18.085 Computational Science and Engineering
Mid-term Quiz 1

Thursday, 19th March 2015, 2:30pm to 4:00pm

There are four questions each worth 20 marks. Please write your answer to each question in the space provided. You may use your class notes, the course book, or index cards. However, you may ***not*** *use wifi-capable devices such as laptops, tablets, or cell phones.*

You may attempt as many questions as you like but ***only your three best answers will count.***

1. (20 marks) Here we show that the Least Squares problem can be solved using the SVD.

   By describing the structure of the matrices involved, what is the SVD of an $m \times n$ ($m > n$) matrix A? What are the normal equations?

   Let $(-1, 2)$, $(0, 1)$, and $(1, 1)$ be data that we wish to fit a line $y = c+dx$ to. That is, we want to solve the following linear system in the least squares sense:

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

   A factorization of the matrix A is given by

$$A = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{2/3} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{V^T}. \tag{1}$$

   Show why (1) is a SVD of A. Show that $A^T A = V^T \Sigma^T \Sigma V$. Find the best fit line, by solving $A^T Ax = A^T b$ with the SVD. Calculate the least squares error.

2. (20 marks) Show that

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + \mathcal{O}(h^2),$$

where h is small. An $n \times n$ discretization of the differential equation $u'(x) = f(x)$ on $[0,1]$ with $u(0) = 0$ is

$$\underbrace{\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & -1 & 1 \end{pmatrix}}_{A_n} \begin{pmatrix} u(h) \\ u(2h) \\ \vdots \\ u((n-1)h) \\ u(nh) \end{pmatrix} = \underbrace{\begin{pmatrix} 2hf(h) \\ 2hf(2h) \\ \vdots \\ 2hf((n-1)h) \\ hf(nh) \end{pmatrix}}_{b},$$

where $h = 1/n$. Calculate the matrix R in a QR factorization of A_3. By setting $f(x) = 1$, what is the solution to $A_n v = b$?

Explain how one can solve $A_n v = b$ immediately by substitution, i.e., no elimination is required.

3. (20 marks) Let A be an $n \times n$ invertible matrix. Suppose $A = LU$ and $A = QR$ is a LU and QR factorization of A, respectively (L, U, Q, and R are square).
(a) Describe the structure of the matrices L, U, Q, and R. What is the standard choice of the diagonal entries of L to make $A = LU$ unique?
(b) Show that $B = UL$ is similar to A.
(c) If it happens that $L = Q$, then explain why A must be an upper-triangular matrix.
(d) Using the fact that $A = LU$, give the LU factorization for A^T.
(e) If A is upper-triangular then how are L and Q related? (Remember that Q is not unique in the QR factorization.)

4. (20 marks) Consider the following system of differential equations

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = 2x - 2y,$$

where $x(0) = 1$ and $y(0) = 1$. Writing this in matrix form, we have

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 1, \qquad y(0) = 1.$$

What are the values of a_{11}, a_{12}, a_{21}, and a_{22}? By drawing the two Gerschgorin circles, explain why the real part of the two eigenvalues of A are not positive. From the Gerschgorin circles, can we determine if $x(t) \to 0$ and/or $y(t) \to 0$ as $t \to \infty$ (explain your answer)? By calculating $\det(A - xI)$, what are the eigenvalues of A? Calculate the eigenvectors of A. Solve for x and y. You are told that a mistake was made and the original system of differential equations should have been

$$\frac{dx}{dt} = -3.1x + y, \qquad \frac{dy}{dt} = 2x - 2.1y, \qquad x(0) = 1, \qquad y(0) = 1.$$

Now does $x(t) \to 0$ and/or $y(t) \to 0$ as $t \to \infty$? Explain your answer.