18.085 Computational Science and Engineering
Problem Set 6
Due in-class on 7th May 2015

Clarification required? Email `ajt@mit.edu`

1. (10 marks) Chebyshev polynomials are given by $T_k(x) = \cos(k \cos^{-1}(x))$.

   (a) Show that they satisfy the three-term recurrence:

   $$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \qquad k \geq 1,$$

   where $T_0(x) = 1$ and $T_1(x) = x$. (This means that they can be easily constructed.)

   (b) Use part (a) to show that $T_k(x)$ is a polynomial of degree k, despite not looking like one.

   (c) Calculate $\max_{x\in[-1,1]} |x^{1000} - x^{1002}|$. This difference is quite small, why does this matter for computations with the monomial polynomial basis?

2. (10 marks) Here is a sound wave:

   $$f(t) = \cos\left(\tfrac{8}{3}\pi t + \tfrac{1}{2}\right) + 2\cos\left(30\pi t + \tfrac{1}{2}\right).$$

   By only evaluating $f(t)$, describe how to **exactly** recover the amplitude, frequency, and phase. (Hint: There is no need to zero pad or use a window function. If the frequencies were all integers, then we know what to do.)

3. (10 marks) Nyquist's sampling rate tells us that we must sample a sound wave at, at least, two points per wavelength if we wish to exactly recover it. In this question we will see why.

   (a) Take 5 equally-spaced points on $[0, 1]$, i.e., 0, 1/4, 1/2, 3/4, and 1. Show that $\cos(\pi x)$ and $\cos(7\pi x)$ evaluate to the same values at these points.

   (b) For what other values of M does $\cos(M\pi x)$ evaluate to the same values as $\cos(\pi x)$ at 5 equally points.

   (c) Give one set of Chebyshev points such that $T_1(x)$ and $T_7(x)$ take the same values.

   (d) Give an argument why part (a) and (b) roughly show that at least 2 points per wavelength is required. (Any intuitive answer is fine here.)

4. (10 marks) Helmholtz equation on the unit square is a partial differential equation given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 1, \qquad (x,y) \in [0,1] \times [0,1].$$

where k is a fixed number (called the wavenumber).

We wish to solve this equation for $u(x,y)$ with conditions $u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0$. The $N \times N$ finite difference discretization of this equation is

$$\underbrace{\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}}_{K_N} X + X \underbrace{\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}}_{=K_N} + k^2 X = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \tag{1}$$

where

$$X = \begin{pmatrix} u(h,h) & \dots & u(h,(N-1)h) \\ \vdots & \ddots & \vdots \\ u((N-1)h,h) & \dots & u((N-1)h,(N-1)h) \end{pmatrix}.$$

(a) Read pages 58 and 59 from the course book.

(b) Explain why the eigenvalue decomposition of K_N is

$$K_N = SDS^{-1},$$

where S is the discrete sine transform matrix (see p. 58 of course book). What is D? (Hint: This question is just like Q3 from pset5.)

(c) Write down the eigenvalue decomposition for $K_N + (k^2/2)I_N$, where I_N is the identity matrix.

(d) [Hard] Describe a procedure to solve (1) in $\mathcal{O}(N^2 \log N)$ operations. (You may assume that Sv and $S^{-1}v$ can be computed in $\mathcal{O}(N \log N)$ operations, where v is a vector.)