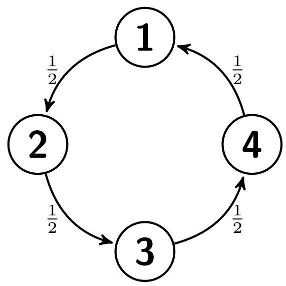
18.085 Computational Science and Engineering
Mid-term Quiz 3

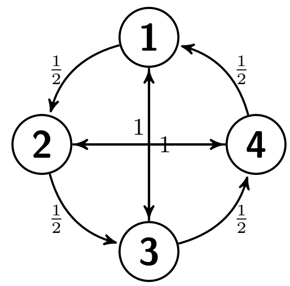
Tuesday, 12th May 2015, 2:35pm to 3:55pm

There are four questions each worth 20 marks. Please write your answer to each question in the space provided. You may use your class notes, the course book, or index cards. However, you may ***not*** *use WiFi-capable devices such as laptops, tablets, or cell phones.*

You may attempt as many questions as you like but ***only your three best answers will count.***

1. (20 marks) Four people are playing Chinese whispers (also called telephone) with four stories simultaneously. Person k invents a story S_k. Person k whispers his/her story to person $k+1$ (and person 4 whispers to person 1). Then, person k whispers his/her neighbor's story to person $k+1$, and so on; passing the four stories around the group.





Each time a person whispers a story to another, the meaning of the story halves. Let C_{jk} be the fraction that the jth person understands story S_k.

(a) Explain why (see left diagram)

$$C = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & 1 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 1 \end{pmatrix}.$$

(b) Persons 1 and 3 and persons 2 and 4 start passing notes between each other. No information is lost when passing notes. (See right diagram.) What is the resulting matrix C now?

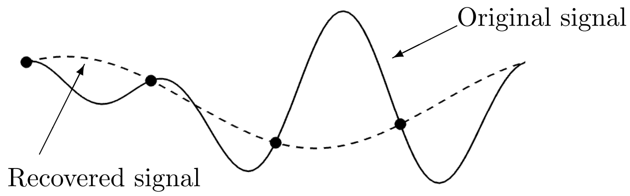
(c) Let C be a circulant matrix. We know that $C = FDF^{-1}$ where F is the DFT matrix and D is a diagonal matrix. Explain why the diagonal entries of D are $(F^{-1}C)e_1$, where $e_1 = (1, 0, \ldots, 0)^T$.

(d) Can a 4×4 circulant matrix be of rank 1? Either give an example or explain why it is not possible.

(e) Give a Chinese whispers scenario or explain why there is not one, where the resulting 4×4 matrix is circulant and of rank 1.

2. (20 marks) Here is a sound wave:

$$f(t) = \cos(6\pi t + 1/2) + \sin(4\pi t).$$



(a) Rewrite $f(t)$ so that $f(t) = A_2 \cos(2\pi 2t + a_2) + A_3 \cos(2\pi 3t + a_3)$. Give the constants A_2, A_3, a_2, and a_3.

(b) State Nyquist's sampling rate, i.e., roughly how many samples per wavelength are required to exactly recover a signal.

(c) For what integers M does $\cos(2\pi Mt + 1/2)$ take the same values as $\cos(6\pi t + 1/2)$ at the points 0, 1/4, 1/2, 3/4, and 1?

(d) Suppose we sample f at 0, 1/4, 1/2, and 3/4. Using part (b), explain what signal the FFT recovers? (Note: This is doing a FFT with $N = 4$ so the recovered signal takes the form $A_0 \cos(a_0) + A_1 \cos(2\pi t + a_1)$.)

(f) The signal has been corrupted at $t = 0$ and we can no longer sample f at $t = 0$. The signal $f(t)$ remains valid for $0 < t \leq 1$. Describe how one could exactly recover the signal $f(t)$ with samples from $[0, 1]$.

3. (20 marks)

(a) Describe the ideas behind the FFT algorithm for $N = 6$.

Here are some things to consider (any explanation will get credit):

- Give the DFT for $N = 6$.
- Draw the $N = 6$ roots-of-unity.
- Show how to decompose a polynomial of degree $N - 1 = 5$.
- Explain how to recombine results together for $N = 6$.

(b) Explain why these ideas do not work for $N = 7$.

4. (20 marks) Consider the following differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + yu = 1,$$

on $[0,1] \times [0,1]$ with boundary conditions $u(0,y) = u(1,y) = u(0,x) = u(1,x) = 0$. This can be discretized as

$$K_n X + X K_n + M X = G, \tag{1}$$

where $X_{jk} = u(jh, kh)$ for $1 \leq j, k \leq n$, M is a diagonal matrix, G is the matrix of all ones, and $h = 1/(n+1)$.

(a) Draw the square $[0,1] \times [0,1]$ and the discretization grid for $h = 1/4$.

(b) Explain why $M = \mathrm{diag}(1/4, 1/2, 3/4)$ when $h = 1/4$?

(c) We now set out to solve (2) using the FFT. Setting $K_n = SDS^{-1}$ show that (1) is equivalent to:

$$(K_n + M)Y + YD = GS, \tag{2}$$

where $Y = XS$.

(d) Explain why (2) can be solved by the following n linear systems:

$$(K_n + M + d_k I_n) y_k = b_k, \qquad 1 \leq k \leq n,$$

where I_n is the identity matrix, d_k is the kth diagonal entry of D, y_k is the kth column of Y, and b_k is the kth column of GS. (Hint: $YD = [\, d_1 y_1 \,|\, \cdots \,|\, d_n y_n \,]$.)

(e) Carefully describe a procedure for solving (1) using the FFT. What is the cost? (You may use the fact that Sv and $S^{-1}v$ cost $\mathcal{O}(n \log n)$ operations using the FFT when v is a vector and one can solve $Ax = b$ in $\mathcal{O}(n)$ cost using elimination when A is tridiagonal.)