Problem sheet 3

① A is of rank 1 because each column is of multiple of column 1.

$$A = \begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & -2 & \sqrt{2}\end{pmatrix}$$

$$Av = \begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & -2 & \sqrt{2}\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\3\end{pmatrix}(v_1 - 2v_2 + \sqrt{2}v_3) = (v_1 - 2v_2 + \sqrt{2}v_3)\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$\Rightarrow$ can compute Av using only 1 vector inner product.

$Av = 0$ if and only if $v_1 - 2v_2 + \sqrt{2}v_3 = 0$

so $v = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ is of the form

$$v = \alpha\begin{pmatrix}2\\1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\\sqrt{2}\end{pmatrix}$$ for any α, β.

Suppose $A = U\Sigma_1 V^T$ is the SVD of A.

Then $A^TA = V\Sigma_1^T\Sigma_1 V$ = eigenvalue decomposition.

~~If A = rank k $\Rightarrow A^TA$ = rank k~~ (since no of non-zero eigenvalues = rank

If A^TA = rank k $\Rightarrow$ A = rank k and no of non-zero Σ = rank.)

(2) $\det(A - xI) = \det\begin{pmatrix} 1-x & -1 \\ 1 & 1-x \end{pmatrix}$

$$= (1-x)^2 + 1$$

$$= +x^2 - 2x + 2$$

$$= +(x-(1+i))(x-(1-i))$$

$\lambda_1 = (1+i)$, $\lambda_2 = (1-i)$

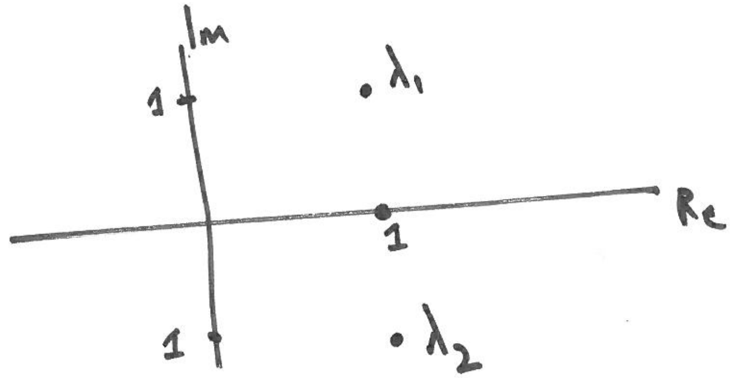
$$B(t) = \begin{bmatrix} (1-t)(1+i)+t & -t \\ -t & (1-t)(1-i)+t \end{bmatrix}, \quad D = \begin{bmatrix} (1+i) & 0 \\ 0 & (1-i) \end{bmatrix}$$

The eigenvalues of $B(t)$ are the roots of

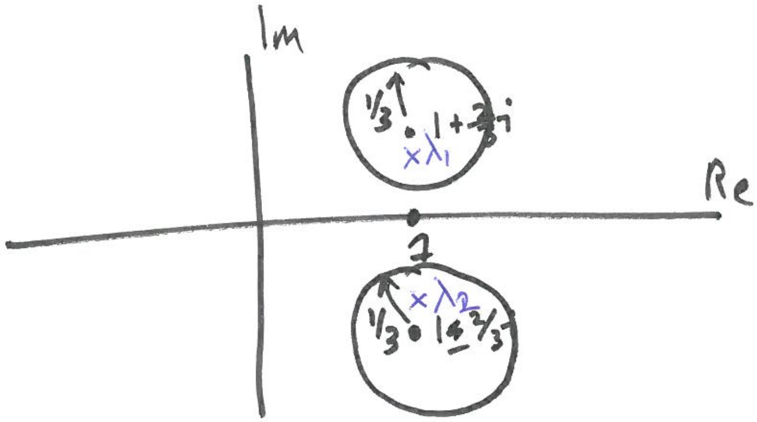
$$\det(B(t) - xI) = x^2 - 2x - 2t + 2$$

$$\Rightarrow \lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4 - 4(2-2t)}}{2}$$

<u>At $t=0$</u>: $B(0) = D$:



At $t = 1/3$: $B(\frac{1}{3}) = 2/3 D + \frac{1}{3} A = \begin{bmatrix} \frac{2}{3}(1+i)+\frac{1}{3} & -1/3 \\ -1/3 & \frac{2}{3}(1-i)+\frac{1}{3} \end{bmatrix}$



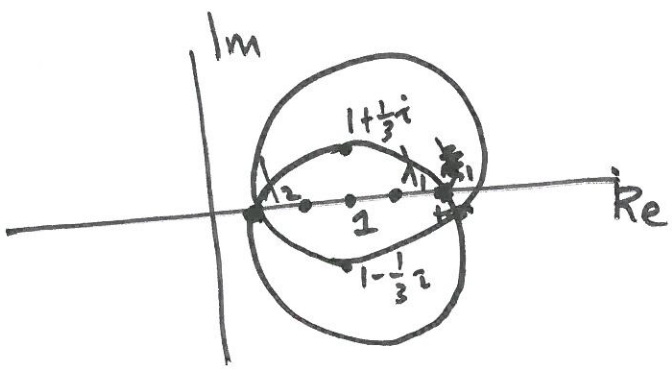
$\lambda_1 = 1 + \frac{\sqrt{3}}{3} i$

$\lambda_2 = 1 - \frac{\sqrt{3}}{3} i$

At $t = 2/3$: $B(2/3) = 1/3 D + 2/3 A = \begin{bmatrix} 1/3(1+i)+2/3 & -2/3 \\ -2/3 & \frac{1}{3}(1-i)+2/3 \end{bmatrix}$

