

18.085 Computational Science and Engineering
Problem Set 4
Due in-class on 16th April 2015

Clarification required? Email ajt@mit.edu

1. (10 marks) Faking results is a big issue in experimental sciences. Here are some experimental results.

Five families of genetically modified mice were allowed to breed together. (There is no interbreeding within families.)

Genetic modification	Original breeding numbers	Corrected
Wildtype	23	22
Big eyes	46	46
Super-tails	12	12
Small ears	16	16
Fast runners	20	20

By using graphs rather than science, explain why the original breeding numbers are fake (or erroneously reported).

Later the results were modified. We are told that “Small ears” and “Fast runners” never breed together and “Wildtype” and “Super-tails” never breed together. We were also told that there were 86 breeding triangles among families, which we believe is in error. Using the corrected numbers, explain why the maximum possible number of triangles among families is 85? (You may use the fact that the maximum of $x^3 + y^3 + z^3$ subject to $x^2 + y^2 + z^2 = 116$ and $x + y + z = 0$ is 510.48 when x, y, z are real.)

2. (10 marks) Let G be a bipartite graph, i.e., its vertices can be separated into two disjoint sets U and V such that every edge connects a vertex in U to one in V . Let A be the adjacency graph.
(a) Show that there is a matrix B such that

$$A = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}.$$

(b) Show that the polynomial $p(x) = \det(A - xI)$ is an even function, i.e., $p(x) = p(-x)$ or an odd function, i.e., $p(x) = -p(-x)$.

(c) Suppose G is graph with adjacency matrix A so that if λ is an eigenvalue of A then so is $-\lambda$. Show that G has no closed walks of odd length.

(A graph with no closed walks of odd length is bipartite.)

(d) By using the eigenvalues of A , describe a test for a bipartite graph. Explain clearly why it is a test.

3. (10 marks) Concisely explain (perhaps with diagrams), the connections between the physics of spring networks and electrical circuits. In each case describe a physical interpretation of $e = b - Ax$, $w = Ce$, and $A^T w = f$. Explain the connection to the least squares problem.
4. (10 marks, challenging, encouraged to work in teams, write up individual answers) Let G be a circuit board with n electrical nodes where every node is connected by a wire to every other node. The manufacturer estimates that with probability $1/2$ a wire is faulty and cannot conduct electricity.
- (a) What is the expected number of wires in G that can carry current? What is the expected number of triangles that have intact wires?
- (b) The manufacturer calls a collection of k nodes a k -subcircuit. A k -subcircuit is *problematic* if each node is connected to every other node in the k -subcircuit (causing short-circuiting) OR if there are no working wires between any of the k nodes (causing poor connectivity of the circuit). Show that if¹

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1,$$

then there is a circuit board with n nodes that has no problematic k -subcircuit.

(c) Vertex i and j are now connected to a device. The manufacturer wants to estimate the electrical connectivity of the circuit board. If i and j are connected by a direct wire that contributes 0.9 to the connectivity, if there is a walk of length 2 that contributes $(0.9)^2$ to the connectivity. (Walks of longer length are not counted.) Calculate the expected value of this truncated Katz score as a measure of the expected electrical connectivity of the circuit.

(d) [Even harder, 2 bonus marks] Give a problem with the standard Katz score (i.e. if we allow walks of any length) as a measure of electrical connectivity in this example.

¹Here, $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the number of distinct ways to select k objects from a total of n objects.