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1. Lurie's generalization of  
Goerss-Hopkins-Miller Thm

"Topological automorphic  
forms"

2. A moduli space to apply Lurie's thm

3. PEL Shimura varieties

4. The spectrum TAF

$E_\infty$  ring Spectrum  $E$  associated to 1 dim'l formal grp htng  
lying in formal completion of a group scheme s.t. localization is iso  
to product  $E_n^{\wedge h G}$

1	2	3
KO	TMF	TAF

Thm: (Goerss-Hopkins-Miller)

$\overline{M}$  = stack of generalized elliptic curves

Then given affine étale open  $\text{Spec } k \xrightarrow{c} \overline{M}$   
the problem of realizing an  $E_\infty$ -ring Spectrum s.t.

$$\pi_0 E = R$$

Formal group associated to  $E = \text{Spf } E^\infty \mathbb{C}P^\infty = \hat{C}$

has a solution in étale topology, i.e.  $\exists$  a sheaf  $\mathcal{O}_{\text{Top}}$

of  $E_\infty$ -ring spectra s.t.  $E = \mathcal{O}^{\text{top}}(\text{spec } R)$  is

weakly even periodic spectrum satisfying 1) & 2)

$\pi_* E = 0$  for  $* \text{ odd}$

$\pi_2 E$  is invertible  $\pi_* E \text{ mod } \pi_2 E$ ,  $\pi_2 E \otimes_{\pi_2 E} \pi_4 E \xrightarrow{\sim} \pi_{2+2} E$

Input for Lurie's generalization:

Local GHM:  $R$  perfect field,  $\text{char } p$ , formal group  $\mathbb{G}/R$  of finite height  $n$ . Then, there

exists a  $n$  weakly even periodic ring spectrum  $E_R$  s.t.

(1)  $\pi_0 E_R = \mathcal{O}_{\text{Def}_R}$  where  $\text{Def}_R$  is formal

Scheme representing deformations.  $= \mathcal{W}(R)[[u_1, \dots, u_{n-1}]]$

(2)  $\mathbb{G}_{E_R} = \text{universal def of } \mathbb{G}$

Def: A  $p$ -divisible group of ht  $n$  over a

scheme  $S$  is a sequence of commutative group schemes  $G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \dots$

$$\text{rk } G_i = p^{i^n}$$

$$0 \rightarrow G_i \rightarrow G \xrightarrow{p^i} G$$

The connected component of the identity is

a formal group  $G_{\text{for}}$

$$1 \leq \text{ht } G_{\text{for}} \leq n$$

Lurie's generalization: 1) Local ring with

residue field perfect of char  $p$

X locally Noetherian, Separated DM stack / Spec A

( $G$  =  $p$ -divisible group of const ht  $n$  and dim)

Over  $\mathbb{X}$

Given formal affine open  $\text{Spf } R \rightarrow \mathbb{X}_A^\wedge$ ,

problem of realizing an  $E^\infty$  ring spectrum

s.t.

$$(1) \pi_0 E_0 = R$$

$$(2) \Gamma_E = \text{Spf}(E^0(\mathbb{P}^\infty)) = f^*(G)$$

(formal group associated is correct)

has a sol'n in étale top i.e.  $\exists$  sheaf  $E_G$  of  $E_\infty$ -ring spectra s.t.  $E = E_G(\text{Spf } R)$  is weakly

even periodic spectrum satisfying (1)(2)

provided the following condition is satisfied:

For some étale cover  $T: X \rightarrow \mathbb{X}$  by sch  $X$

for every pt  $x \in X_{\text{pt}}^\wedge$ , the map

$$X_x^\wedge \xrightarrow{\cong} \text{Def } \pi^* G_x$$

Classifying def  $(\pi^* G)$  |  $X_x^\wedge$  of  $\pi^* G_x$

Need 2 conditions (1) objects that  $\mathcal{X}$  parameterizes

should have canonically attached p-divisible groups

(2) local geom of  $\mathcal{X}$  should correspond exactly to deformations of  $G$

- in dim'l abelian scheme  $A$  has a p-divisible group  $\{A[p^i]\}$  of ht  $2n$  and  $\dim n$

To achieve (1) need to cut down dim of  $p$ -divisible group to  $\dim l$ .

