Problem Set 2

1.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 29 \end{pmatrix}$$

$$R^T R = \begin{pmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 29 \end{pmatrix}$$

Hence $A^T A = R^T R$.

The problem to solve is

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}}_{b}$$

Solving normal equations:

$$R x = Q^T b$$

$$\Rightarrow \begin{pmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 4\sqrt{3} \\ \frac{3}{2}\sqrt{2} \end{pmatrix}$$

$\therefore$

$$\Rightarrow \quad d = 3/2, \quad c = -1/2.$$

So best-fit line is $y = -\frac{1}{2} + 3/2\, X$.

2. A is psychologically lower triangular. Thus

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 \\ \boxed{1} & 2 \\ 0 & 1 \end{pmatrix}$$ I want a zero here.

so
$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,
$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \sqrt{2}/2 \\ 0 & \boxed{1} \end{pmatrix}$$ I want a zero here now.

Very hard to figure out Q, but I only want R! $\|Qx\|_2 = \|x\|_2$ for any orthogonal matrix. Lets make

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \text{ then } R = \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \boxed{\sqrt{3/2}} \\ 0 & 0 \end{pmatrix}$$

length of $\begin{pmatrix} \sqrt{2}/2 \\ 1 \end{pmatrix}$

3. The LU decomposition of T_n is

$$T_n = \underbrace{\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}}_{U}$$

First ~~$~~ I solve

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow y_1 = \frac{1}{16}, \quad y_2 = y_1 + \frac{1}{16} = \frac{1}{8}$$

$$y_3 = y_2 + \frac{1}{16} = \frac{3}{\cancel{8}16}.$$

Now solve

$$\begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/16 \\ 1/8 \\ 3/16 \end{pmatrix}$$

$$\Rightarrow x_3 = \frac{3}{16}, \quad x_2 = x_3 + \frac{1}{8} = 5/16$$

$$x_1 = x_2 + \cancel{5/16}\ 1/16 = 3/8$$

In general; $T_n = LU$

Solve $Ly = b$ $\quad (y_1, y_2, \ldots$ in order$)$

Solve $Ux = y$ $\quad (x_n, x_{n-1}, \ldots$ in order$)$

BOUNDARY ROW

$$\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & \ddots & \\ & & & -1 & 1 \end{pmatrix} y = b$$

L

is like solving

$$-\frac{du}{dx} = f$$

$$u(0) = 0$$

because

$$\frac{du}{dx} \approx \frac{u(x) - u(x-h)}{h}$$

$$\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix} x = y$$

BOUNDARY

is like solving

$$\frac{dv}{dx} = u$$

$$v(1) = 0$$

because

$$\frac{dv}{dx} \approx \frac{v(x+h) - v(x)}{h}$$

Therefore, Solve $Ly = b$, the $Ux = y$ is like doing

$$\boxed{\begin{array}{ll} \text{Set} & v(x) = u'(x) \\ \text{Solve} & -v'(x) = f(x) \quad v(0) = 0 \\ \text{Now solve} & u'(x) = v(x) \quad u(1) = 0 \end{array}}$$

THIS DOES SOLVE $-u''(x) = f(x)$, $u'(0) = 0$, $u(1) = 0$

4. (a) $(Q_2^T Q_1)^T (Q_2^T Q_1) = Q_1^T \underbrace{Q_2 Q_2^T}_{=I} Q_1$

$$= Q_1^T Q_1$$

$$= I$$

so $Q_2^T Q_1$ = orthogonal

(b) $Q_1 R_1 = Q_2 R_2 \Rightarrow Q_2^T Q_1 = R_2 R_1^{-1}$

(because $Q_2^{-1} = Q_2^T$)

(c) D = upper-triangular and orthogonal.

Let $D = [d_1 | \dots | d_n]$

$d_1 = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $\|d_1\|_2 = 1 \Rightarrow d_1 = \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$d_1^T d_2 = 0 = d_{11} d_{21} = \pm d_{21} \Rightarrow d_{21} = 0$

so $d_2 = \begin{pmatrix} 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Keep going.... $\Rightarrow D = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$.

(d)

$$B = Q_2^T Q_1 = \text{orthogonal}$$
$$= R_2^{-1} R_1 = \text{upper-triangular}$$

so $B = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$

(e) ~~B~~ I was given two QR's and showed that $Q_2^T Q_1 = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \Rightarrow Q_1 = Q_2 \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

and $R_2^{-1} R_1 = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \Rightarrow R_1 = R_2 \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

~~So I get to select~~

Any $B = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$ gives me a QR

there are 2^n ~~choose~~ ways to pick $\pm$ in B

$\Rightarrow 2^n$ QR factorizations for $n \times n$ matrix.