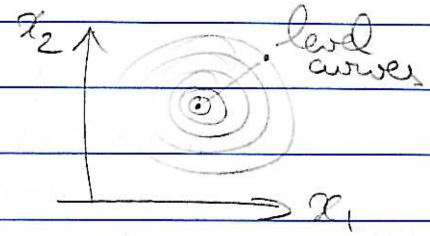
$$|\nabla_x\tau(x,y)| = \frac{1}{c(x)}, \qquad \tau(y,y) = 0$$

ex $|\nabla_x\tau| = 1 \qquad \tau(y,y) = 0$

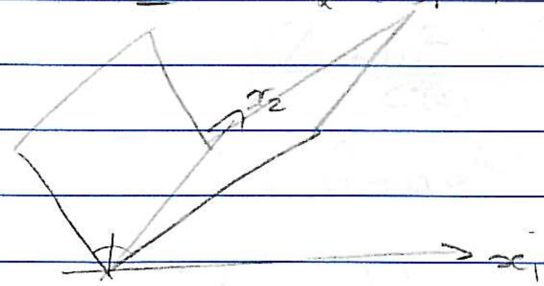
$\Rightarrow \tau(x,y) = |x-y|$ distance function

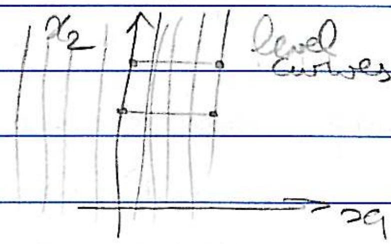
$\nabla_x\tau = \frac{x-y}{|x-y|}$ obeys $|\nabla_x\tau| = 1$





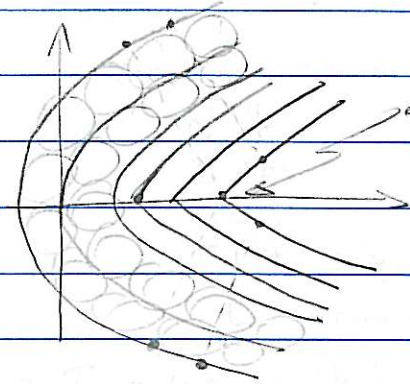
<u>ex</u> $|\nabla_x \tau(x)| = 1$ $\quad$ $\tau(x) = 0$ when $x_1 = 0$





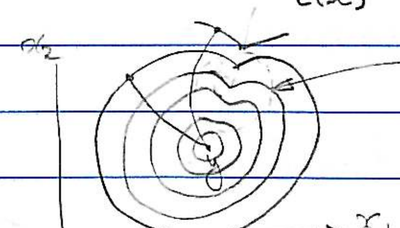
<u>ex</u>. $|\nabla_x \tau(x)| = 1$ $\quad$ $\tau(x) = 0$ along $x_1 = x_2^2$

Huygen's principle.



- discontinuity of $\nabla\tau$.
- jump in the point on the parabola w.r.t. to which the distance is measured.
- when normal lines cross. (rays).

<u>ex</u>. $|\nabla_x \tau(x)| = \dfrac{1}{c(x)}$ $\quad$ locally looks like $\dfrac{|x-y|}{c(y)}$



region where c(x) is large

<u>ex</u>. (1D) $\begin{cases} \left|\dfrac{d\tau}{dx}\right| = \dfrac{1}{c(x)} \\ \tau(y,y) = 0 \end{cases} \Rightarrow \tau(x,y) = \displaystyle\int_x^y \frac{1}{c(z)}\,dz$ ✓

Rmk: Geom. optics

$$G(x,y,t) = a(x,y)\,\delta(t-\tau(x,y))$$
$$\Rightarrow \hat{G}(x,y,\omega) = \int e^{i\omega t} a(x,y)\,\delta(t-\tau(x,y))\,dt$$
$$= a(x,y)\, e^{i\omega\tau(x,y)}$$

(generalization of $\dfrac{e^{i\omega|x-y|}}{4\pi|x-y|}$)

Green's function for the Helmholtz equation:
$$\left(\Delta + \frac{\omega^2}{c^2(x)}\right)\hat{G}(x,y,\omega) = \delta(x-y)$$

Try $\hat{G} = a e^{i\omega\tau}$ and equate like-powers of ω
$\Rightarrow$ get (2), (1).

Asymptotic matching as $\omega \to \infty$:
High-frequency approximation to the actual $\hat{G}$.
(more accurate as $\omega \to \infty$)

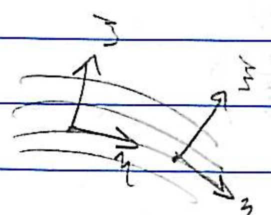
Smoothness in t	$\Leftrightarrow$ Asymptotics as $\omega \to \infty$
$\delta''(t)$	ω^2
$\delta'(t)$	ω
$\delta(t)$	1
$H(t)$ (step)	$1/\omega$
t_+ (ramp)	$1/\omega^2$

$\Rightarrow \tau$ also called a phase

Rmk: Characteristic surfaces:

- wave eq. $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$, $\quad -\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + |\nabla\phi|^2 = 0$ (Ham-Jac.)

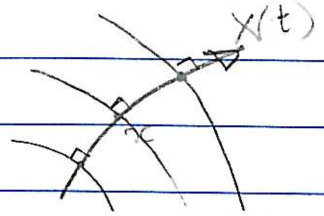
- Helmholtz, $\left(-\frac{\omega^2}{c^2} - \Delta\right) u = 0$, $\quad -\frac{\omega^2}{c^2} + |\nabla\phi|^2 = 0$
then $\phi = \omega\tau$ (eikonal)



Hw.: Show that under $(x,y) \to (\phi,\eta)$, such that $\phi = \omega\tau$, there exist (approx.) solutions of HE that oscillate in ξ and are const. in η, locally: $\boxed{\frac{\partial^2 u}{\partial \xi^2} + \omega^2 u = 0}$ and $\boxed{\frac{\partial^2 u}{\partial \eta^2} = 0}$ (approx.)

(*) Back to p. 2

Method of characteristics
Ray method for determining τ:
curves along which the traveltime is calculated.



$$|\nabla_x \tau| = \frac{1}{c(x)}$$

consider $X(t)$: ray
↳ parameter

$$\frac{d}{dt}\tau(X(t)) = \nabla\tau(X(t)) \cdot \dot{X}(t)$$

$\dot{X}(t) \,/\!/\, \nabla\tau$
$\Rightarrow X(t)$ perpendicular to level lines of τ

locally polar coordinates

Choose $\boxed{\dot{X}(t) = c(X(t)) \cdot \frac{\nabla\tau}{|\nabla\tau|}}$

$$\dot{X}(t) = c^2(X(t)) \cdot \nabla\tau(X(t)) \qquad \text{ODE}$$

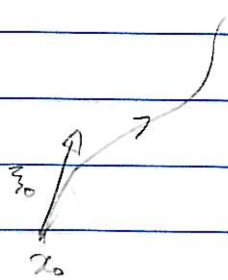
then $\nabla\tau \cdot \dot{x} = c^2 |\nabla\tau|^2 = 1.$

$\Rightarrow \tau(X(t)) - \tau(X(t_0)) = t - t_0$
and τ is identified with t

$X(t)$ still depends on τ though.
Record direction:
Put $\xi(t) = \nabla\tau(X(t))$ (other possible choice: $\frac{\nabla\tau}{|\nabla\tau|}$)

Then
$$\begin{aligned}
\dot{\xi}(t) &= \nabla\nabla\tau(X(t)) \cdot \dot{X}(t) \\
&= \nabla\nabla\tau \cdot \nabla\tau \; c^2 \qquad (\text{at } X(t)) \\
&= \frac{1}{2} c^2 \nabla \cdot |\nabla\tau|^2 \\
&= \frac{1}{2} c^2 \nabla(c^{-2}) \\
&= -\frac{1}{2} c^{-2} \nabla(c^2) \\
&= -\frac{1}{2} |\nabla\tau|^2 \nabla(c^2) \\
&= -\frac{1}{2} |\xi(t)|^2 \nabla(c^2)(X(t))
\end{aligned}$$

$\boxed{\text{indep. of } \tau}$



$$\Rightarrow \begin{cases} \dot{X}(t) = c^2(X(t))\; \xi(t) & X(0) = x_0 \\ \dot{\xi}(t) = -\frac{\nabla c^2}{2}(X(t))\; |\xi(t)|^2 & \xi(0) = \xi_0 \end{cases}$$

$\Rightarrow$ Solve for the rays about y.
τ = arclength along the rays.

