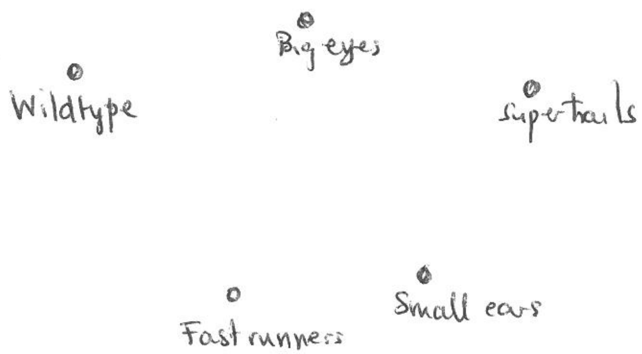
Problem Sheet 4

1. By the handshake lemma $\sum_{i=1}^{n} \deg(\text{vertex } i) = \text{even}$,

Here, thinking of this as the graph:

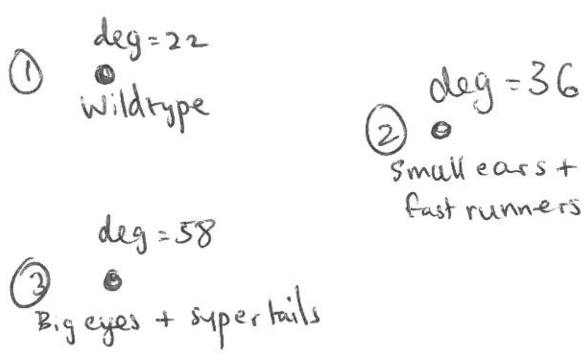


we have $\sum_i \deg(\text{vertex } i) = \text{odd} \Rightarrow$ false.

See breedingtriangles.m on course webpage for a solution to the 2nd part of Q1. The hint did not work out.

We can merge "Big eyes" and "super tails" as well as "Small ears" and "Fast runners" since there is no breeding between those families.

We have:



This gives us a 3×3 adjacency matrix $A = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$

A = symmetry
$\Rightarrow$ real eigenvalues.

2a)

$$A = \begin{array}{c} \\ u \\ v \end{array} \begin{array}{cc} u & v \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}} \end{array}$$

Because no edges between U and $\cancel{V}$ U OR V and V.

Moreover $A^T = A$, so $C = B^T$.

(b) $p(x) = \det(A - xI) = (-1)^{n+m}(x - \lambda_1) \cdots (x - \lambda_{n+m})$, where $n = |U|$, $m = |V|$.

If λ is an eigenvalue, then $\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. $\left(= \begin{bmatrix} B^T v_2 \\ B v_1 \end{bmatrix} \right)$

and

$$\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} -B^T v_2 \\ B v_1 \end{bmatrix} = \begin{bmatrix} -\lambda v_1 \\ \lambda v_2 \end{bmatrix} = -\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\therefore$ λ is an eigenvalue $\Leftrightarrow$ $-\lambda$ is an eigenvalue.

$\therefore$ Can pair up all non-zero eigenvalues as $(\lambda, -\lambda)$.

$$\Rightarrow \quad p(x) = (-1)^{n+m} x^k (x^2 - \tilde{\lambda}_1^2) \cdots (x^2 - \lambda_N^2)$$

$\therefore$ either $p(x) = -p(-x)$ if k is odd

or $p(x) = p(-x)$ if k is even.

(c) # closed walks of length k = $\sum_{k=1}^{n} \lambda_i(A)^k = 0$ if k = odd, because λ = eigenvalue then so is $-\lambda$.

$\cancel{= \sum_{k=1}^{p} \lambda_i(A)^k + \sum_{k=p}^{1}}$

(d) <u>TEST</u>: A graph G is ~~bag~~ bipartite $\Leftrightarrow$ eigenvalues of adjacency matrix are symmetric (i.e. if λ = eigenvalue then so is $-\lambda$).

This is a test because only bipartite graphs have this property. I'm going to just compute eigenvalues of A, order them and check if every λ has a corresponding $-\lambda$.

3) <u>Springs</u>: n = pins, m = springs, A = adjacency matrix

$$\begin{tikzcd}
x_1, \dots x_n \text{ (displacements)} \arrow[d, "\underline{e} = b - A\underline{x}"'] & & A^T w = f \text{ balancing equation} \\
e_1, \dots, e_m \text{ (elongations)} \arrow[rr, "w = C\underline{e}", "\text{HOOKE'S LAW}"'] & & w_1, \dots w_m \text{ (internal forces)} \arrow[u, "A^T w"']
\end{tikzcd}$$

$\underline{b}$ = elongation caused by external factors on the springs

$\underline{f}$ = external forces on the nodes.

Electrical networks

$n =$ electrical nodes

$M =$ wires.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
x_1 \dots x_n \text{ (potentials)} \arrow[d, "\underline{e} = \underline{b} - A\underline{x}"'] & A^T w = f \text{ (Kirchhoff's current law)} \\
e_1 \dots e_m \text{ (potential differences)} \arrow[r, "w = C\underline{e}", "\text{OHM'S LAW}"'] & w_1 \dots w_m \text{ (current along wires)} \arrow[u, "A^T w"']
\end{tikzcd}$$

$\underline{b}$ = potential difference caused by external factors such as a battery

$\underline{f}$ = external current on the nodes such as a current source.

Both case we have

$$-A^T C A \underset{\sim}{x} = f - A^T C b$$

letting ~~$g = A^T C g$~~ $f = A^T C g$ we have

$$A^T C A x = A^T C (b - g)$$

This is a weighted least square problem (weight normal equations)

4 a) $\mathbb{E}[\#\text{edges}] = n\frac{1}{2} = n/2$

$\mathbb{E}[\#\text{triangles}] = \binom{n}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{n(n-1)(n-2)}{48}$

(b) $P[$G has a probdematic ~~wor~~ k-subcircuit$]$

$$= 1 - P[\text{G does not have a problematic k-subcircuit}]$$

$$= 1 - \underbrace{\binom{n}{k}}_{\text{no. of k-subcircuits}} \cdot \underbrace{\left(\frac{1}{2}\right)^{\binom{k}{2}}}_{\text{chance of all wires the same}} \cdot \underbrace{2}_{\text{can be all wires or no wires}} = 1 - \binom{n}{k} 2^{1-\binom{k}{2}} > 0$$

If probability $> 0 \Rightarrow$ there is at least one graph that has no problematic k-subcircuits.

(c) $E[\text{connectivity}] = E[\text{connectivity due to } i \leftrightarrow j]$

$+ \sum_{\substack{s \neq i \\ s \neq j}} E[\text{connectivity due to } i \leftrightarrow s \leftrightarrow j]$

$$= \frac{0.9}{2} + (0.9)^2 \underbrace{(n-2)}_{\substack{s \neq i,\, s \neq j \\ \text{no of via} \\ \text{nodes}}} \cdot \underbrace{\frac{1}{4}}_{\substack{\text{both wire} \\ \text{in fact}}}$$

(d) It is unbounded.