Solutions (by Alex Townsend)

Problem Set 1.

1. By Taylor

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2 u''(x) + \frac{8h^3}{6} u'''(x) + \cdots$$
$$u(x+h) = u(x) + h u'(x) + h^2/_2\, u''(x) + h^3/_6\, u'''(x) + \cdots$$

$$\therefore\ u(x+2h) - 2u(x+h) + u(x) = u(x) + 2hu'(x) + 2h^2u''(x) + \frac{8}{6}h^3 u'''(x) + \cdots$$
$$- 2u(x) - 2hu'(x) - h^2 u''(x) - 2h^3/_6\, u'''(x) - \cdots$$
$$+ u(x)$$
$$= 0u(x) + 0u'(x) + h^2 u''(x) + h^3 u'''(x) + \cdots$$

$$\therefore\ u''(x) \approx \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2} + \underbrace{h u'''(x) + \cdots}_{O(h)}$$

2. $$B_4 = \begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix}$$

<u>Elimination</u>:

$$\begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix} \xrightarrow{②+①} \begin{pmatrix} 1 & -1 & & \\ 0 & 1 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix} \xrightarrow{③+②} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} ④+③$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore\ \det(B_4) = 1 \cdot 1 \cdot 1 \cdot 0 = 0 \Rightarrow B_4$ is <u>not</u> invertible.

For any $n \geq 4$

$$B_{n+1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Therefore, columns are linearly dependent $\Rightarrow$ B_{n+1} is <u>not</u> invertible.

Suppose $u(x)$ is a solution to $-u''(x) = f(x)$, $u'(0) = 0$, $u'(1) = 0$.

Then $u(x) + c$, where c is a constant is also a solution. Since,

$$-(u(x)+c)'' = -u''(x) = f(x)$$

and $(u(x)+c)'\Big|_{x=0} = u'(0) = 0$, $(u(x)+c)'\Big|_{x=1} = u'(1) = 0$

3 a) let $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $B = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

LOTS OF OTHER EXAMPLES HERE

then $AB = \begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$, $BA = \begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$.

(b) $A = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, then $A^2 = -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(c) $B = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$ then $B^2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

4. K_n^{-1} is symmetric. First we seek a formula for K_{ij} where $i \geq j$.

From MATLAB experiments

$$(K_n^{-1})_{ij} = j(n-i+1)\,\det(K_n), \qquad i > j$$

$$= \frac{j(n-i+1)}{n+1}, \qquad i > j$$

Hence, by symmetry

$$(K_n^{-1})_{ij} = \begin{cases} \dfrac{j(n-i+1)}{n+1}, & i \geq j, \\[2ex] \dfrac{i(n-j+1)}{n+1}, & i < j. \end{cases}$$

To solve $K_n x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, note that

$$v = K_n^{-1}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1^{st} \text{ column of } K_n^{-1} = \frac{1}{n+1}\begin{pmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 2 \\ 1 \end{pmatrix}.$$

This just solved $-u''(x) = 0$, $u(0) = 1$, $u(1) = 0$. (see class 2 notes)

This has the solution $u(x) = 1-x$.

This corresponds to the discrete solution of

$$\frac{1}{n+1}\begin{pmatrix} n \\ n-1 \\ \vdots \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{n}{n+1} \\ \frac{n-1}{n+1} \\ \vdots \\ \frac{2}{n+1} \\ \frac{1}{n+1} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{n+1} \\ 1-\frac{2}{n+1} \\ \vdots \\ 1-\frac{n-1}{n+1} \\ 1-\frac{n}{n+1} \end{pmatrix}.$$

[You can also do this with delta functions]

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