Rmk: τ = action for the Fermat principle

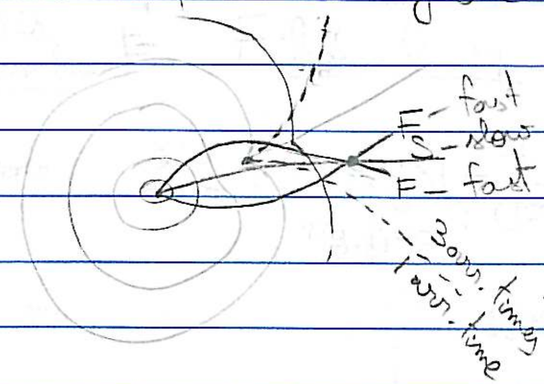
Rmk: PDE in x : Eulerian viewpoint
2 ODEs in t : Lagrangian viewpoint.

Rmk. Rays are charact. curves for the eikonal equation. They are bicharact curves for the WE.

Rmk. $H(x,\xi) = \frac{1}{2} c^2(x) |\xi|^2$ (optics, acoustics)

then
$$\begin{cases} \dot{X}(\sigma) = \nabla_\xi H(X(\sigma), \xi(\sigma)) \\ \dot{\xi}(\sigma) = -\nabla_x H(X(\sigma), \xi(\sigma)) \end{cases}$$
Hamiltonian system

Rmk. Validity of τ? $\Omega(y)$
$= \{ x :$ one and only one ray links x to $y \}$



— multipathing.

In general, need τ_j, as many functions as there are arrival times
→ multiple-valued solution to
$$|\nabla \tau| = 1/c \quad \text{(locally)}$$
Otherwise, τ records the 1st arrival.
(in which sense? viscosity solution → PDE)

Rmk. What about a? Have $2\nabla a \cdot \nabla \tau - a \Delta \tau = 0$
Called a transport equation: along rays
$$\frac{d}{dt} a(X(t)) = \dot{X}(t) \cdot \nabla a(X(t))$$
$$= c^2(X(t)) \nabla \tau \cdot \nabla a(X(t))$$
$$= c^2(X(t))\, a(X(t))\, \underbrace{\Delta \tau(X(t))}$$

ex: express in terms of $X, \xi, \frac{\partial X}{\partial x_0}, \frac{\partial \xi}{\partial x_0}$

⇒ ODE along rays. (do properly next time)

Next time: a, caustics, or analytic extension, blowup, remainder R, then GRT migration. Talk about wavefronts

11/12/09 Geometrical optics: amplitude

	Wave equation	Helmholtz eq.
eq.	$\left(\frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right)u = 0$	$\left(\Delta + \frac{\omega^2}{c^2(x)}\right)u = 0$

Lax

$G(x,y,t) \simeq a(x,y)\,\delta(t-\tau(x,y))$

Prog. wave expansion

$t-\tau(x,y)$ is a wavefront

Eikonal eq

$\Phi(x,y,\omega) \simeq a(x,y)\,e^{i\omega\tau(x,y)}$

geom. optics

τ is a phase

Char. eq.

Ham-Jacobi here	$\xleftrightarrow{\text{Osher}}$	$\lvert\nabla_x \tau(x,y)\rvert = \frac{1}{c(x)}$, $\tau(y,y)=0$

connection

$\tau(X(t)) - \tau(X(0)) = t$ $\qquad X(0)=y$

τ is traveltime along rays

$\nabla\tau = \xi$ called slowness vector

τ is the action for the Fermat principle.

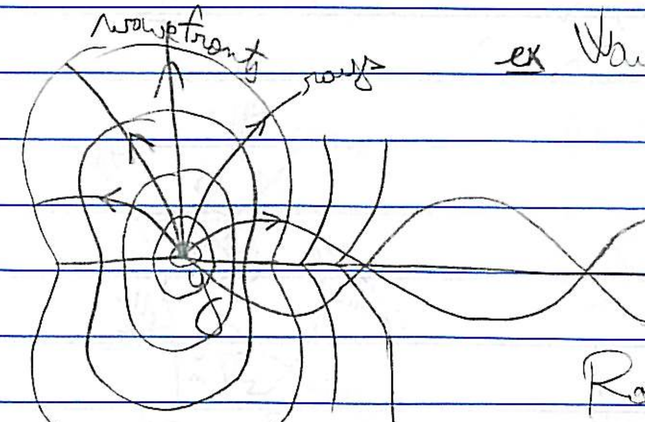
Bichar. eq.

Ray equations:

$$H(x,\xi) = \frac{1}{2}c^2(x)|\xi|^2$$

$$c^2(X(t))\,\xi(t) = \dot{X}(t) = \nabla_\xi H(X(t),\xi(t)), \quad X(0) = x_0$$

$$-\frac{1}{2}\nabla c^2(X(t))\,|\xi(t)|^2 = \dot{\xi}(t) = -\nabla_x H(X(t),\xi(t)), \quad \xi(0) = \xi_0$$



ex Waveguide:

Rays ⊥ Wave fronts: only in an isotropic medium simple geom. optics

Validity: no multiple arrivals. / τ records only 1^{st} arrival
otherwise branches of a multivalued function.
$\nabla\tau$ discontinuous. $\xi = \nabla\tau$ not defined uniquely

Viscosity solution: $|\nabla \tau_\varepsilon(x)|^2 + \varepsilon\, \Delta \tau_\varepsilon(x) = 0$
then let $\varepsilon \to 0$.

Eikonal eq. valid for $|y-x|$ not too large.
Ray equations are valid for all times.

⚠ when using τ, simple geom. optics.

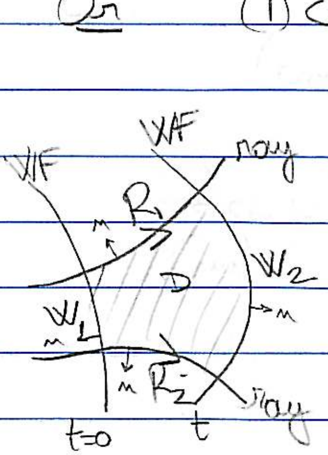
Amplitude: $\quad 2\, \underline{\nabla a \cdot \nabla\tau} + a\, \Delta\tau = 0. \qquad (1)$

dir. der. of a in the dir. of $\nabla\tau = \xi$

$$\frac{d}{dt}\, a(x(t),y) = \dot{x}(t)\cdot \nabla a \qquad (\text{Fix } y)$$
$$= c^2\, \xi \cdot \nabla a$$
$$= c^2\, \nabla\tau \cdot \nabla a$$
$$= -\frac{c^2}{2}\, a\, \Delta\tau \quad \text{by } (1)$$

$\to$ transport equation along rays.

Or $\quad (1) \iff \nabla\cdot(a^2 \nabla\tau) = 0 \qquad$ div. form.
$\left(2\, a \nabla a \cdot \nabla\tau + a^2 \Delta\tau = 0 \right)$



Use the div. thm in  : (ray tube)

$$\iint_D \nabla\cdot(a^2\nabla\tau)\, dA \qquad (\text{surface})$$
$$\|$$
$$\oint_{\partial D} a^2 \nabla\tau \cdot n\, ds \qquad (\text{line-circulation})$$

$$\partial D = W_1 \cup W_2 \cup R_1 \cup R_2$$

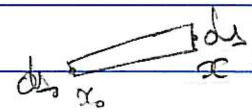
$$n \cdot \nabla\tau = 0 \quad \text{on } R_1, R_2$$

$$n \cdot \nabla\tau = \begin{cases} -|\nabla\tau| & \text{on } W_1 \\ +|\nabla\tau| & \text{on } W_2 \end{cases}$$

$$\Rightarrow \int_{W_1} a^2 |\nabla\tau|\, ds = \int_{W_2} a^2 |\nabla\tau|\, ds$$

$$\int_{W_1} \frac{a^2}{c}\, ds = \int_{W_2} \frac{a^2}{c}\, ds$$

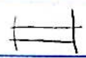
limit $|W_1|, |W_2| \to 0$, $\quad \dfrac{a^2(x)}{a^2(x_0)} \dfrac{c(x_0)}{c(x)} = \dfrac{ds}{ds_0}$



$$a(x) = a(x_0) \cdot \sqrt{\frac{c(x_0)}{c(x)} \frac{ds}{ds_0}}.$$

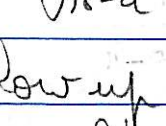
$\to$ conservation of energy
(flow down tube)

ex. Plane waves: $\dfrac{ds}{ds_0} = 1 \qquad a = \text{const.}$



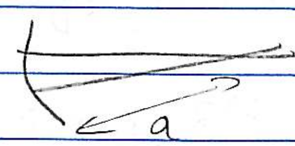
Circ. waves / cylindrical: $\dfrac{ds}{ds_0} = r, \quad a \sim 1/\sqrt{r}$

