Turning derivatives into differences

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Here is how to derive the first- and second-order difference formulas.

1 Forward difference

The Taylor series of $u(t+h)$ about t tells us that

$$u(t+h) = u(t) + hu'(t) + \frac{h^2}{2}u''(t) + \frac{h^3}{6}u'''(t) + \dots \tag{1}$$

Rearranging for $u'(t)$ we have

$$u'(t) = \frac{1}{h}\left[u(t+h) - u(t) - \frac{h^2}{2}u''(t) - \frac{h^3}{6}u'''(t) + \dots\right].$$

Taking the first two terms:

FIRST-ORDER FORWARD DIFFERENCE FORMULA:

$$u'(t) \approx \frac{u(t+h) - u(t)}{h}$$

Error $= \mathcal{O}(h)$

2 Backward difference

For the backward difference, consider Taylor series of $u(t-h)$ about t. We have

$$u(t-h) = u(t) - hu'(t) + \frac{h^2}{2}u''(t) - \frac{h^3}{6}u'''(t) + \dots \tag{2}$$

Rearranging for $u'(t)$ we have

$$u'(t) = \frac{1}{h}\left[u(t) - u(t-h) + \frac{h^2}{2}u''(t) - \frac{h^3}{6}u'''(t) + \dots\right].$$

Taking the first two terms:

FIRST-ORDER BACKWARD DIFFERENCE FORMULA:

$$u'(t) \approx \frac{u(t) - u(t-h)}{h}$$

Error $= \mathcal{O}(h)$

3 Central difference

Subtract equation (2) from (1). We obtain

$$u(t+h) - u(t-h) = 2hu'(t) + \frac{h^3}{6}u'''(t) + \dots$$

Rearranging for $u'(t)$ we have

$$u'(t) = \frac{1}{2h}\left[u(t+h) - u(t-h) + \frac{h^3}{6}u'''(t)\right].$$

Taking the first two terms:

FIRST-ORDER CENTRAL DIFFERENCE FORMULA:

$$u'(t) \approx \frac{u(t+h) - u(t-h)}{2h}$$

Error $= \mathcal{O}(h^2)$

4 Second order central difference

Take the Taylor series of $u(t+h)$ about t, which is given by

$$u(t+h) = u(t) + hu'(t) + \frac{h^2}{2}u''(t) + \frac{h^3}{6}u'''(t) + \frac{h^4}{24}u'''(t) + \dots$$

This can be rearranged to find an expression for $u''(t)$ as follows:

$$u''(t) = \frac{2}{h^2}\left[u(t+h) - u(t) - hu'(t) - \frac{h^3}{6}u'''(t) - \frac{h^4}{24}u''''(t) - \dots\right] \tag{3}$$

Now, take the Taylor series of $u(t-h)$ about t, which is given by

$$u(t-h) = u(t) - hu'(t) + \frac{h^2}{2}u''(t) - \frac{h^3}{6}u'''(t) + \frac{h^4}{24}u'''(t) - \dots$$

This can be rearranged for $u''(t)$ as follows:

$$u''(t) = \frac{2}{h^2}\left[u(t-h) - u(t) + hu'(t) + \frac{h^3}{6}u'''(t) - \frac{h^4}{24}u'''(t)\dots\right] \tag{4}$$

Add together (3) and (4). We obtain

$$2u''(t) = \frac{2}{h^2}\left[u(t+h) + u(t-h) - 2u(t) - \frac{h^4}{12}u''''(t) + \dots\right]$$

Thus, taking the first three terms:

SECOND-ORDER CENTRAL DIFFERENCE FORMULA:

$$u''(t) \approx \frac{u(t+h) - 2u(t) + u(t-h)}{h^2}$$

Error $= \mathcal{O}(h^2)$