Homework problems, Nov 03 version

1. Consider the wave equation with $c = 1$. In this exercise we prove that waves cannot propagate at a speed faster than $c = 1$. Consider that the initial conditions u_0 and u_1 are zero inside of the ball $B(x_0, t_0)$ of center x_0 and radius t_0. We aim to show that $u(x, t) = 0$ inside the cone

$$C = \bigcup_{0 \le t \le t_0} B(x_0, t_0 - t)$$

   (a) Find the conserved energy E(t) for this wave equation. It is an integral of some quantity over space.

   (b) Consider this integral over $x \in B(x_0, t_0 - t)$ instead of the whole space, i.e., focus on the energy inside this ball at time t. Call this restricted energy $e(t)$. What is $e(0)$?

   (c) Find an expression for de/dt as an integral over the *boundary* $\partial B(x_0, t_0 - t)$, i.e., over the sphere of center x_0 and radius $t_0 - t$. [Hints: for any smooth family of volumes $V(t)$,

   $$\frac{d}{dt} \int_{V(t)} f(x, t) dx = \int_{\partial V(t)} f(x, t) dS_x + \int_{V(t)} \frac{\partial f}{\partial t}(x, t) dx.$$

   Then use the wave equation, and integrate by parts.]

   (d) Perform the proper majoration to find that $de/dt \le 0$. [Hint: $2ab \le a^2 + b^2$. Think about how a normal derivative relates to the gradient.]

   (e) Argue that $e(t) = 0$ for all times $0 \le t \le t_0$.

2. Solve the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u = 0,$$

   by the method of characteristics, i.e., find an adequate change of the independent variables that simplifies this equation to the point that you can solve it explicitly. (Klein-Gordon is the equation that governs particles of spin 0 in quantum field theory.)

3. Calculate the Fourier transforms in x of the Green's functions $g(x, t)$ (for the wave equation) and $\phi(x, \omega)$ (for the Helmholtz equation). You may encounter distributions that require careful treatment; be as descriptive as possible.

4. Generalize the reflection and transmission coefficients to the case of plane waves in two spatial dimensions, and a linear interface. Formulate your result as a function of the angle of incidence θ that the wave number makes with the direction normal to the interface.

5. This problem involves a bibliographical search. Provide one proof that some scattering series akin to the Born series seen in class, converges under a weak scattering assumption. You may start with volume 1 of "Methods of modern mathematical physics" by Reed and Simon. Added hint: you can also look up the book by Colton and Kress, "Inverse acoustic and electromagnetic scattering theory".

6. This problem is not for the faint of heart, but if you need a challenge, here it is. In class we saw a perturbative result concerning the accuracy of the Born approximation under a weak scattering assumption. Strengthen this result in *some* direction of your choosing, perhaps by working with different norms of the reflectivity $V(x)$. You are allowed to make smoothness assumptions on the incident field.

7. The treatment of reverse-time migration seen in class involves data $u(r, t)$ for an interval in time t, and along the surface $z = 0$ in r. Consider instead the *snapshot* setup, where t is fixed, and there are receivers everywhere in the domain of interest. (So we have full knowledge of the wavefield at some time t.) Repeat the analysis of the imaging operator, adjoint to the forward operator that forms snapshot data from singly scattered waves. In particular, find what the adjoint-state wave equation becomes in this case.

8. We saw in class an argument of constrained optimization that fully recovered the adjoint-state method from equating the variations of a Lagrangian to zero. This argument used the time-dependent wave equation; adapt it to the Helmholtz equation instead.

9. (Stationary phase) Consider
$$I(\beta) = \int e^{i\beta\phi(x)} a(x) dx,$$
where ϕ is C^∞ and has a single stationary phase point: $\nabla\phi(x^*) = 0$. Assume that the Hessian of ϕ is nondegenerate at x^*. Look up[1] a formula that gives the leading asymptotic behavior of $I(\beta)$ as $\beta \to \infty$. Make sure you understand and explain what all the quantities mean in this formula. Then reproduce a proof of this result. (Without proof, counts for a half exercise, with proof, counts for a full exercise).

10. We know that
$$(2\pi)^{-d} \int e^{i(x-y)\cdot k} dk = \delta(x - y),$$
and we'll try to get a feeling for this formula by stationary phase. The following steps require to have done the previous question.

   (a) Why does the phase $\phi(k) = (x - y) \cdot k$ NOT satisfy the assumptions of the previous question? (Nevertheless, the stationary phase condition $x - y = 0$ is quite useful as it corresponds precisely to the locus of singularity of $\delta(x - y)$.)

   (b) A direct application of the theorem will not work, but we can write
   $$\int \left[(2\pi)^{-d} \int e^{i(x-y)\cdot k} dk \right] f(x) dy = f(y).$$
   Now use the Fourier inversion formula $f(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} \hat{f}(\xi) d\xi$ and gather all the phase factors. We'll consider the stationary phase formula in the *pair* of variables (x, k), by keeping y and ξ as fixed parameters. Identify the stationary phase point(s) and argue that the assumptions for validity of the stationary phase formula are now satisfied.

   (c) Note that we don't really have a large β here, but that could be fixed by rescaling frequency $k = \beta k'$ and considering $\hat{f}$ compactly supported at large $|\xi|$. Notwithstanding, use the stationary phase formula in what remains of the integral above, and compare it to the result $f(y)$ that you are supposed to obtain.

---

[1] On the web or in a book, starting from "stationary phase" as a keyword. For a book, I like Elias Stein's "Harmonic Analysis", chapter 8

(d) Optional: what does the stationary phase condition tell you, physically?