Spherical waves: $\dfrac{dA}{dA_0} = r^2, \quad a \sim 1/r$



(compare to $\dfrac{e^{i\omega|x-y|}}{4\pi|x-y|}$)

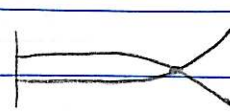
Focus / Caustic:



$: \; a \sim \dfrac{1}{\sqrt{r-a}}$

$\to$ blow up of the amplitude

or



$\to$ brighter spots

$$G(x,y,t) = a(x,y)\,S(t-\tau(x,y)) + R(x,y,t)$$

Closure. (ω^2, S'') $\quad |\nabla\tau| = \frac{1}{c(x)}$ $\quad\Rightarrow$ eq. for τ

(ω^1, S'): $\quad 2\nabla a\cdot\nabla\tau + a\Delta\tau = 0$ $\quad\Rightarrow$ eq. for a

(ω^0, S): $\quad \Delta a\, S + \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)R = 0.$

$\Rightarrow$ eq. for R.

Ex. • R is no worse than discontinuous if $S=\delta$
• $\int dx\int dt\, |R(x,y,t)|^2 \leq C.\|G_0\|^4$

Hint: energy estimate, | $\frac{\partial w}{\partial t} - Lw = f$
$\Rightarrow \frac{dE}{dt} = 2(f,w)$
$E(t) = \langle w,w\rangle$.

Concl: R is smoother than S.

Forward / Imaging operators

$$u_{inc}(x,t) = \iint G(x,y,t-s)\, f(y,s)\, dy\, ds$$

⚠ $f(y,s) = \delta(y-x_0)\delta(t)$

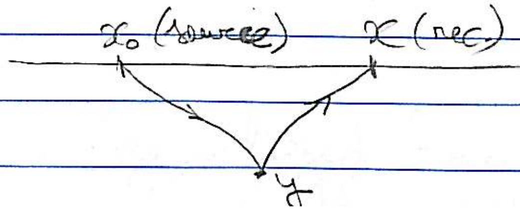
$$= G(x,x_0,t)$$

$$u_{x,B}(x,t) = \iint G(x,y,t-s)\, V(y)\, \frac{\partial^2 u_{inc}}{\partial t^2}(y,s)\, dy\, ds$$

with $V(y) = \frac{2\delta c}{c^3} = m$

$$= \int dy\, V(y) \int ds\, G(x,y,t-s)\, \frac{\partial^2 G}{\partial t^2}(y,x_0,s)\, ds$$

⚠ $G(x,y,t) \simeq a(x,y)\,\delta(t-\tau(x,y))$ $\quad$ ⚠ simple geom. optics



$$u_{s,B}(x,t) = \int dy\, V(y)\, K(x,y,t) \qquad = FV(x,t)$$

kernel $$K(x,y,t) = \int G(x,y,t-s)\, \frac{\partial^2 G}{\partial t^2}(y,x_0,s)\, ds$$

$$\simeq \int a(x,y)\, a(y,x_0)\, \delta(t-s-\tau(x,y)) \times \delta''(s-\tau(y,x_0))\, ds$$

$$= \boxed{a(x,y)\, a(y,x_0)\, \delta''(t-\tau(x,y)-\tau(y,x_0))}$$

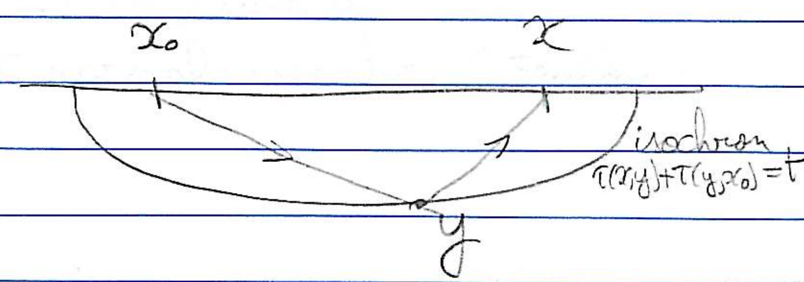
unclear

Possible when $s = t-\tau(x,y)$
and $s = \tau(y,x_0)$
do not intersect transversally $\Rightarrow$ $\nabla_x \tau(x,y) \neq -\nabla_x \tau(y,x_0)$

no forward scattering $\Leftarrow$ simple geom. optics

$$\Rightarrow u_{s,B}(x,t) = \underbrace{\frac{\partial^2}{\partial t^2}}_{\text{filtering}} \int \underbrace{a(x,y)\, a(y,x_0)}_{\text{weighting}}\, \underbrace{\delta(t-\tau(x,y)-\tau(y,x_0))\, V(y)\, dy}_{\text{integral over isochron}}$$

$\{y : t = \tau(x,y) + \tau(y,x_0)\}$



ellipse/ellipsoid when $c_0 = \text{const}$

$$\tau(x,y) = \frac{|x-y|}{c_0}$$

$\rightarrow$ Generalized Radon trf.,
Kirchhoff modeling

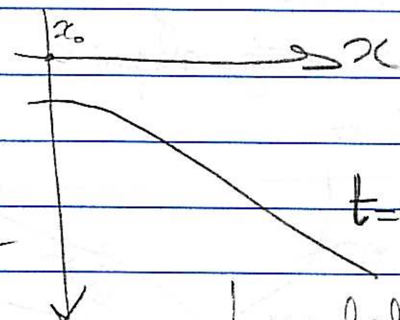
Adjoint: $\langle d, FV \rangle_{(x,t)} = \langle F^* d, V \rangle_{(y)}$

$$\iint d(x,t) \left(\int V(y) K(x,y,t)\, dy \right) dx\, dt$$

$$\int V(y) \left(\iint d(x,t) K(x,y,t)\, dx\, dt \right) dy$$

$$\Rightarrow \quad F^* d(y) = \iint K(x,y,t)\, d(x,t)\, dx\, dt$$

$$= \boxed{\iint a(x,y)\, a(y,x_0)\, \delta'\left(t - \tau(x,y) - \tau(y,x_0)\right) d(x,t)\, dx\, dt}$$



fix x_0, y:

$t = \tau(x,y) + \tau(y,x_0)$

hyperbola / hyperboloid when $c = \text{const}$
(moveout curve)

$\rightarrow$ GRT as well
Kirchhoff migration

Next: multi-offset / sources
GRT normal operator (Beylkin)

11/17/09.

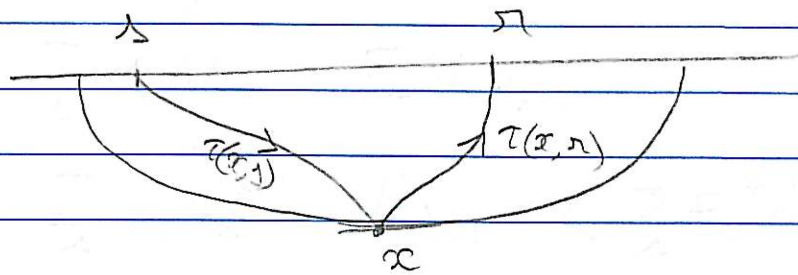
$\left\{ \begin{array}{ll} r = \text{receiver} & s = \text{source, fixed} \\ t, t' = \text{time} & x = \text{subsurface position} \end{array} \right.$

$$u_{s,B}(r,t) = \int K(r,x;t)\, V(x)\, dx$$

$$\text{with } K(r,x,t) = \int dt'\, G(r,x,t-t')\, \frac{\partial^2 G}{\partial t^2}(x,s,t')$$

