$\boxed{1}$

(a) $c_{jk} = 2^{-p}$, where $p = \#$ of persons between person j and k.

This is because each communication loses a factor of $1/2$ and ~~th~~ this is multiplicative.

(b)
$$C = \begin{bmatrix} 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 & 1 \\ 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 & 1 \end{bmatrix}$$

(c) $(F^{-1}C)e_1 = (DF^{-1})e_1 = D(F^{-1}e_1)$

Now, ~~F~~ the 1^{st} column of F^{-1} is $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\Rightarrow (F^{-1}C)e_1 = D\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} =$ diagonal entries.

(d) Rank 1:
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

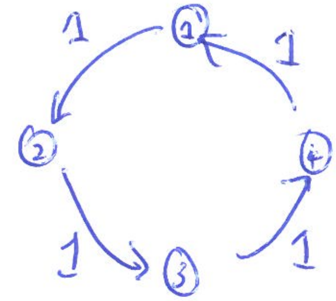
Rank 4
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank 2
$$\begin{bmatrix} 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 & 1 \\ 1 & 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 & 1 \end{bmatrix}$$

Rank 3:
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(e) Take $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (for example)

- ~~Everyone talk~~
  Corresponding to:



$\boxed{2}$

(a) $\sin(4\pi t) = \cos(4\pi t + \pi/2)$

$\therefore \quad f(t) = \cos(2\pi(2)\pi t + \pi/2) + \cos(\cancel{6\pi t}\ 2\pi(3)t + 1/2)$

$\therefore \quad A_2 = 1, \ a_2 = \pi/2, \ A_3 = 1, \ a_3 = 1/2.$

(b) ~~St~~ Nyquist's sampling rate: Need at least 2 pts per wavelength to exactly recover a signal.

(c) Points are $k/4 \quad 0 \le k \le 4$. AND cosine 2π-periodic:

so $\cos(6\pi k/4 + 1/2) = \cos(6\pi k/4 + 1/2 + 2\pi L k)$ for any integer k.

$$= \cos(2(3+4L)\pi k/4 + 1/2)$$

$\therefore \quad M = 3 + 4L$ for any integer L.

e.g. $M = -9, -5, -1, 3, 7, \ldots$

(d) $\sin(4\pi k/4) = \sin(k\pi) = 0 \quad 0 \le k \le 4.$

$$\therefore \quad f(k/4) = \cos(6\pi k/4 + 1/2) = \cos(-2\pi k/4 + 1/2)$$
$$= \cos(2\pi k/4 - 1/2)$$

Recover signal is $\cos(2\pi t - 1/2)$, i.e. $A_0 = 0, \ a_0 = 0$
$A_1 = 1, \ a_1 = -1/2.$

~~(d) Nyquist's sampling rate: Need at least two points samples per wavelength to exactly recover a signal.~~

e) Easy question : Since $f(x)$ is periodic, sample f at

k/N, $1 \le k \le N$ instead.

Procedure :

1 Sample f at ~~β~~ k/N, $1 \le k \le N$.

2 ~~Main~~ Apply FFT to the vector $f(1), f(1/N), f(2/N), \dots f(N-1/N)$.

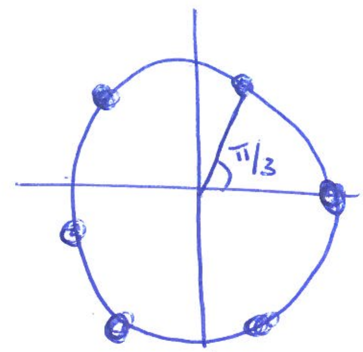
~~3~~ N must be large enough. Here, $N \geq 7$,

(anything more than 7 is fine)

$\boxed{3}$

Give the DFT for $N=6$:

$$X_k = \sum_{k=0}^{5} c_k \, e^{-2\pi i n k/6}, \qquad 0 \le k \le 5$$



$$w = e^{-2\pi i/6} = \frac{1}{2} + -\frac{\sqrt{3}}{2} i$$ so equivalent to X_0, X_1, X_2

In matrix form

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

DECOMPOSE POLYNOMIAL:

$$p_5(z) = \sum \quad c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5$$

$$= \underbrace{c_0 + c_2 z^2 + c_4 z^4}_{p_{even}(z^2)} + z\underbrace{(c_1 + c_3 z^2 + c_5 z^4)}_{p_{odd}(z^2)}$$

Equivalently, one could do:

$$p_5(z) = \underbrace{c_0 + c_3 z^3}_{p_1(z^3)} + z\underbrace{(c_1 + c_4 z^3)}_{p_2(z^3)} + z^2\underbrace{(c_2 + c_5 z^3)}_{p_3(z^3)}$$

MAIN IDEA: The DFT of size $N=6$ can be broken up into 2 DFTs of size $N=3$.