Approach of (Gorbunov - Mahowald - Ranicki)

is to assume there are endo's of  $A$   
Splitting of a 1-dim'l Summand  
canonically.

(2) Given def of 1-dim'l Summand,  
(an complete it to a def of  $A$ )  
Uniquely

$\frac{1}{2}$  of 2) is Serre-Tate Thm

PEL Shimura Varieties have this  
property

PEL Shimura Varieties.

Shimura Varieties are higher dim  
analogues of modular curves

P  
polanization

E  
Endomorphism  
Structure

L  
Level  
Structure

ht of summand provides 1 dim  
is n Summand of  $A(p)$

ensure summand  
controls whole  
of  $A(p)$

used to cut down ~~XX~~

Conn Comp of Moduli  
Stack from infinite  
to finite

Consider case where  $A$  has complex  
multip

choose an imaginary quadratic extension  
of  $\mathbb{Q}$  s.t.  $p$  splits in  $F$

$O_F$  its ring of integers

$$F \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$$

$\Rightarrow$  Have idempotent  $\wedge_{\text{not}} = 0, 1$

Complex conj takes  $e \mapsto -e$

Functors: loc Noeth  $\xrightarrow{\quad}$  groupoids  
schemes

$S^p F \mathcal{O}/p$

$X(X) = (A, \lambda, i, \text{level structure})$

abelian prime-to-  
scheme  $P$   
 $\dim n$  polarization

$$A \xrightarrow{\lambda} A^*$$

$$\downarrow \\ \text{Pic}^0 A$$

which is a

Symmetric isogeny <sup>from</sup>  
ker rk prime to p, anizing ample line bundle

$i : \mathcal{O}_F \rightarrow \text{End } A$

Conjugate commutes with  $\lambda$

Level structure given by

Orbit of  $s_{\text{thg}}$  under

$$K^0 \subseteq \text{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p^{\oplus n})$$

Morphisms are isos preserving level structure

$$(f^* \lambda_A f = m \lambda_A)$$

Endomorphism structure:

$\text{End } A[\tilde{p}]$  is  $p$ -complete so it's a  $\mathbb{Z}_p$ -alg

$$\mathcal{O}_F \xrightarrow{i} \text{End } A \longrightarrow \text{End } A(p)$$

$\downarrow$

$$\mathcal{O}_F \otimes \mathbb{Z}_p$$

112 idempotent  $\epsilon$

$$\mathbb{Z}_p \times \mathbb{Z}_p$$

Get  $\epsilon A(p)$  Assume this is  
dim 1

We don't yet know height

Polarization

$$\lambda : A(p) \xrightarrow{\sim} A^*(p) \quad (\text{A prime to } p)$$

$$\lambda e = (1 - e^\vee) \lambda$$

$\lambda$  is sum of

$$\varepsilon A(p) \cong (1-e^\vee) A^\vee(p) \cong ((1-\varepsilon) A(p))^\vee$$

$$(1-\varepsilon) A(p) = (\varepsilon A(p))^\vee$$

Rmk: Taking duals does not preserve dimension, but it does preserve ht.

\* ht is additive

$$\Rightarrow \text{ht } A(p) = \frac{1}{2}(2n) = n$$

Thm The functor  $\mathcal{X}$  is representable by a stack.

$\text{Sh}(K^p)$  of  $\dim n-1$  over

$\text{Spf } \mathbb{Z}_p$

• If  $K^p$  is sufficiently small, then representable by a quasi-proj scheme

Luit's

→  
—  
thm

Sheaf  $E_G$

Def TAF :=

$E_G(\text{Sh}(K^p)_p^\vee)$

For TMF,  $\pi_* E_G(U = \text{Spf } R) =$

$$\begin{cases} 0 & \text{if } * \text{ odd} \\ H^0(U, W_f^{*, \otimes}) & \text{for } * \text{ even} \end{cases}$$

and there is a similar descent  
SS

$\text{Sh}(K^p)^{(n)}(\mathbb{F}_p)$  = finite set  
of points <sub>hGal</sub>

Thm  $TAF_{K(n)} \cong \left( \prod_{E_n}^{h\text{Aut}(A, \lambda, i, n)} \right) \cup \left( \prod_{(A, \lambda, i, n)} \text{Sh}(K^p)^{(n)}(\mathbb{F}_p) \right)$