⚠ $f(x,t) = \delta(x-s)\,\delta(t)$

⚠ $G(x,y,t) = a(x,y)\,\delta(t-\tau(x,y))$
Simple geom optics



$$\Rightarrow K(r,x,t) = a(x,s)\, a(x,r)\, \delta''(t-\tau(x,r)-\tau(x,s))$$

$$\text{and } \int \delta'' f = \int \delta f'' \quad \forall f$$

$$u_{s,B}(r,t) = \underbrace{\frac{\partial^2}{\partial t^2}}_{\text{filter}} \int \underbrace{a(x,s)\, a(x,r)}_{\text{weight}}\, \delta(t-\tau(x,r)-\tau(x,s))\, V(x)\, dx$$

integral over $\{x : t = \tau(x,r)+\tau(x,s)\}$
isochron (one for each r, t)

$\Rightarrow$ generalized Radon transform along ellipse / ellipsoid when $c_0 = \text{const.}$ $\tau(x,y) = \frac{|x-y|}{c_0}$

$\Rightarrow$ Kirchhoff modeling

Adjoint: see p. 6 prev lecture
Kirchhoff migration

Multiple sources: $u_{x,B}(r,s;t) = F_s(V(x))(r,t)$

Modeling: $FV = \{ F_s V \ ; \ s \text{ sources} \}$.

$$\langle d, FV \rangle_{(r,t;s)} = \langle F^* d, V \rangle_{(x)}$$

$$\| $$

$$\sum_{r,s,t} d(r,s,t) (F_s V)(r,t)$$

$$\| $$

$$\sum_s \langle d(\cdot,s,\cdot), F_s V \rangle_{(r,t)}$$

$$\| $$

$$\sum_s \langle F_s^* d, V \rangle_{(x)} = \langle \sum_s F_s^* d, V \rangle_{(x)}$$

$$\Rightarrow \boxed{F^* = \sum_s F_s^*}$$

Imaging operator: sum over redundant coord. / sources:
= stack

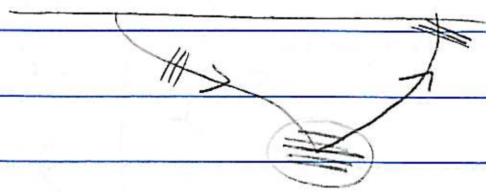
Other ways to cut up data space
gather eg common source (r,s)
common midpoint $\left(\frac{r-s}{2}, \frac{r+s}{2}\right)$
offset

$\{ F_s^* d \}$: prestack migration
$\sum_s F_s^* d$: poststack migration

Algo for K mig:
loop over s
loop over x
loop over r
$f(\ldots t = T(x,r) + T(x,s))$

Kinematics of migration (instead of studying the normal op like before)

$$u_{sc,B}(r,t) = \int K(r,x,t)\, V(x)\, dx$$



Goal: understand relationship between singular/oscillatory behaviors of $u_{sc,B}$ and V (microlocal)

(eg $\delta(x) = \int 1\, e^{i\xi x}\, d\xi$ $\Rightarrow$ very rich high freq. content)

a) Singular supports: $V(x) = \delta(x - x_0)$

$$\Rightarrow u_{sc,B}(r,t) = \text{amp} \times \delta(t - \tau(x_0, r) - \tau(x_0, s))$$

map $x_0 \to \{(r,t) : t = \tau(x_0, r) + \tau(x_0, s)\} = S_{x_0}$

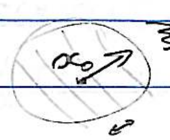
Thm. $\text{sing supp}(u_{sc,B}) \subseteq \bigcup_{x_0 \in \text{sing supp} V} S_{x_0}$.

But: find a way to index x: $\in C_0^\infty$ no confusion with Ham. flow

b) (Local) Oscillations: $V(x) = \chi(x - x_0)\, e^{i\xi \cdot x}$

$\lambda = \frac{2\pi}{|\xi|}$ tiny

diam(χ) small



(Complex OK)

→ study oscillations in $u_{sc,B}$

→ compare them to $\psi(r,t) = e^{i(\omega t + \xi_r r)}$ (detector)

Find conditions (on ξ, ξ_r, ω) under which

$\langle \bar{\Psi}, FV \rangle_{(r,t)}$ is large

[or in other words find ξ, ξ_r, ω s.t. $\hat{K}(\xi; \xi_r, \omega)$ is large (F.T in all arguments)]

$$= \iint_{r,t} e^{-i(\omega t + \xi_r r)} \int_x \mathrm{amp}(-)\, \chi_1 \chi_2\, e^{i \xi x}\, \delta''(t - \tau(r,x) - \tau(r,s))\, dx\, dr\, dt.$$

int by parts in t: $\frac{\partial^2}{\partial t^2}\left(e^{-i\omega t} \chi_2(t-t_0, r-r_0)\right)$

$$\simeq -\omega^2 e^{-i\omega t} \chi_2(t-t_0, r-r_0)$$

$\Rightarrow$ replace δ'' by δ

$$\iint_{r\,x} e^{i\xi x} e^{-i\xi_r r} e^{-i\omega(\tau(r,x) + \tau(x,s))} \left[\mathrm{amp}\, \chi_1 \chi_2\right] dr\, dx$$

Stationary phase: (check nondegen Hessian: see homework!)

- in x: $\xi - \omega D_x \tau(r,x) - \omega D_x \tau(x,s) = 0$.

$$\boxed{\xi = \omega\left(D_x \tau(r,x) + D_x \tau(x,s)\right)}$$

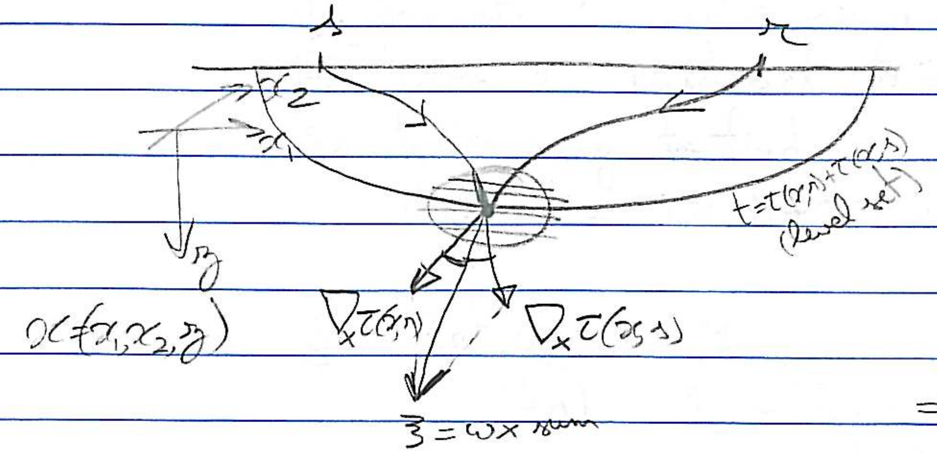
⚠ x chosen s.t. $D_x \tau(r,x) + D_x \tau(x,s) \neq 0$.

- in r: $\xi_r + \omega D_r \tau(x,r) = 0$.

$$\eta \cdot \left(\qquad \right) = 0$$

$$\boxed{\begin{pmatrix} \xi_r \\ \omega \end{pmatrix} \cdot \begin{pmatrix} \eta \\ \eta \cdot D_r \tau(x,r) \end{pmatrix} = 0}$$

• in x: (model space)



ξ = (co)normal to reflector

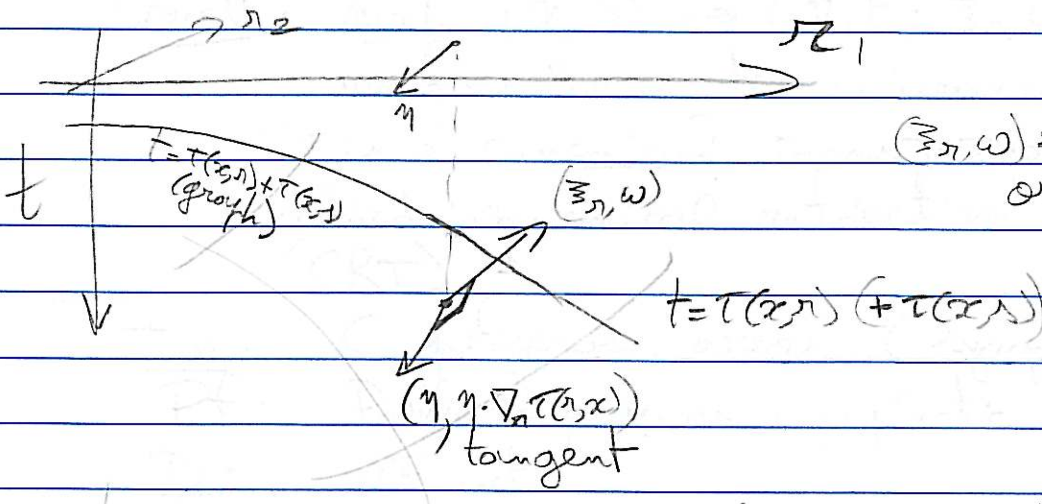
$|\nabla_x \tau(x,s)| = |\nabla_x \tau(x,r)|$
$(= 1/c(x))$

