THIRD PROBLEM SESSION EXERCISES

The purpose of these exercises is to provide a proof of the Shannon Sampling Theorem discussed in class. We introduce a few extra concepts first. For a given $h > 0$, which will be fixed throughout, let $x_j = hj$ for all integers $j$. The semidiscrete Fourier transform (SFT) of a function $u$ on $\mathbb{R}$ with sufficiently fast decay is defined to be

$$\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} u(x_j), \quad k \in [-\pi/h, \pi/h]$$

and the inverse semidiscrete Fourier transform (ISFT) is

$$u(x_j) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k) \, dk.$$ 

We will take as given that these two transforms are right and left inverses of one another if $u$ decays fast enough and $\hat{v}$ is sufficiently regular, although this can be an extra exercise if you want to be complete. We emphasize that we will consistently use the notation $\hat{u}$ for the continuous Fourier transform of $u$, which is

$$\hat{u}(k) = \int_{\mathbb{R}} e^{-ixk} u(x) \, dx,$$

while the SFT of $u$ is $\hat{v}(k)$.

(1) Let $q(x) = \frac{\sin(\pi x/h)}{\pi x/h}$.

Note that $q(x)$ is a smooth function. Let $\chi_{[-\pi/h, \pi/h]}$ be the indicator function of the interval $[-\pi/h, \pi/h]$. Show that the continuous Fourier transform of $q$ is given by

$$\hat{q}(k) = h \chi_{[-\pi/h, \pi/h]}(k).$$

(2) For $u$ decaying sufficiently fast, let $\hat{v}$ be the SFT of $u$. Show that $\chi_{[-\pi/h, \pi/h]}(k) \hat{v}(k)$ is the continuous Fourier transform of

$$p(x) = \sum_{j=-\infty}^{\infty} u(x_j) q(x - x_j).$$
(3) For \( u \) sufficiently regular and decaying sufficiently fast, \( \hat{u} \) the continuous Fourier transform of \( u \), and \( \hat{v} \) the SFT of \( u \), prove the Poisson summation formula

\[
\hat{v}(k) = \sum_{j=-\infty}^{\infty} \hat{u}(k + j \frac{2\pi}{h}), \quad k \in [-\pi/h, \pi/h].
\]

Hint: Start by taking the ISFT.

(4) Combine the previous parts to prove the Shannon Sampling Theorem stated in class:

**Theorem 1.** If \( u \) is band limited in the interval \([-\pi/h, \pi/h]\) (which means \( \hat{u} \) is supported in that interval) and decays sufficiently fast at \( \infty \) then

\[
u(x) = p(x)
\]

where \( p(x) \) is as defined in question (2). Also, if \( f \) and \( g \) both satisfy the same hypotheses then

\[
\int_{\mathbb{R}} f(x) \overline{g(x)} \, dx = h \sum_{j=-\infty}^{\infty} f(x_j) \overline{g(x_j)}.
\]

When \( u \) is not band limited it is possible to use the Poisson summation formula to characterize the error incurred when approximating \( u \) by \( p \) in terms of the regularity of \( u \). This error is known as aliasing and corresponds to the mistaken identification of higher frequency parts of the signal \( u \) as having lower frequencies because they are not sampled densely enough.