Mid-term quiz 1

① The SVD of $A = U \Sigma_1 V^T$

A = general matrix ($m \times n$, $m > n$)

U = $m \times m$ orthogonal matrix

Σ_1 = $m \times n$ diagonal matrix
(diagonal entries are real and in ~~decrea~~ nonincreasing order)

V = $n \times n$ orthogonal matrix

Least squares problem can be solved by normal equation

$$\overset{m\times n}{A}\,\overset{n\times 1}{x} = b \xrightarrow[\text{by}]{\text{solved}} A^T A x = A^T b$$

NORMAL EQN

(1) is an SVD because U = orthogonal

Probably is needed:

~~NOT NEEDED FOR FULL MARKS~~

that is $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1$

$\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$

$\left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\sqrt{\frac{2}{3}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = 1$

$\frac{1}{\sqrt{3}} \cdot -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = 0$

$\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{6}} = 0$

$-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{6}} = 0$

$V =$ orthogonal. (~~clear~~ identity = orthogonal)

$\Sigma =$ diagonal and $\sqrt{3} > \sqrt{2}$.

$$A^T A = (U \Sigma V^T)^T (V \Sigma V^T) = V \Sigma^T \underbrace{U^T U}_{=I} \Sigma V^T$$

as $U =$ orthogonal

$$= V \Sigma^T \Sigma V^T$$

In our case $V = I$ so

$$A^T A = \Sigma^T \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \cancel{A^T \begin{pmatrix} c \\ d \end{pmatrix} =}$$

$$A^T \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$\therefore$ NORMAL EQN: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$$\Rightarrow c = 4/3, \ d = -1/2$$

LEAST SQUARES ERROR:

error at $(-1,2) = 1/6$

$(-1,2)$: $4/3 + \frac{1}{2} = 11/6$ ~~error at (-1,2)~~

$(0,1)$: $4/3 - \frac{1}{2} \cdot 0 = 4/3$ error at $(0,1) = 1/3$

$(1,1)$: $4/3 - \frac{1}{2} \cdot 1 = 5/6$ error at $(1,1) = \frac{1}{6}$

$$\therefore \|Ax - b\|_2^2 = (1/6)^2 + (1/3)^2 + \left(\frac{1}{6}\right)^2 \approx$$

Using calculator: $\|Ax - b\|_2 = 0.4082\ldots$

(2)

$$u(x+h) = u(x) + hu'(x) + 2hu''(x) + \dots$$
$$u(x-h) = u(x) - hu'(x) \cancel{+} - 2hu''(x) + \dots$$

$$\Rightarrow \quad \frac{u(x+h) - u(x-h)}{2h} = \cancel{2h}\frac{1}{2h}\left[2hu'(x) + 4hu''(x) + \dots\right]$$

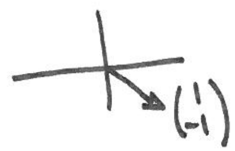
$$= u'(x) + 2hu''(x) + \dots$$

$$\therefore \quad u'(x) \approx \frac{u(x+h) - u(x-h)}{2h} + O(h)$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

<u>Step 1:</u> Put a zero in (2,1) entry of A_3. Rotate $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \quad \cancel{A_3} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>Step 2:</u> Put a zero in (3,2) entry.  , [Rotate 45° anticlockwise]

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

and
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & -\sqrt{2}/2 \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix}}_{\underline{\underline{R}}}$$

Set $f(x) = 1$

Since $u'(x) = 1$, $u(0) = 0$. $\Rightarrow$ $u(x) = x$

I guess
$$V = \begin{pmatrix} u(h) \\ \vdots \\ u(nh) \end{pmatrix} = \begin{pmatrix} h \\ \vdots \\ nh \end{pmatrix}$$

Check it is a solution:

eq 1 : $u(2h) = 2h = 2hf(h)$ ~~$= 2h$~~ ✓

eq 2 : $-u(h) + u(3h) = 2h = 2hf(2h)$ ✓

$\vdots$

eq $(n-1)$: $-u((n-2)h) + u(nh) = 2h = 2hf((n-1)h)$ ✓

eq n : $-u((n-1)h) + u(nh) = h = hf(nh)$ ✓

Since A_n is invertible $\Rightarrow$ v is only solution.

Bonus part: 1^{st} equation shows that $u(2h) = 2hf(h)$

3^{rd} equation show that $-u(2h) + u(4h) = 2hf(3h)$

If n = even: ① Solve for $u(2h), u(4h), \ldots, u(nh)$ by forwards subsitution.

② Use last equation to solve for $u((n-1)h)$.

