Solutions pset 6

1 a) let ~~$\theta = \cos$~~ $x = \cos(\theta)$, then.

$$T_{k+1}(x) = T_{k+1}(\cos\theta) = \cancel{\$} \cos((k+1)\cos^{-1}(\cos\theta))$$
$$= \cos((k+1)\theta)$$
$$= 2\cos(\theta)\cos(k\theta) - \cos((k-1)\theta) \quad \text{(By trig identity.)}$$
$$= 2x\cos(k\cos^{-1}(x)) - \cos((k-1)\cos^{-1}(x))$$
$$= 2x\, T_k(x) - T_{k-1}(x).$$

$$T_0(x) = \cos(0) = 1, \quad T_1(x) = \cos(\cos^{-1}(x)) = x.$$

b) Well, T_0 and T_1 = polynomial so

$T_2(x) = 2x\, T_1(x) - T_0(x)$ is a polynomial

then so is $T_3(x) = 2x\, T_2(x) - T_1(x) \ldots$

and then $T_4, T_5, \ldots$

(making this precise with induction is fine).

(c)

$$f(x) = x^{1000} - x^{1002}$$

$$\frac{df}{dx} = 1000x^{999} - 1002x^{1001} = x^{999}(1000 - 1002x^{2})$$

max of $f \Rightarrow \frac{df}{dx} = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{\frac{1000}{1002}}$

$$\therefore f(x) = \left(\frac{1000}{1002}\right)^{500} - \left(\frac{1000}{1002}\right)^{501} \approx 0.000735.$$

If the difference is small, then not really a basis on a computer.

2. Just take equally spaced samples from $[0,3]$ instead of $[0,1]$. Now everything has integer number of cycles over the interval $[0,3]$. Now, do exactly as in class.

3a)

~~$\cos(\pi x) = 2 = \cos(7\pi$~~

Equally-spaced points ~~at~~ are $k/4$ $\quad k = 0,1,2,3,4$.

$$\cos(k/4\,\pi) = \cos(-k/4\,\pi) \qquad \text{because } \cos \text{ is even}$$

$$= \cos(-k/4\,\pi + 2k\pi) \qquad \cos \text{ is } 2\pi\text{-periodic}$$

$$= \cos(7k/4\,\pi)$$

(b) So $\cos(k/4\,\pi) = \cos(-k/4\,\pi + 2jk\pi)$ for any j

$$= \cos((8j-1)k/4\,\pi) \quad \text{any } j$$

also $\cos(k/4\,\pi) = \cos(k/4\,\pi + 2jk\pi)$ for any j.

$$= \cos((8j+1)k/4\,\pi).$$

$\therefore$ $M = \cancel{8}\,(2j - 1/4)$ or $(2j + 1/4)$ for any j.

(c) The chebyshev points of size 5.

d) If we have less than 2 points per wavelength, then we ~~could be~~ cannot tell the difference between the signal and a lower frequency signal.

4 a) I did not read them.

b)

$$\begin{bmatrix} \sin((k+1)\pi/(N+1)) \\ \sin(2(k+1)\pi/(N+1)) \\ \vdots \\ \sin(N(k+1)\pi/(N+1)) \end{bmatrix}$$

is an eigenvector for $k = 0, 1, \ldots, N-1$.

This is the k^{th} column of the DST matrix.

So $K_N S = S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$

By book $\lambda_n = 2 - 2\cos(n\pi/N+1)$.

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \qquad \boxed{K_n = SDS^{-1}}$$

eigenvalue decomposition

(c) Shifted by identity so

$$K_N + \frac{k^2}{2} I_N = S\left(D + \frac{k^2}{2} I_N\right) S^{-1}$$

eigenvalues are shifted by $k^2/2$.

(d) We want to solve

$$k^2 I_N + K_N X + X K_N = F$$

$$\Rightarrow \left(K_N + \frac{k^2}{2} I_N\right) X + X\left(K_N + \frac{k^2}{2} I_N\right) = F$$

$$\Rightarrow S\left(D + \frac{k^2}{2} I_N\right) S^{-1} X + X S\left(D + \frac{k^2}{2} I_N\right) S^{-1} = F$$

$$\Rightarrow \left(D+\tfrac{k^2}{2}I_N\right)S^{-1}XS + S^{-1}XS\left(D+\tfrac{k^2}{2}I_N\right) = S^{-1}FS$$

let $Y = S^{-1}XS$, ~~then~~ and $G = S^{-1}FS$, then

$$\Rightarrow \left(D+k^2/_2 I_N\right)Y + Y\left(D+k^2/_2 I_N\right) = G$$

so we have

$$\cancel{(\lambda_j + k^2/_2)(\lambda_k + k^2/_2)Y_j}$$

$$(\lambda_j + k^2/_2)Y_{jk} + Y_{jk}(\lambda_k + k^2/_2) = G_{jk}.$$

$$\therefore \quad Y_{jk} = \frac{G_{jk}}{\left[(\lambda_j + k^2/_2) + (\lambda_k + k^2/_2)\right]} \qquad \text{EASY.}$$

Finally, $\boxed{X = SYS^{-1}}$

<u>Cost</u>: If Sv is $O(N\log N)$ then SA, where A = matrix is $O(N^2\log N)$.

We are doing ~~#~~ $S{\cdot}A$ type operations $\Rightarrow O(N^2\log N)$ cost.