$\lambda_1 = 1 + \frac{\sqrt{3}}{3}$

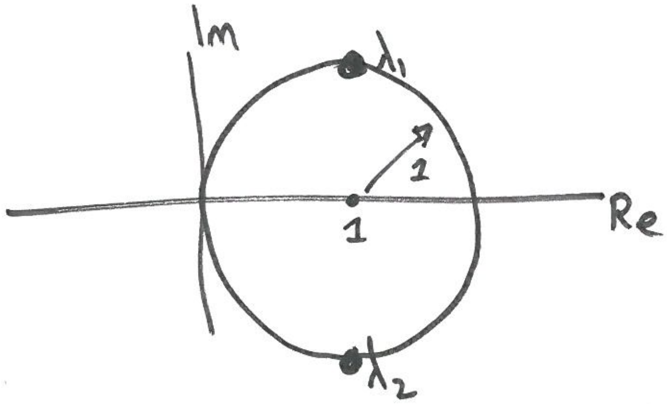
$\lambda_2 = 1 - \sqrt{3}/3$



At $t = 1$: $B(1) = A = \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}$

$\lambda_1 = (1+i)$

$\lambda_2 = (1-i)$



(3) $A = LL^T$ is +ve semi-def because

$$x^T L L^T x = (L^T x)(L^T x) = \|Lx\|_2^2 \geq 0$$

~~and if $x \neq 0$~~

(b) If $A = LL^T$, then

$$\begin{aligned} \det(A) &= \det(LL^T) \\ &= \det(L)\det(L^T) \\ &= \det(L)\det(L) \\ &= \det(L)^2 \end{aligned}$$

(c) $K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$

let ~~$x_1, x_2, x_3 \neq 0$~~:

$$\underset{\neq 0}{[x_1, x_2, x_3]} K_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1\, x_2\, x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2$$

$$= 2\left[x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right]$$

$$= 2\left[\tfrac{1}{2}x_1^2 + \left(\tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - x_1x_2\right) + \left(\tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2 - x_2x_3\right) + \tfrac{1}{2}x_3^2\right]$$

$$= 2\left[\tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_1 - x_2)^2 + \tfrac{1}{2}(x_2 - x_3)^2 + \tfrac{1}{2}x_3^2\right]$$

$$> 0$$

From class:

$$K_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}^2\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{3/2} & 0 \\ 0 & -\frac{2}{3}\sqrt{3/2} & \frac{2\sqrt{3}}{3} \end{bmatrix}}_{= L}\underbrace{\begin{bmatrix} \sqrt{2} & -\sqrt{2}/2 & 0 \\ 0 & \sqrt{3/2} & -\frac{2}{3}\sqrt{3/2} \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}}_{= L^T}.$$

(d) ① Calculate $A = LU$ using elimination.
(since A = positive definite, diagonal entries of U are +ve.)

② ~~Scale out~~ let $D = \text{diag}(U)$, D = diagonal with +ve entries

Set $L = LD^{1/2}$, $U = D^{1/2}U$.

[One may also show that $A = LU$ exists and that elimination (without pivoting) does <u>not</u> fail.]

(4)

$$\underbrace{C_1 = \begin{pmatrix} 0 & -1/2 \\ 1 & -2 \end{pmatrix}}_{\text{from class.}} \qquad \frac{1}{2}p(x) = x^2 + 2x + \frac{1}{2}$$

$$\det(C_1 - xI) = \frac{1}{2}p(x) \Rightarrow \text{eigenvalues} = \frac{\sqrt{2}}{2} - 2$$
$$= -\frac{\sqrt{2}}{2} - 2.$$

$$C_2 = C_1^T = \begin{pmatrix} 0 & 1 \\ -1/2 & -2 \end{pmatrix}$$ also <u>works</u>.

$$\det(P^{-1}C_2P - xI) = \det(P^{-1}C_2P - xP^{-1}P) \quad \text{as } P^{-1}P = I$$
$$= \det(P^{-1}(C_2 - xI)P)$$
$$= \det(P^{-1})\det(C_2 - xI)\det(P)$$
$$= \det(C_2 - xI) \qquad \text{as } \det(P^{-1}) = \frac{1}{\det(P)}$$
$$= \frac{1}{2}p(x)$$

<u>One way is:</u>

$$C = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} + \begin{pmatrix} -a_0 - 1 \\ -a_1 \\ \vdots \\ -a_{n-2} \\ -a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$$

If $C_0 = Q + uv^T, = Q_0 R_0$

Then $C_1 = R_0 Q_0 = Q_0^T Q_0 R_0 Q_0$ (as $Q_0^T Q_0 =$ identity

$$= Q_0^T C_0 Q_0$$

$$= Q_0^T (Q + uv^T) Q_0$$

$$= \underbrace{Q_0^T Q Q_0}_{\text{orthogonal}} + \underbrace{(Q_0^T u)(Q_0^T v)^T}_{\text{rank } 1}.$$