

Problem Set 2

1.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 29 \end{pmatrix}$$

$$R^T R = \begin{pmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 29 \end{pmatrix}$$

Hence $A^T A = R^T R$.

The problem to solve is

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}_x = \underbrace{\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}}_b$$

Solving normal equations:

$$R^T x = Q^T b$$
$$\Rightarrow \begin{pmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 4\sqrt{3} \\ \frac{3}{2}\sqrt{2} \end{pmatrix}$$

\therefore ~~$c = -1/2$~~

$$\Rightarrow d = 3/2, \quad c = -1/2.$$

So best-fit line is $y = -\frac{1}{2} + \frac{3}{2}x$.

(1)

2. A is psychologically lower triangular. Thus

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 \\ \boxed{1} & 2 \\ 0 & 1 \end{pmatrix} \rightarrow \text{I want a zero here.}$$

$$\text{so } \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 3/2\sqrt{2} \\ 0 & \sqrt{2}/2 \\ 0 & \boxed{1} \end{pmatrix} \rightarrow \text{I want a zero here now.}$$

Very hard to figure out Q, but I only want R! $\|Qx\|_2 = \|x\|_2$ for any orthogonal matrix. Lets make

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \text{ then } R = \begin{pmatrix} \sqrt{2} & 3/2\sqrt{2} \\ 0 & \boxed{\sqrt{3/2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \text{length of } (\sqrt{2}/2) \\ 1 \end{pmatrix}$$

3. The LU decomposition of T_n is

$$T_n = \underbrace{\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & & -1 & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}}_U$$

First I solve

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow y_1 = \frac{1}{16}, y_2 = y_1 + \frac{1}{16} = \frac{1}{8}$$

$$y_3 = y_2 + \frac{1}{16} = \frac{3}{16}$$

Now solve

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/16 \\ 1/8 \\ 3/16 \end{pmatrix}$$

$$\Rightarrow x_3 = \frac{3}{16}, x_2 = x_3 + \frac{1}{8} = \frac{5}{16}$$

$$x_1 = x_2 + \frac{1}{16} = \frac{3}{8}$$

In general; $T_n = LU$

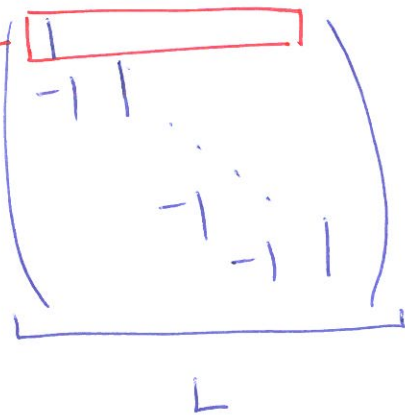
Solve $Ly = b$

(y_1, y_2, \dots) in order

Solve $Ux = y$

(x_n, x_{n-1}, \dots) in order

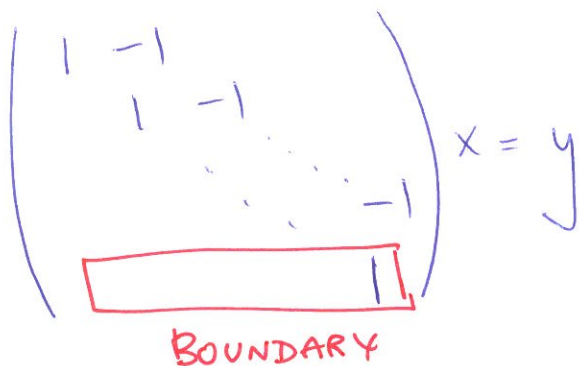
BOUNDARY ROW



$y = b$ is like solving
 $-\frac{du}{dx} = f$
 $u(0) = 0$

because

$$\frac{du}{dx} \approx \frac{u(x) - u(x-h)}{h}$$



$x = y$ is like solving
 $\frac{dv}{dx} = u$
 $v(1) = 0$

because

$$\frac{dv}{dx} \approx \frac{v(x+h) - v(x)}{h}$$

Therefore, Solve $Ly = b$, the $Ux = y$ is like doing

Set $v(x) = u'(x)$
Solve $-v'(x) = f(x)$ $v(0) = 0$
Now solve $u'(x) = v(x)$ $u(1) = 0$

THIS DOES SOLVE $-u''(x) = f(x)$, $u'(0) = 0$
 $u(1) = 0$

$$4. (a) \quad (Q_2^T Q_1)^T (Q_2^T Q_1) = Q_1^T \underbrace{Q_2 Q_2^T}_{=I} Q_1$$

$$\cong Q_1^T Q_1$$

$$= I$$

so $Q_2^T Q_1 = \text{orthogonal}$

$$(b) \quad Q_1 R_1 = Q_2 R_2 \Rightarrow Q_2^T Q_1 = R_2 R_1^{-1}$$

(because $Q_2^{-1} = Q_2^T$)

(c) $D = \text{upper-triangular and orthogonal.}$

$$\text{let } D = [d_1 | \dots | d_n]$$

$$d_1 = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } \|d_1\|_2 = 1 \Rightarrow d_1 = \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$d_1^T d_2 = 0 \Rightarrow d_{11} d_{21} = \pm d_{21} \Rightarrow d_{21} = 0$$

so $d_2 = \begin{pmatrix} 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Keep going... $\Rightarrow D = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \dots & \\ & & & \pm 1 \end{pmatrix}$

(d) $B = Q_2^T Q_1 = \text{orthogonal}$
 $= R_2^{-1} R_1 = \text{upper-triangular}$

so $B = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$

(e) ~~§~~ ~~§~~ I was given two QR's and showed
 that $Q_2^T Q_1 = \begin{pmatrix} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix} \Rightarrow Q_1 = Q_2 \begin{pmatrix} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix}$
 and $R_2^{-1} R_1 = \begin{pmatrix} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix} \Rightarrow R_1 = R_2 \begin{pmatrix} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix}$

~~§~~ I get to ~~select~~

Any $B = \begin{pmatrix} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix}$ gives me a QR

there are 2^n ~~choose~~ ways to pick \pm in B

$\Rightarrow 2^n$ QR factorizations for $n \times n$ matrix.