EQUIVALENTLY: THE DFT of size $N=6$ can be broken up into 3 DFTs of size $N=2$.

MAIN IDEA needs the fact that

$$\left(e^{-2\pi i n k/6}\right)^{2} = e^{-2\pi i n k/3}$$

ROOTS OF UNITY OF $N=3$ (REPEATED TWICE)

<u>EQUIVALENTLY:</u>

$$\left(e^{-2\pi i n k/6}\right)^{3} = e^{-2\pi i n k/2}$$

ROOTS OF UNITY of $N=2$ (REPEATED $3\times$)

$\Rightarrow$ $p_{even}(z_k^2)$, $p_{odd}(z_k^2)$, ~~$p\,z_k$~~ are DFTs

<u>Equivalently:</u> $p_1(z_k^3)$, $p_2(z_k^3)$, and $p_3(z_k^3)$ are DFTs

Full credit also if they go for the matrix factorization:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^8 & w^{12} & w^{16} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{\text{identity } 3\times 3}{I_3} & I_3 \\ I_3 & -I_3 \end{bmatrix}}_{\text{combine.}} \underbrace{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & w & \\ & & & & & w^2 \end{bmatrix}}_{Z.} \underbrace{\begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & w^2 & w^4 & & & \\ 1 & w^4 & w^8 & & & \\ & & & 1 & 1 & 1 \\ & & & 1 & w^2 & w^4 \\ & & & 1 & w^4 & w^8 \end{bmatrix}}_{\text{DFT of size } N=3}$$

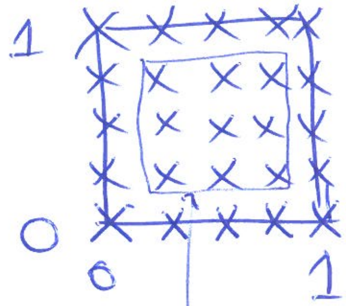
<u>EQUIVALENTLY:</u>

$$= \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & w^2 I_3 & w^4 I_3 \\ I_3 & w^4 I_3 & w^8 I_3 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & w & & \\ & & & & 1 & \\ & & & & & w^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & & & & \\ 1 & -1 & & & & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & & & & 1 & 1 \\ & & & & 1 & -1 \end{bmatrix}$$

(b) $N=7$ is a prime number $\Rightarrow$ no factorization.

$\boxed{4}$

(a)



5×5 grid.

Edge values are known to be 0.

X is 3×3 and contains the values of solution at these grid points.

(b) MX represents yu. So take y on the grid i.e. $1/4, 1/2, 3/4$ (excluding endpoints)

~~MX = va~~ If X = values of u on grid, then MX = values of yu on grid.

(c)
$$K_n X + X(SDS^{-1}) + MX = G$$
$$\therefore \quad K_n[XS] + [XS]D + M[XS] = GS \quad \text{(multiply } S \text{ on the right)}$$

Setting $Y = XS$ we have
$$K_n Y + YD + MY = GS$$
$$\Rightarrow \quad (K_n + M)Y + YD = GS.$$

(d)

$$YD = \begin{bmatrix} d_1 y_{11} & d_2 y_{12} & \cdots & d_n y_{1n} \\ d_1 y_{21} & d_2 y_{22} & & d_n y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 y_{n1} & d_2 y_{n2} & \cdots & d_n y_{nn} \end{bmatrix} = \left[d_1 y_1 \middle| \cdots \middle| d_n y_n \right]$$

$\therefore$ ~~$K_n + M$~~

$\therefore$ To solve for k^{th} column of Y we have

$$(K_n + M) y_k + d_k y_k = \underbrace{b_k}_{k^{th} \text{ column of } GS}$$

$$\therefore \underbrace{(K_n + M + d_k I_n)}_{\text{tridiagonal}} y_k = b_k .$$

(e)

1. Compute GS $\qquad O(n^2 \log n)$
2. Solve $\underbrace{(K_n + M + d_k I_n)}_{\text{tridiagonal}} y_k = b_k \quad 1 \le k \le n$ $\qquad O(n^2)$
3. Compute $X = YS^{-1}$ $\qquad O(n^2 \log n)$

$O(n^2 \log n)$