③ Now solve for $u((n-1)h), u((n-3)h), \ldots, u(h)$ using back~~ward~~ substitution.

(3)

$A = LU$

$L =$ unit lower triangular ~~(L = unit lower tr~~

$U =$ upper triangular

$A = QR$

$Q =$ orthogonal

$R =$ ~~tri~~ upper triangular

Since L has ~~B~~ ones of diagonal (and A is invertible)

$A = LU$ is unique.

b) $B = UL = \underbrace{L^{-1}L}_{\text{identity}}UL = L^{-1}AL \Rightarrow$ A is similar to B.

(c) $D = L = Q =$ orthogonal and lower triangular (with ones on the diagonal)

$$D = \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ * & * & 1 & & \\ \vdots & \vdots & & \ddots & \\ & & & & 1 & \\ * & * & & & * & 1 \end{pmatrix}$$

2^{nd} from last col is orthogonal to last col $\Rightarrow$

$$D = \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ \vdots & * & 1 & & \\ \vdots & \vdots & & \ddots & \\ & & & & 1 & \\ * & * & \cdots & * & 0 & 1 \end{pmatrix}$$

3^{rd} from last col is orthogonal to 2^{nd} from last and last $\Rightarrow$

3^{rd} from last col of $D = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Repeating this argument we have

$$D = \text{identity} = L = Q.$$

$$\therefore A = U = R = \text{upper triangular}.$$

(d) $A = LU \Rightarrow A^T = U^T L^T$

but $U^T \neq$ unit lower triangular.

let $D = \text{diag}(U^T)$

$\Rightarrow U^T D^{-1} =$ unit lower triangular

so $A^T = \underbrace{(U^T D^{-1})}_{\text{unit lower triangular}} \underbrace{(D L^T)}_{\text{upper triangular}}.$

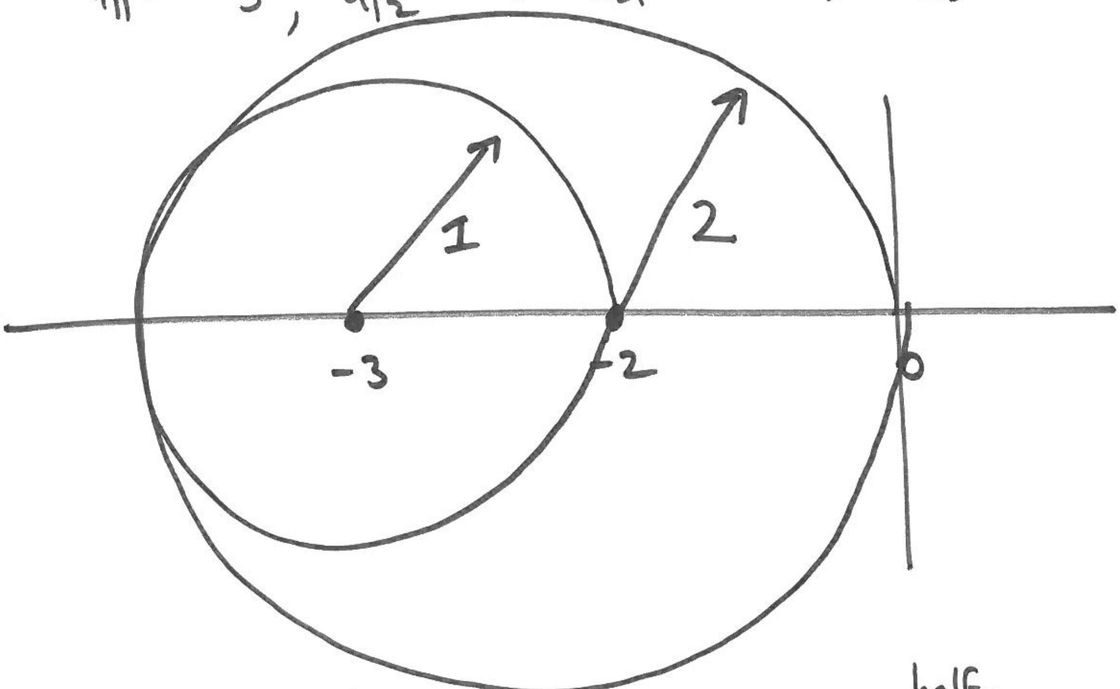
BONUS PART: If $A =$ upper-triangular

then $A = LU$, $L =$ identity.

and $A = QR$, $Q = \text{diag}(\pm 1\text{'s})$

so $L = Q \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

④ $a_{11} = -3$, $a_{12} = 1$ $a_{21} = 2$, $a_{22} = -2$



The circles are not in +ve real ~~part~~ half-plane $\Rightarrow$ eigenvalues ~~not strictly~~ has nonpositive real part.