$\Rightarrow$ $\boxed{\xi \text{ bisector of } \nabla_x \tau(x,r) \text{ and } \nabla_x \tau(x,s)}$

Snell's law of reflection

Also, ξ normal to isochron!
(again, $\nabla_x\tau + \nabla_x\tau \neq 0$)

• in r: (data space)



moveout curve/surface

(ξ_r,ω) = (co)normal to oscillations in data

(ξ_r,ω) $\perp$ to all tang. vectors

$\Rightarrow$ $\boxed{(\xi_r,\omega) = \text{(co)-normal to moveout curve/surface}}$

How to det. (ξ_r,ω)?
Fix s. Consider (x,ξ).

① connect x_s and x by a ray, find $\nabla_x \tau(x,s)$
② from ξ construct take-off dir. $-\nabla_x \tau(x,r)$
③ trace ray to find r ($z=0$ crossing),
then $t = \tau(x,r) + \tau(x,s)$

④ Det ω and ξ_r:
4a $\quad \xi = \omega(\nabla_x \tau(x,r) + \nabla_x \tau(x,s))$
$\Rightarrow \omega = |\xi| / |\nabla_x\tau + \nabla_x \tau|$ (denom. $\neq 0$)
4b $\quad \xi_r + \omega \nabla_r \tau(x,r) = 0$
$\Rightarrow \xi_r = -\omega \nabla_r \tau(x,r)$

Concl: map $(x,\xi) \to (r,t;\xi_r,\omega) = C(x,\xi)$
removes the multi-valuedness of $x \to (r,t)$
It is a canonical transformation,
called canonical reflection transformation
(CRT)

Thm. C preserves areas (symplectomorphism)
(Pf: Pullback preserves symplectic 2-form)
Def. WF set
Thm $WF(u_{x,\xi}) \subseteq C \circ WF(V)$
Rmk. C is a global object - extends
beyond caustics, when multipathing.

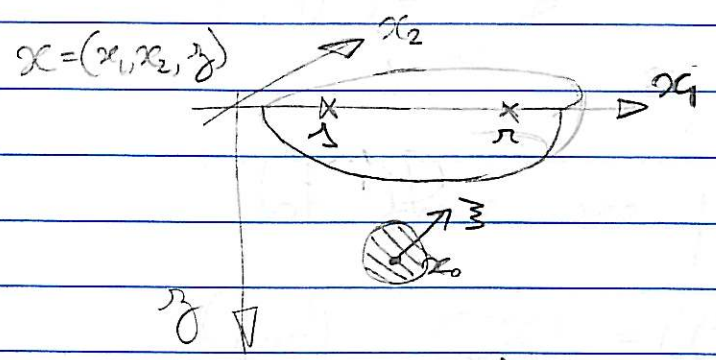
11/19/09 Kinematics.

Forward / demigration:

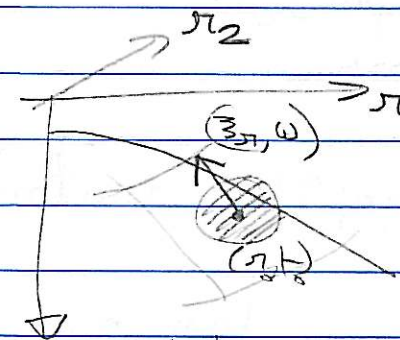
$$F V(r,t) = \int a(x,r)\, a(x,s)\, \delta''(t - \tau(x,r) - \tau(x,s))\, V(x)\, dx$$

Migration:

$$F^* d(x) = \iint a(x,r)\, a(x,s)\, \delta''(t - \tau(x,r) - \tau(x,s))\, d(r,t)\, dr\, dt$$



model



data

Local oscillations : $(x_0, \xi) \to \left(\begin{pmatrix} r_0 \\ t_0 \end{pmatrix}, \begin{pmatrix} \xi_r \\ \omega \end{pmatrix} \right)$ (map in phase-space)

trial. $V(x) = \chi(x - x_0) e^{i \xi \cdot x}$, $\chi \in C_0^\infty$

test $\psi(r,t) = \chi(r - r_0)\, \chi(t - t_0)\, e^{i(\xi_r r + \omega t)}$

then $\langle \psi, FV \rangle_{(r,t)} = \iint \overline{\psi}(r,t)\, FV(r,t)\, dr\, dt$

↳ ⚠ continuous r, t.

Rmk: V complex : OK by linearity.

Assume $\frac{2\pi}{|\xi|}, \frac{2\pi}{\left|\binom{\xi_r}{\omega}\right|} \ll \operatorname{diam}(\operatorname{supp}(\chi))$

$$\langle \psi, FV\rangle_{(r,t)} = \iiint e^{-i(\xi_r r + \omega t)} e^{i\xi x} \chi(x-x_0)\chi(r-r_0)\chi(t-t_0)$$

$$\times [\text{amp}]\ \delta''(t-\tau(x,r)-\tau(x,s))\ dx\,dr\,dt$$

Int by parts: $\frac{\partial^2}{\partial t^2}\left(e^{-i\omega t}\chi(t-t_0)\right)$

$$\simeq -\omega^2 e^{-i\omega t}\chi(t-t_0)$$

$$\langle \psi, FV\rangle = \iint e^{i(\xi\cdot x - \xi_r r - \omega(\tau(x,r)+\tau(x,s)))}[\chi][\text{amp}]\ dx\,dr$$

Stat. phase: see p. 4 last lecture. (2 pages here)

Concl. $C_F(x,\xi) = \left(\begin{pmatrix} r\\ t\end{pmatrix}, \begin{pmatrix}\xi_r\\ \omega\end{pmatrix}\right)$ is a map.

- not defined at x s.t. $\nabla_x\tau(x,r)+\nabla_x\tau(x,s) = 0$
- also,
  - ray from s may not reach x

— ——— x ——— r

(go off to China)

Called canonical reflection transformation

(Surjectivity: perhaps not either)

Rmk. Global object: generalizes to multivalued traveltimes

Rmk. (Hw) (tough) For $\left(\begin{pmatrix} r\\ t\end{pmatrix}, \begin{pmatrix}\xi_r\\ \omega\end{pmatrix}\right) \in \text{Ran}\, C_F$,

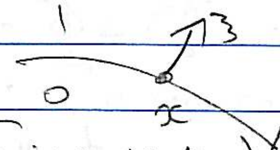
$$C_{F^*}\left(\begin{pmatrix} r\\ t\end{pmatrix}, \begin{pmatrix}\xi_r\\ \omega\end{pmatrix}\right) = (x,\xi) \quad \text{as above}$$

$$\Rightarrow \boxed{C_{F^*} = C_F^{-1}}$$

$\to$ resolution of ~~seis.~~ lin. inv. prob, high freq.

Real definition of C_F: uses wavefront sets (microlocal analysis)

Def. (later) $WF(V) = \{ (x,\xi) \in \mathbb{R}^{2n} :$ "V is singular at x in the direction ξ" $\}$.



($)

Rmk. $\text{Sing supp } V = \pi \, WF(V) = \{ x : \exists \xi : (x,\xi) \in WF(V) \}$

Def. $C_F = WF(\underbrace{K(r,t;x)}_{\text{kernel of } F})$

$= \{ (x,\xi; \binom{r}{t}, \binom{\xi_r}{\omega}) :$ "K is singular at $(\binom{r}{t}; x)$ in the direction $(\binom{\xi_r}{\omega}; \xi)$" $\}$

Def. Composition of WF / relations

$A \subseteq X$, $B \subseteq X \times Y$

$B \circ A = \{ y \in Y : \exists x \in A : (x,y) \in B \}$.

Thm. (Composition of distributions).

