18.085 Computational Science and Engineering
Problem Set 5
Due in-class on 30th April 2015

Clarification required? Email `ajt@mit.edu`

1. (10 marks) The FFT algorithm is based on the three facts from class. What are the three facts when $N = 3^k$? Explain how the facts give you a fast algorithm? (Hint: Break $p(z)$ into three pieces and understand how roots-of-unity look after cubing. Then, put everything together.)

2. (10 marks) [A fast polynomial multiplication algorithm.] Using FFTs of size N where N is a power of 2, explain how to multiply $q_1(z) = z + z^3$ and $q_2(z) = 1 + z + z^2$ together using the FFT. (Hint: It is easy to multiply together polynomials when you have the values of p and q on the same grid.)

3. (5 marks) Consider the $n \times n$ matrix C_n corresponding to the finite difference discretization of $-u''(x) = f(x)$ with $x \in [0, 1]$ and $u(0) = u(1)$.

   (a) Show that $v_k = (1, w^k, w^{2k}, \ldots, w^{(n-1)k})^T$ is an eigenvector of C_n, where $w = e^{2\pi i/n}$. What is the corresponding eigenvalue?

   (b) Write down the eigenvalue decomposition of C_n.

4. (15 marks) A doctor wants to model the concentration of testosterone, $f(t)$, in an athlete's blood over a one-day period. The doctor straps a small device around the athlete's wrist that is designed to take measurements at $t_0, \ldots, t_{N-1}$ ($t_n = n/N$) equally-spaced time points. The doctor plans to use the model

$$f(t) \approx p(t) = \sum_{k=0}^{N-1} c_k e^{2\pi i k t}, \tag{1}$$

   where the coefficients $c_0, \ldots, c_{N-1}$ are to be calculated so that $f(t_n) = p(t_n)$, i.e.,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} p(t_n) e^{-2\pi i k t_n}, \qquad 0 \le k \le N-1.$$

   (a) Show that $p(t + 1) = p(t)$. Why is $p(t + 1) = p(t)$ a reasonable assumption? Also, give an interpretation of c_0.

To the doctors disappointment the measurements were actually taken at $s_n = t_n + \epsilon_n$, where $|\epsilon_n| \leq 0.01N^{-1}$ is a small 1% error. We wish to design a fast algorithm for computing $c_0, \ldots, c_{N-1}$ from the non-uniform samples $s_1, \ldots, s_N$.

(b) What matrix-vector product do we want to calculate for the doctor?

(c) What is the Taylor series expansion of e^{-x} about $x = 0$? How many terms do you need to have an error of less than 10^{-3} when $|x| < 0.1$? By writing $e^{-2\pi i k s_n}$ as $e^{-2\pi i k t_n} e^{-2\pi i k \epsilon_n}$, approximate $e^{-2\pi i k s_n}$ by a Taylor expansion.

(d) [Start thinking very hard now] By employing the Taylor expansion in part (c) to every entry of the matrix in part (b), derive a fast algorithm for computing $c_0, \ldots, c_{N-1}$ to cheer up the disappointed doctor. You will need to very clearly explain why it is a fast algorithm. You do not need to explain how the FFT works, just how you can still exploit it. (You just made yourself a non-uniform FFT! People write research papers on this.)