~~Not~~ Not quite: $x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$y(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$

since ~~##~~ $Re(\lambda_1) \le 0$, $Re(\lambda_2) \le 0$ (the eigenvalues could have has a zero real part)

$\Rightarrow x(t) \not\to 0 \quad y(t) \not\to 0$ (perhaps)

$$\det(A - xI) = \det\begin{pmatrix} -3-x & 1 \\ 2 & -2-x \end{pmatrix} = (x+3)(x+2) - 2$$

$$= x^2 + 5x + 6 - 2$$

$$= x^2 + 5x + 4$$

$$= (x+1)(x+4)$$

$\Rightarrow$ eigenvalues of $A = -1$ and -4.

Now

$\underline{\lambda_1 = -1}$:

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2v_1 + v_2 &= 0 \\ 2v_1 - v_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} v_1 &= 1/2 \\ v_2 &= 1 \end{aligned}$$

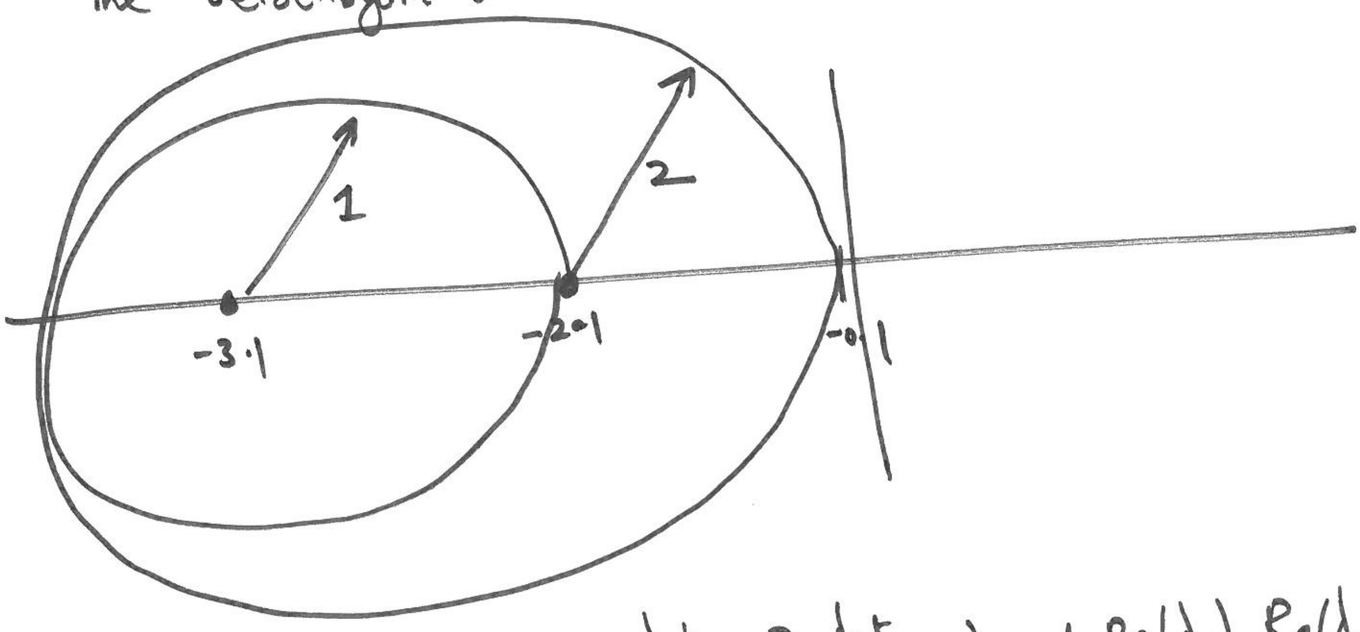
$\underline{\lambda_2 = -4}$

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4v_1 \\ -4v_2 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} v_1 + v_2 &= 0 \\ 2v_1 + v_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 1 \\ v_2 &= -1 \end{aligned}$$

<u>BONUS PART</u>:

The Gerschgorin discs are now:



So

$$\left.\begin{aligned} x(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ y(t) &= Ce^{\lambda_1 t} + De^{\lambda_2 t} \end{aligned}\right\} \text{ and } Re(\lambda_1), Re(\lambda_2) < 0$$

$\Rightarrow x(t), y(t) \to 0$ as $t \to \infty$.