Assume C_F does not have elements of the kind $(x,\xi; 0,0)$ or $(0,0, \binom{r}{t}, \binom{\xi_r}{\omega})$

amount to $\nabla_x \tau + \nabla_x \tau \neq 0$

Then $WF(FV) \subseteq \underbrace{WF(K)}_{C_F} \circ WF(V)$

(see Hörmander Vol 1)

Cor. $WF(F^*FV) \subseteq WF(V)$ (normal operator is microlocal)

<u>Def.</u> WF(f)

Fix x; <u>Def.</u> the singular fiber $\Sigma_x(f)$ as follows:

$v \in \left(\Sigma_x(f)\right)^c$ when there exist:
- $\phi \in C_0^\infty$: $\phi(x) \neq 0$,
- Γ a conical neighborhood of v,

such that

$$\left|\widehat{\phi f}(\xi)\right| \leq C_N (1+|\xi|)^{-N} \quad \forall \xi \in \Gamma, \ \forall N \in \mathbb{Z}_0^+.$$

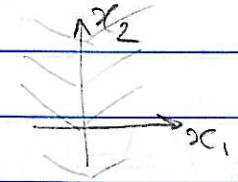
Then $WF(f) = \{(x,\xi) : \xi \in \Sigma_x(f)\}$.

<u>Ex.</u> $f \in C_0^\infty$; $WF(f) = \phi$.

$f(x) = \delta(x)$; $WF(f) = \{(0,\xi), \ \xi \in \mathbb{R}^n\}$.

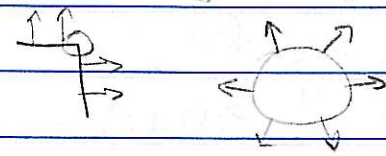
$f(x) = |x-a|$; $WF(f) = \{(a,\xi), \ \xi \in \mathbb{R}^n\}$

$f(x) = |x_1|$; ($x \in \mathbb{R}^2$)



$WF(f) = \left\{\left(\begin{pmatrix}0\\x_2\end{pmatrix}, \begin{pmatrix}\xi_1\\0\end{pmatrix}\right), \ x_2, \xi_1 \in \mathbb{R}\right\}$

etc.



<u>Next</u>: Normal operator.

<u>Rmk</u>: Canonical = symplectic = preserves areas

12/01/09 Microlocal analysis: formalize ideas such as
- reflector, reflection
- mapping of singularities
- like the identity
- directions of singularities
- imaging vs. inversion.

Map $(x, \xi) \to \left(\binom{x_r}{t}, \binom{\xi_r}{\omega} \right)$

(model) (data)

Obtained by stationary phase.

$\to$ Fourier analysis of singularities

Wavefront sets.

$k \leftrightarrow \xi$.

Thm Let $u \in C_0^\infty(\mathbb{R}^n)$, and $\hat{u}(k) = \int e^{-ixk} u(x)\,dx$

Then $|\hat{u}(k)| \le C_N (1+|k|^2)^{-N}$ $\quad N = 1, 2, 3, \dots$

Pf. $k = \beta \hat{k}$, $\quad \beta = |k| \quad \hat{k} = k/|k|$

$x \cdot k = \beta \phi(x)$ with $\phi(x) = \hat{k} \cdot x$

$\nabla \phi = \hat{k} \neq 0$

Stat. phase $|\hat{u}(k)| \le C_N \beta^{-N}$.

$$\frac{(I - \Delta)}{1+|k|^2} e^{-ixk} = e^{-ixk}$$

$$\hat{u}(k) = \int \left(\frac{I - \Delta}{1+|k|^2} \right)^N \left[e^{-ix\cdot k} \right] u(x)\,dx$$

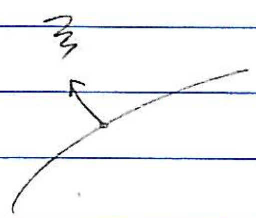
$$= (1+|k|^2)^{-N} \int e^{-ix\cdot k} (I-\Delta)^N u(x)\,dx$$

$$|\hat{u}(k)| \le (1+|k|^2)^{-N} \cdot \underbrace{\| (I-\Delta)^N u \|_{L^1}}_{\|u\|_{W^{1,N}} = C_N} \quad \square$$

Def. Let $u \in \mathcal{E}'(\mathbb{R}^n)$ or $\mathcal{D}'(\mathbb{R}^n)$

$\text{Sing supp}(u) = \{x \in \mathbb{R}^n : \nexists$ open neighborhood V of x s.t. $u \in C^\infty(V)\}$.

Now $WF(u)$: set of couples $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.



- $x \in$ sing supp u.
- "u is singular in the direction ξ" (estimate $(1+|\xi|^2)^{-N}$ fails.)

Def. (conic neighborhood)

X is a conic neighborhood of $Y \subseteq \mathbb{R}^n$ if

- X is open
- $X \supseteq Y$
- if $\xi \in X$ then $\lambda\xi \in X$ for all $\lambda > 0$.

Def. (Singular cone).

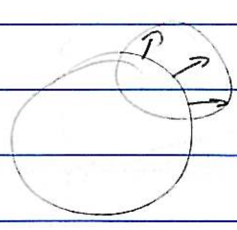
$\Gamma(u) = \{\eta \in \mathbb{R}^n \setminus \{0\} : \nexists$ conic neighborhood W of η s.t. for $\xi \in W$,

$|\hat{u}(\xi)| \le C_N (1+|\xi|^2)^{-N}$ $N = 1, 2, 3, \ldots\}$

= "directions in which u is nonsmooth, globally."

Lemma Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{E}'(\mathbb{R}^n)$ or $\mathcal{D}'(\mathbb{R}^n)$.

Then $\Gamma(\phi u) \subseteq \Gamma(u)$.



Def (Singular fiber) For $x \in X \subseteq \mathbb{R}^n$,

$$\Gamma_x(u) = \bigcap_{\phi} \Gamma(\phi u) : \quad \phi \in C_0^\infty(X),\ \phi(x) \neq 0.$$

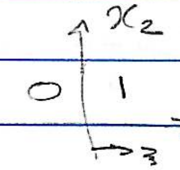
Alt. def, $\Gamma_x(u) = \lim_{j \to \infty} \Gamma(\phi_j u)$ $\quad \phi_j \in C_0^\infty(X),\ \phi_j(x) \neq 0,$ $\operatorname{supp} \phi_j \to \{x\}.$

Def. (Wavefront set)

$$WF(u) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \text{ s.t. } \xi \in \Gamma_x(u)\}.$$

Rmk $\Pi_{\mathbb{R}^n} WF(u) = \operatorname{sing\,supp}(u)$

$\Pi_{\mathbb{R}^n \setminus \{0\}} WF(u) = \Gamma(u)$

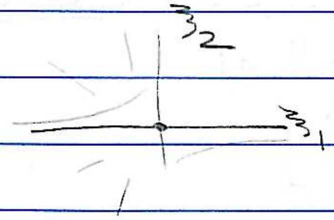


Ex. $u = H(x_1) = \begin{cases} 1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$

$$WF(u) = \left\{ \left(\begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \right) : x_2 \in \mathbb{R},\ \xi_1 \in \mathbb{R} \setminus \{0\} \right\}$$

Because: $\phi u \in C_0^\infty$ when $\operatorname{supp} \phi \cap \{x_1 = 0\} = \emptyset$

$\Rightarrow \operatorname{sing\,supp} = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$



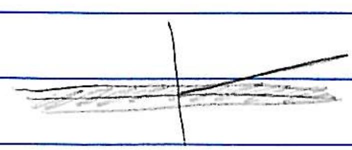
$$\hat{u}(\xi) = \int e^{-ix \cdot \xi} H(x_1)\, dx_1\, dx_2$$

$$= \frac{1}{i\xi_1} \delta(\xi_2)$$

$\Rightarrow \Gamma(u) = \left\{ \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} : \xi_1 \in \mathbb{R} \setminus \{0\} \right\}$

$$\widehat{\phi u}(\xi) = \hat{\phi} * \hat{u}(\xi)$$

$\Rightarrow \Gamma_x(u) \subseteq \Gamma(u)$



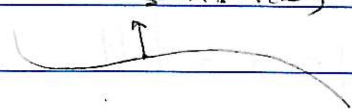
Ex. $u = \delta(x) = \delta(x_1)\delta(x_2)$ $\qquad x \in \mathbb{R}^2$

$$WF(u) = \left\{ \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right) : \xi_1, \xi_2 \in \mathbb{R} \setminus \{0\} \right\}$$

Because
- $\phi u = 0$ when $\phi(0) = 0$
  $\Rightarrow$ singsupp $(u) = \{0\}$.
- $\hat{u}(\xi) = 1$
  $\Rightarrow \Gamma(u) = \mathbb{R}^2 \setminus \{0\}$.
- $\phi u(x) = \phi(0)\delta(x)$
  $\widehat{\phi u}(\xi) = \phi(0)$
  $\Rightarrow \Gamma_x(u) = \mathbb{R}^2 \setminus \{0\}$ when $\phi(0) \neq 0$

Ex. $u = H(\psi(x))$ $\qquad \psi$ = level set function, assume $\nabla\psi \neq 0$
$\qquad H$ = Heaviside

$\xi = \lambda\nabla\psi(x)$



$$WF(u) = \{ (x, \lambda\nabla\psi(x)) : \psi(x) = 0, \ \lambda \neq 0 \}.$$

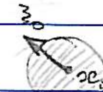
Break?

