18.085 Computational Science and Engineering
Problem Set 3
Due in-class on 19th March 2015

Clarification required? Email `ajt@mit.edu`

1. (10 marks) Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 2 & -4 & 2\sqrt{2} \\ 3 & -6 & 3\sqrt{3} \end{pmatrix}.$$

What is the rank of A? Write down the matrix A as a low rank representation. How would you efficiently compute the the matrix-vector product Av using this representation? For what vectors v, do we have $Av = 0$?

Consider a general $m \times n$ ($m > n$) matrix A. Suppose $A^T A$ is of rank k, what is the rank of A?

2. (10 marks) Let A be the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Calculate the eigenvalues λ_1 and λ_2 of A. Set $D = \mathrm{d}iag(\lambda_1, \lambda_2)$ to be a diagonal matrix of eigenvalues and let $B(t) = (1-t)D + tA$, $0 \leq t \leq 1$. By considering $\det(B(t) - xI)$, calculate expressions for the eigenvalues of $B(t)$. Draw four diagrams: The Gerschgorin circles of $B(t)$ for $t = 0, 1/3, 2/3, 1$.

3. (10 marks) Let A be a $n \times n$ symmetric matrix.
(a) If $A = LL^T$ (L is lower-triangular, but not necessarily unit lower-triangular), then show that A is positive semi-definite.
(b) If $A = LL^T$, then what is $\det(A)$?
(c) Show that K_3 is positive definite. Calculate a matrix L (using elimination or otherwise) such that $K_3 = LL^T$.
(d) (Bonus part, useful, 2 marks) More generally, suppose that A is positive definite. Show that there is an L such that $A = LL^T$. (Hint: Given A is positive definite, find a procedure to calculate L such that $A = LL^T$.)

4. (10 marks) Let $p(x) = 2x^2 + 4x + 1$. Write down a matrix C_1 so that $\det(C_1 - xI) = \frac{1}{2}p(x)$. Calculate the eigenvalues of C_1. Find a different matrix C_2 so that $\det(C_2 - xI) = \frac{1}{2}p(x)$. Show that $\det(P^{-1}C_2P - xI) = \frac{1}{2}p(x)$ for any 2×2 invertible matrix P.

   Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The roots of $p(x)$ are given by the eigenvalues of

$$C = \begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}.$$

   Write $C = Q + uv^T$, where Q is orthogonal and uv^T is of rank 1. Show that this structure is preserved in the QR algorithm, i.e., if $C = C_0 = Q_0 R_0$ is orthogonal plus rank 1 then $C_1 = R_0 Q_0$ is orthogonal plus rank 1.

   (Bonus part, extremely hard, 5 marks) Can you use this to derive a fast way to calculate the roots of a polynomial?