(Pf. deform to $H(x_1)$.)

Conclusion, $(x_0, \xi_0) \in WF(u)$ if $\langle e^{ix\xi_0}, \phi_{x_0} u \rangle$ is "large" when
- ϕ_{x_0} is loc. near x_0
- ξ_0 is very large

not $(1+|\xi_0|^2)^{-N}$ $\forall N$

ξ_0



Rmk. $\mathcal{F}V(r,t) = \int K(x,r,t)\, V(x)\, dx$

Check $\left\langle \phi_{\binom{r_0}{t_0}} e^{i(r\xi_r+\omega t)}, K\, \phi_{x_0} e^{ix\xi} \right\rangle$

$$= \left\langle \phi_{x_0}\phi_{\binom{r_0}{t_0}} e^{i(x\xi+r\xi_r+\omega t)}, K \right\rangle$$

$$= \mathcal{F}_{(x,r,t)}\left(\phi_{x_0}\phi_{\binom{r_0}{t_0}} K\right)(\xi,\xi_r,\omega)$$

Def. $C_\mathcal{F}$ = "set of $\left(x, \binom{r}{t}; \xi, \binom{\xi_r}{\omega}\right)$ s.t. the above is large"

$= WF(K)$

= "set of ( ) s.t. K is singular at $\left(x, \binom{r}{t}\right)$ in the direction $\left(\xi, \binom{\xi_r}{\omega}\right)$"

Rmk. Not a map, but a <u>relation</u>

Def. Composition of relations

$A \subseteq X$, $B \subseteq X \times Y$

$B \circ A = \{ y \in Y : \exists x \in A : (x,y) \in B \}$.

Thm. (Hörmander) Assume the elements of $C_\mathcal{F}$ obey $\xi = 0 \iff \binom{\xi_r}{\omega} = 0$. Then

$$\underbrace{WF(\mathcal{F}V)}_{\substack{\text{using}\\ \text{kernel } K}} \subseteq \underbrace{WF(K)}_{C_\mathcal{F}} \circ WF(V).$$

$$\underbrace{\qquad\qquad}_{\text{reflections}} \qquad \underbrace{\qquad\qquad}_{\substack{\text{canonical}\\ \text{reflection trf.}}} \qquad \underbrace{\qquad\qquad}_{\text{reflectors}}$$

Rmk. Simple geom optics $\Rightarrow \left\{ \begin{array}{l} C_F \text{ is a graph, } ((\vec{t})(\vec{z}, \vec{\omega})) = C_F(\vec{x}, \vec{\xi}), \\ C_{F^*} = C_F^{-1} \text{ on Ran } C_F, \end{array} \right.$

when $F =$ Kirchhoff modeling

$F^* =$ — migration

Then

$$WF(F^*FV) \subseteq WF(F^*) \circ WF(FV)$$
$$\subseteq WF(F^*) \circ WF(F) \circ WF(V)$$
$$\subseteq WF(V)$$

$\Rightarrow$ The normal operator F^*F does not move singularities. "like the identity".

Def. An operator T s.t. $WF(Tu) \subseteq WF(u)$ is called a microlocal operator

Rmk. The normal operator needs not be invertible

Inclusion but not identity of WF sets:

Imaging but not (accurate) inversion

Idealized (linearized) migration problem: given d, find $\tilde{V}$: $WF(\tilde{V}) \subseteq WF(V)$.

Next:
- Diff. op. are microlocal.
- ΨDO are microlocal
- The normal op. is —
- Change of variables for WF sets. $\rightarrow$ cotang. bundle
- Can. rel. preserves areas $\rightarrow$ symplectic

12/03/09

High-frequency imaging

⚠ Simple geom. optics

$$WF(d) = WF(FV) \subseteq WF(F) \circ WF(V) \qquad V \xrightarrow{F} d$$

$$WF(F^*d) = \boxed{WF(F^*FV) \subseteq WF(F^*) \circ WF(F) \circ WF(V)}$$

$$\boxed{\subseteq WF(V)}$$

$$WF(V) \underset{WF(F^*)}{\overset{WF(F)}{\rightleftarrows}} WF(d)$$

Q. $WF(F^*FV) \subseteq WF(V)$.

(conditions under which)

if Hörmander applies, and if $C_{F^*} = C_F^{-1}$

Can write kernel of F^*F as (change var.)

$$N(x,y) = \int e^{i(x-y)\cdot\xi}\, a(x,\xi)\, d\xi. \qquad \simeq \delta(x-y)$$

a ↳ smooth $\qquad \simeq 1$

(singular at $x=y$)

$$F^*F\, V(x) = \int N(x,y)\, V(y)\, dy$$

$$= \iint e^{i(x-y)\cdot\xi}\, a(x,\xi)\, V(y)\, dy\, d\xi$$

$$= \int e^{ix\cdot\xi}\, a(x,\xi)\, \hat{V}(\xi)\, d\xi. \qquad \simeq V(x)$$

→ called a pseudodifferential operator.

Ex. $a(x,\xi) = (2\pi)^{-n} \sum_{|\alpha| \le N} a_\alpha(x)(i\xi)^\alpha \qquad \alpha = (\alpha_1, \dots, \alpha_n)$

(polynomial of $i\xi$)

$$(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n}$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n$$

N = degree of the polynomial

Then $\int e^{ix\cdot\xi} (2\pi)^{-n} \sum_\alpha a_\alpha(x)(i\xi)^\alpha \hat{V}(\xi)\, d\xi$

$= \sum_\alpha a_\alpha(x) \, (2\pi)^{-n} \int e^{ix\cdot\xi} (i\xi)^\alpha \hat{V}(\xi)\, d\xi$

$= \sum_\alpha a_\alpha(x) \, \nabla^\alpha V(x)$

where $\nabla^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$

$\rightarrow$ differential operator.

<u>Thm</u> If $a(x,\xi) = \sum_\alpha a_\alpha(x)(i\xi)^\alpha$, $a_\alpha(x) \in C_0^\infty(\mathbb{R}^n)$

and $Tu(x) = \int e^{ix\cdot\xi} a(x,\xi)\, \hat{u}(\xi)\, d\xi$

then $WF(Tu) \subseteq WF(u)$, when $u \in C_0^\infty(\mathbb{R}^n)$

<u>Rem.</u> $\Gamma(u)$ = "directions of slow decay of $\hat{u}(\xi)$"

$\Gamma_x(u) = \lim_{j\to\infty} \Gamma(\phi_j u)$, $\operatorname{supp} \phi_j \to \{x\}$

$WF(u) = \{(x,\xi) : \xi \in \Gamma_x(u)\}$.

<u>Pf. of thm</u> Check $|\xi| \to \infty$ in

$T_j(\xi) = \int e^{-ix\cdot\xi} \phi_j(x)\, Tu(x)\, dx$

$= \int e^{-ix\cdot\xi} \phi_j(x) \sum_\alpha a_\alpha(x) \nabla^\alpha u(x)\, dx$

$= \sum_\alpha \int u(x) (-1)^{|\alpha|} \nabla^\alpha \left(e^{-ix\cdot\xi} \phi_j(x)\, a_\alpha(x) \right) dx$

$= \sum_\alpha \int u(x) \sum_\gamma \beta_{\alpha\gamma}(x)\, e^{-ix\cdot\xi}\, \xi^\gamma\, dx$

where $\beta_{\alpha j}(x) =$ li. co. of ϕ_j and a_α and their derivatives $\in C^\infty(\mathbb{R}^n)$.

$$\operatorname{supp} \beta_{\alpha j}(x) \subseteq \overline{\operatorname{supp} \phi_j(x)}$$

$$= \sum_\gamma \xi^\gamma \int e^{-ix\cdot\xi} \left[\sum_\alpha \beta_{\alpha\gamma}(x) \right] u(x)\, dx.$$

Fix $x \in \operatorname{supp}(u)$, let $\xi \notin \Gamma_x(u)$.

then $\exists\, j > 0 : \left| \int e^{-ix\xi} \left[\sum_\alpha \beta_{\alpha\gamma}(x) \right] u(x)\, dx \right| \leq C_M (1+|\xi|^2)^{-M} \quad \forall M$

$$\Rightarrow \left| \sum_{|\gamma| \leq N} \xi^\gamma \int (\quad - \quad) \right|$$

$$\leq \sum_{|\gamma| \leq N} C_M \xi^\gamma (1+|\xi|^2)^{-M} \quad \forall M$$

$$\leq C'_M (1+|\xi|^2)^{-M+N/2} \quad \forall M$$

$$\leq C'_{M+N/2} (1+|\xi|^2)^{-M}$$

$$\Rightarrow \xi \notin \Gamma_x(Tu)$$

Concl: $(x,\xi) \notin WF(u) \Rightarrow (x,\xi) \notin WF(Tu)$

$(x,\xi) \in WF(Tu) \Rightarrow (x,\xi) \in WF(u)$

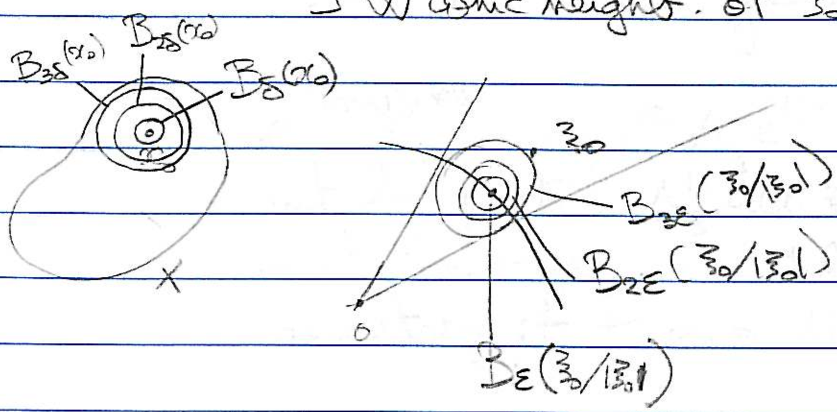
$WF(Tu) \subseteq WF(u)$

$\square$.

Thm Let $Tu(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\,d\xi$
with $|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|^2)^{(m-|\alpha|)/2}$
and a compactly supported in x.
Then $\forall u \in C_0^\infty(\mathbb{R}^n)$,
$$WF(Tu) \subseteq WF(u).$$

Ex. $T = F^*F$

Pf. Assume $(x_0,\xi_0) \notin WF(u)$, show $(x_0,\xi_0) \notin WF(Tu)$
$\Rightarrow$ $\exists X$ neighb. of x_0,
$\exists W$ conic neighb. of ξ_0,



$\widehat{\phi u}(\xi)$ decays fast for $\text{supp}\,\phi \subseteq X$, $\xi \in W$

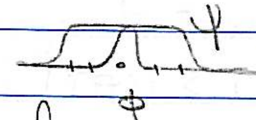
Let $\phi \in C_0^\infty(B_\delta(x_0))$, $\phi(x_0) \neq 0$.
$\psi \in C_0^\infty(B_{3\delta}(x_0))$, $\psi(x) = 1$, $x \in B_{2\delta}(x_0)$

Estimate $\widehat{\phi Tu}(\xi)$, $\xi \in B_\varepsilon(\xi_0/|\xi_0|)$
Write $u = (1-\psi)u + \psi u$
Claims: 1) $\phi T(1-\psi)u \in C^\infty$
2) $\widehat{\phi T \psi u}(\xi)$ decays fast.
for $\frac{\xi}{|\xi|} \in B_\delta\left(\frac{\xi_0}{|\xi_0|}\right)$

1) $\phi T(1-\psi)u(x)$

$$= \phi(x)\int e^{ix\xi}\, a(x,\xi)\int e^{-iy\xi}\,(1-\psi(y))\,u(y)\,dy\,d\xi$$

$$= \int d\xi \int dy\; a(x,\xi)\,\phi(x)(1-\psi(y))\,e^{i(x-y)\cdot\xi}\,u(y)$$



$|x-y| > \delta$ for $x \in \operatorname{supp}\phi$,
$y \in \operatorname{supp}(1-\psi)$

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2}\,\Delta_\xi\, e^{i(x-y)\cdot\xi} \qquad |x-y|^{-2} \text{ harmless}$$

int by parts: $|\Delta_\xi^M a(x,\xi)| \le (1+|\xi|^2)^{-M}$

$\to$ int. converges with M large enough.

Same story if P is a diff. op.

$P\phi T(1-\psi)u$ introduces powers of ξ in the integrand
$\to$ undone by $(1+|\xi|^2)^{-M}$

2). $$\widehat{\phi T\psi u}(\eta) = \iint a(x,\xi)\phi(x)\,\widehat{\psi u}(\xi)\,e^{ix\cdot(\xi-\eta)}\,dx\,d\xi$$

$\eta = \tau\theta'$, $\theta' \in B_\varepsilon(\xi_0/|\xi_0|)$

$\xi = \tau\theta$

$$\iint a(x,\tau\theta)\phi(x)\,\widehat{\psi u}(\tau\theta)\,e^{i\tau x\cdot(\theta-\theta')}\,dx\;\tau^n\,d\theta.$$

- Over $|\theta-\theta'| > \varepsilon$:

$$\tau^{-2}(\theta-\theta')^{-2}(-\Delta_x)\,e^{i\tau x\cdot(\theta-\theta')} = e^{i\tau x\cdot(\theta-\theta')}$$

6

Int. by parts: get τ^{-2M}

Δ_x on a and ϕ is fine

$\Longrightarrow$ fast decay in τ

- Over $|\theta-\theta'| \leq \varepsilon$, $\theta \in B_{2\varepsilon}(\xi_0/|\xi_0|) \subseteq W$

Since $X \times W$ is disjoint from $WF(u)$,

$$|\widehat{\psi u}(\tau\theta)| \leq C_N \tau^{-N}$$

$\longrightarrow$ fast decay in τ $\square$

Open problems:

① Compute F^*F from $a(x,\xi)$

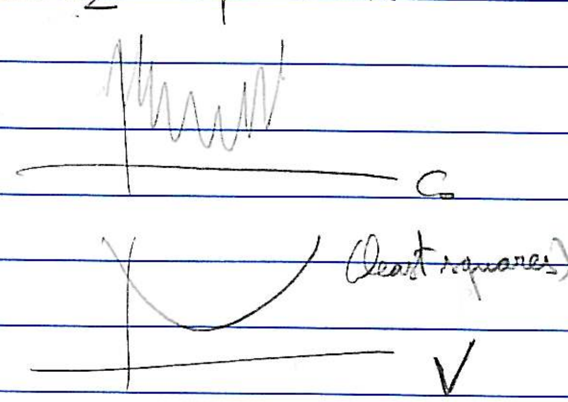
and $(F^*F+\delta)^{-1} u(x) = \int e^{ix\cdot\xi} b(x,\xi)\, \hat{u}(\xi)\, d\xi$

$\rightarrow$ get b from a numerically, in compressed form.

② Obtain c_0, model velocity estimation.

$$\min_{c_0, V} \| d - F^*[c_0] V \|_2 \quad \text{prestack}$$

$\rightarrow$



$\rightarrow$ new objectives

3 Compressed sensing:

$$f_s(x,t) \rightarrow F_j(x,t) = \sum_s a_{js} f_s(x,t)$$

$$d(r,s,t) \rightarrow D_j(r,t) \hat{=} \sum_s a_{js} d(r,s,t)$$

Recover V from D_j : not mere stack.
Use sparsity ideas:

$$\min \sum_k |\langle V, \psi_k \rangle| \text{ s.t. } \text{predicted} = \text{actual}_j$$